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Application of Expectation Maximisation Algorithm on Mixed Distributions

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Abstract

Mixed distributions are a statistical tool used for modelling a range of phenomena in fields as diverse as marketing, genetics, medicine, artificial intelligence, and finance. A mixture model is capable of describing a quite complex distribution of data, often in situations where a single parametric distribution is unable to provide a satisfactory result. The Expectation Maximisation (EM) algorithm is an iterative maximum likelihood method typically used to estimate parameters in incomplete data problems, such as mixtures. This thesis presents a thorough analysis of mixture modelling and estimation via the EM algorithm for normal, Weibull, exponential, gamma, loglogistic, and uniform component distributions. Full derivations of relevant EM equations are provided, including censored EM equations for exponential and Weibull component distributions.

Goodness-of-fit tests assess how well an hypothesised statistical model fits a set of observations. This thesis considers two goodness-of-fit testing frameworks, the first being formal hypothesis based testing, the second being model selection via information criteria. It has been empirically justified that critical values for Kolmogorov-Smirnov, Kuiper, Cramér-von Mises, and Anderson-Darling goodness-of-fit tests don't exhibit the same parameter independent properties as single distributions. Critical values are in fact parameter dependent, as well as being dependent on sample size, significance level, and truncation level. A comprehensive analysis is also provided of model selection via information criteria, for the Akaike information criterion, and Bayesian information criterion. Goodness-of-fit testing in this manner was found to be more appropriate for mixture modelling.

The work culminates with the application of previously discussed statistical methodology to an analysis of limit-order inter-arrival times, and mid-price waiting times on the London Stock Exchange. It is reasoned that censored mixtures which include a Weibull component most appropriately model this data.

Declaration

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name, in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name, for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint-award of this degree.

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Signature: .. .. Date: ......12-04-2019......

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Chapter 1

Introduction

Through experimentation and observation, scientists seek to better understand phenomena of interest. In the modern world, the requirement to accurately collect and analyse data is ever increasing. A diverse range of disciplines have therefore become heavily reliant on statistical analysis, some include; machine learning, engineering, business, medicine, and finance. Critical to scientific research in such fields, statistical models are often employed as an analysis tool. These models can be used to describe stochastic phenomena, which are defined by a degree of associated uncertainty. Regardless of the amount of information collected, the outcome of stochastic processes cannot be predicted with absolute certainty. For such systems, one can only specify probabilities of certain outcomes. The notion of randomness fits within the stochastic domain. On the other hand, phenomena with associated outcomes which can be determined with absolute certainty, are referred to as deterministic. **This thesis deals exclusively with stochastic processes**. Data is the primary resource employed by scientists who endeavour to extract information and describe stochastic structures, so that predictions regarding the future behaviour of the system can be made. Statistical modelling can reveal relationships between aspects of stochastic phenomena, and reliably support inferences about unknown aspects of the structure. Recent developments in vogue fields such as machine learning and modern computing have exacerbated the requirement for statistical analysis to be undertaken, and modelling to be well understood.

1.1 Continuous Probability Distributions

The fundamental role of a statistical model is to describe the process in which the observed data is sampled. This is undertaken by use of probability distributions. Stochastic processes have probabilistic outcomes, and probability distributions are used to represent and assign each outcome a probability of occurrence. Such distributions describe the behaviour of a random variable, which takes values of specified outcomes of random experiments, on a given sample space. Examples of sample spaces include; the positive real numbers $\mathbb{R}^+$, i.e. when dealing with the height of members of a population, a set of discrete outcomes, i.e. $\{1, 2, 3, 4, 5, 6\}$ when considering rolling a die, or a non-numerical sample space, i.e. {heads, tails} when considering tossing a coin (although often non-numerical sample spaces are mapped to numerical outcomes). These examples allude to the existence of two categories of probability distributions, namely, continuous and discrete. As the name suggests, discrete probability distributions describe the behaviour of discrete outcomes. A random variable is categorised as discrete if it has a finite, or countably infinite number of outcomes, each with an associated non-zero probability of occurrence. In comparison, continuous probability distributions describe continuous random variables, which have an uncountably infinite number of possible values. **This thesis deals exclusively with continuous probability distributions.** The distribution of probability in the continuous framework is defined by the probability density function, or more commonly the pdf, denoted as $f(x)$, for a corresponding random variable X. Because of the infinite number of possible outcomes, the probability that a continuous random variable takes an exact value is zero. Because of this, the value of the pdf at a specific point gives only the relative likelihood of the random variable taking that value, and no information on the absolute likelihood. Rather, integrating $f(x)$ between two points a and b ($a \leq b$) defines the probability that the random variable will take a value within the interval $[a,\, b]$,

$$\mathbb{P}(a \leq x \leq b) = \int_a^b f(x)\,dx \;. \tag{1.1}$$

From this definition, the pdf is non-negative everywhere, and the normalisation condition yields that the integral over the entire sample space must equal one. This is conceptualised to mean the random variable must take a value somewhere in the sample space Ω,

$$\int_\Omega f(x)\,dx \;=\; 1 \;. \tag{1.2}$$

The probability of random variable X taking any value less than or equal to x can be defined by the cumulative distribution function, or more commonly the cdf, denoted as $F(x)$. The cdf and pdf are related by

$$F(x) = \mathbb{P}\,(X \leq x) = \int_{-\infty}^x f(t)\,dt \qquad \forall \quad x \in \Omega \;. \tag{1.3}$$

From the normalisation condition, it can be shown that

$$\lim_{x \to +\infty} F(x) \;=\; 1 \;, \quad \text{and} \quad \lim_{x \to -\infty} F(x) \;=\; 0 \;,$$

the pdf also takes values on $\mathbb{R}^+$, hence the cdf is a non-decreasing function taking values on $[0, 1]$. The probability distributions of interest to this work are parametrised by a vector $\boldsymbol{\psi}$ that is confined to a known parameter space $\boldsymbol{\Omega}$, i.e. ($\boldsymbol{\psi} \in \boldsymbol{\Omega}$). A pdf including parametrisation will be denoted $f(\boldsymbol{x} \,|\, \boldsymbol{\psi})$. To find parameter estimates of a pdf, given a data sample, the associated pdf parameter vector must be identifiable, in the sense that distinct parameter values $\boldsymbol{\psi}$ determine a distinct density $f(\boldsymbol{x} \,|\, \boldsymbol{\psi})$. More formally, if parameter vector $\boldsymbol{\psi}$ for pdf $f(\boldsymbol{x} \,|\, \boldsymbol{\psi})$ is identifiable then

$$\boldsymbol{\psi}^1 \neq \boldsymbol{\psi}^2 \quad \Rightarrow \quad f(x \,|\, \boldsymbol{\psi}^1) \neq f(x \,|\, \boldsymbol{\psi}^2) \qquad \forall \;\; x \;. \tag{1.4}$$

Throughout this work repeated reference will be made to finding estimates for a specified parameter vector. We denote the estimate of a parameter vector for a given data sample $\boldsymbol{x}$, by $\hat{\boldsymbol{\psi}}$, for an associated pdf $f(\boldsymbol{x} \,|\, \hat{\boldsymbol{\psi}})$.

1.2 Finite Mixture Models

Finite mixture models are a class of statistical model that leverage *mixed distributions* to describe a given dataset. Often such datasets exhibit sub-populations within the overall population, making the use of a mixed probability distribution appropriate. For a random variable X with corresponding data vector $\boldsymbol{x} = (x_1, ..., x_n)$, a g-component mixed probability distribution (where $g \in \mathbb{Z}^+$) is defined by pdf

$$f(\boldsymbol{x} \,|\, \boldsymbol{\psi}) = \sum_{i=1}^{g} \pi_i f_i(\boldsymbol{x} \,|\, \boldsymbol{\theta}_i) \;, \tag{1.5}$$

where $f_i(\boldsymbol{x} \,|\, \boldsymbol{\theta}_i)$ defines the i^{th} component probability distribution, which is weighted by a mixture proportion π_i. Intuitively a mixed distribution is conceptualised as the "mixture" of a finite number of component probability densities, with contribution to the overall mixture quantified by a given mixture proportion. Normalisation requires that

$$\sum_{i=1}^{g} \pi_i = 1 \quad \Rightarrow \quad \pi_g = 1 - \sum_{i=1}^{g-1} \pi_i \;, \tag{1.6}$$

hence, the mixture proportion π_i represents the probability that a given datum $x_j \in \boldsymbol{x}$, for $j = 1, ..., n$, was sampled from, and corresponds to the i^{th} component of the mixture. The mixture model is parametrised by the vector $\boldsymbol{\psi} = (\boldsymbol{\pi},\, \boldsymbol{\xi})$, where $\boldsymbol{\pi} = (\pi_1, ..., \pi_{g-1})$ is a vector containing the g - 1 mixture proportions (or weights), and $\boldsymbol{\xi} = (\boldsymbol{\theta_1}, ..., \boldsymbol{\theta_g})$ contains the parameter vectors for each component $f_i(\cdot \,|\, \boldsymbol{\theta}_i)$ of the mixture, for $i = 1, ..., g$.

Data describing the height of members of a population is a typical example for motivating the use of mixture models. It is known that a single Gaussian distribution well describes the height of both the adult male and female sub-populations. From data obtained from the Australian Government Bureau of Statistics [1] these distributions are parametrised by $\mu_M = 175.6\,\text{cm}$ (mean parameter) and $\sigma_M = 7.42\,\text{cm}$ (standard deviation parameter) for men, and $\mu_F = 161.8\,\text{cm}$ and $\sigma_F = 7.11\,\text{cm}$ for women. The distribution of heights for the overall population can be described by a Gaussian mixed distribution, with two distinct mixture components. Mixture proportions $\pi_M = \pi_F = 0.5$ may be chosen because the percentage of males and females in the overall population is roughly even. From Fig. 1.1 it can be seen that the resultant distribution of height exhibits a characteristic flat region at highest probability density.

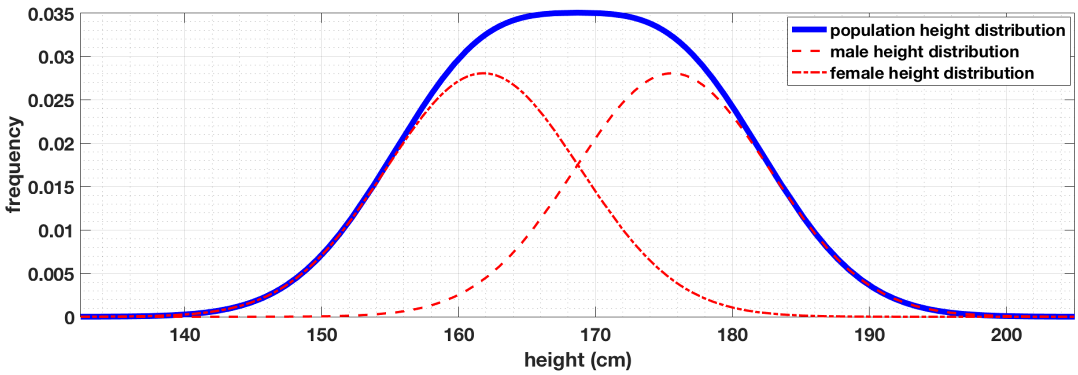


Figure 1.1: Two-component normal mixture model describing the height of the Australian adult population.

Generally, variations to the parameter vector of the component distributions can change the shape of the resultant mixed distributions, for example, if the mixture components were more highly separated, a dip would be evident in the middle region of the pdf, or if one mixture proportion dominated the second, the distribution would be skewed to one side. This thesis provides a thorough treatment of finite mixture models in Chapter 2.

1.3 Likelihood Function

The notion of likelihood is best introduced by comparison to the more familiar concept of probability. It is known that probability refers to the plausibility of a random variable taking a specific value, given a set of model parameters. Likelihood on the other hand describes the plausibility of a model taking certain parameters, given an observed dataset. Probability can be quantified by the pdf, $f(\boldsymbol{x}\,|\,\boldsymbol{\psi})$, which is a function of the data $\boldsymbol{x}$, for a fixed model parameter vector $\boldsymbol{\psi}$. The likelihood function, denoted $L(\boldsymbol{\psi}\,|\,\boldsymbol{x})$, is a function of the model parameters $\boldsymbol{\psi}$, for a fixed data sample $\boldsymbol{x}$. The likelihood function describes the "likelihood" of a model to take certain model parameters given the observed data. For this reason, the likelihood function is a fundamental concept central to much of statistical parameter estimation. Consider a data sample $\boldsymbol{x} = (x_1, ..., x_n)$, the likelihood function is defined as

$$L(\boldsymbol{\psi}\,|\,\boldsymbol{x}) = \prod_{j=1}^{n} f(x_j\,|\,\boldsymbol{\psi}) \ . \tag{1.7}$$

Because the data sample $\boldsymbol{x}$ is distributed identically and independently (i.i.d) and described by a random variable X, $\prod_{j=1}^{n} f(x_j\,|\,\boldsymbol{\psi})$ is the joint probability density function which describes the relative likelihood that data sampled according to the random variable X is given as $\boldsymbol{x}$. Commonly the parameter vector of a specific pdf, $f(\boldsymbol{x}\,|\,\boldsymbol{\psi})$ is fixed, usually known a priori, giving the interpretation of $\prod_{j=1}^{n} f(x_j\,|\,\boldsymbol{\psi})$ as the relative likelihood of a particular data sample $\boldsymbol{x} = (x_1, ..., x_n)$ being observed. However, it is equally valid to consider a sample of data which is observed, hence fixed, and for the function to be interpreted as the relative likelihood of the parameter vector taking particular values. This is what the likelihood function does.

1.4 Maximum Likelihood Estimation

As mentioned previously, data is the primary resource scientists have at their disposal to construct statistical models. Given a set of observed data, if not known *a priori*, one can begin to estimate the nature of the probability distribution describing the data. This is usually undertaken by postulating a probability distribution $f(\boldsymbol{x} \,|\, \boldsymbol{\psi})$, and finding a parameter vector estimate $\hat{\boldsymbol{\psi}}(\boldsymbol{x})$, given the data sample $\boldsymbol{x}$. Many estimation methods exist, including the "moment estimator", "bayes estimator", and the method of "maximum likelihood estimation". **This thesis deals exclusively with maximum likelihood estimation as a tool for parameter estimation**, because it has many desirable attributes [2–10]. Three of the most important properties, namely consistency, efficiency, and uniqueness, were proven by Lehmann and Casella [11]. Kizilersü et al. [5] presented the analogous proof for the incomplete-data case. The basic principle of maximum likelihood estimation (commonly abbreviated as MLE) is to select parameters that maximise the likelihood function given a set of data. That is, the parameter vector $\hat{\boldsymbol{\psi}}(\boldsymbol{x})$ which maximises the likelihood is the parameter combination with the greatest likelihood to have produced the data. The mathematical construction of maximum likelihood estimation is

$$\hat{\boldsymbol{\psi}}(\boldsymbol{x}) = \underset{\boldsymbol{\psi} \in \Omega}{\operatorname{argmax}} \; L(\boldsymbol{\psi} \,|\, \boldsymbol{x}) \tag{1.8}$$

$$= \underset{\boldsymbol{\psi} \in \Omega}{\operatorname{argmax}} \; \prod_{j=1}^{n} f(x_j \,|\, \boldsymbol{\psi}) \; . \tag{1.9}$$

This expression involves the product of n terms, which is both difficult to compute for large sample sizes, and difficult to work with numerically. By introducing the log-likelihood, many of these difficulties can be avoided,

$$\log L(\boldsymbol{\psi} \,|\, \boldsymbol{x}) = \log \prod_{j=1}^{n} f(x_j \,|\, \boldsymbol{\psi}) \tag{1.10}$$

$$= \sum_{j=1}^{n} \log f(x_j \,|\, \boldsymbol{\psi}) \; . \tag{1.11}$$

Because the natural logarithm is a monotonically increasing function, parameter vector $\hat{\boldsymbol{\psi}}(\boldsymbol{x})$ which maximises the likelihood function will also maximise the log-likelihood function. Hence, maximum likelihood estimation can be reformulated as

$$\hat{\boldsymbol{\psi}}(\boldsymbol{x}) = \underset{\boldsymbol{\psi} \in \Omega}{\operatorname{argmax}} \; \log L(\boldsymbol{\psi} \,|\, \boldsymbol{x}) \tag{1.12}$$

$$= \underset{\boldsymbol{\psi} \in \Omega}{\operatorname{argmax}} \; \sum_{j=1}^{n} \log f(x_j \,|\, \boldsymbol{\psi}) \; . \tag{1.13}$$

The first partial derivatives of $\log L(\boldsymbol{\psi} \,|\, \boldsymbol{x})$ with respect to each parameter $\boldsymbol{\psi} = (\psi_1, ..., \psi_m)$, evaluated at $\hat{\boldsymbol{\psi}}(\boldsymbol{x})$, will vanish. Hence, $\hat{\boldsymbol{\psi}}(\boldsymbol{x})$ is a solution to the maximum likelihood equations,

$$\frac{\partial}{\partial \psi_k} \log L(\boldsymbol{\psi} \,|\, \boldsymbol{x}) = \frac{\partial}{\partial \psi_k} \sum_{j=1}^{n} \log f(x_j \,|\, \boldsymbol{\psi}) = 0 \; , \qquad \forall \; k = 1, ..., m \; . \tag{1.14}$$

Parameter estimates are solutions to this set of simultaneous nonlinear equations. In some cases maximum likelihood equations exist in closed form and an analytical solution can be found. In other cases a numerical approach may be required to determine the parameter estimates. This may be an iterative scheme, a direct search for the global maximum of the log-likelihood function, or a numerical root finding algorithm like Newton-Raphson.

The conditions and assumptions which are required to be met such that the maximum likelihood estimator tends toward the true parameters efficiently for large sample sizes was outlined in [5,11].

1.5 Expectation Maximisation Algorithm

An example of an iterative method for computing maximum likelihood estimates is the Expectation Maximisation algorithm, or more commonly the EM algorithm, which is typically applied to incomplete-data problems involving missing or unobservable data [12, 13]. Consider an observed data vector $\boldsymbol{x} = (x_1, ..., x_n)$, corresponding to a proposed probability density function $f(\boldsymbol{x} \,|\, \boldsymbol{\psi})$, where $\boldsymbol{\psi} = (\psi_1, ..., \psi_m)$ is a vector of parameters restricted to the parameter space Ω, required to be estimated from the data. The observed data vector $\boldsymbol{x}$ is viewed as incomplete. **This thesis is concerned with incompleteness that manifests from mixed distributions**, where it is unknown which mixture component an individual datum belongs to. Let $\boldsymbol{y}$ denote the complete data vector, and $\boldsymbol{z}$ denote the additional unobserved or missing data. Let the probability density function for the complete data vector $\boldsymbol{y}$ be denoted $f_c(\boldsymbol{y} \,|\, \boldsymbol{\psi})$, and the corresponding log-likelihood function given as

$$\log L_c(\boldsymbol{\psi} \,|\, \boldsymbol{y}) = \log f_c(\boldsymbol{y} \,|\, \boldsymbol{\psi}) \; . \tag{1.15}$$

As the name suggests, the EM algorithm alternates between an expectation, or E step, and a maximisation, or M step, and approaches the problem of dealing with the incomplete log-likelihood function indirectly by iterating in terms of the complete log-likelihood function. As the missing data is unobserved, it is replaced by its conditional expectation given $\boldsymbol{x}$, in terms of the current estimate of the unknown parameters $\boldsymbol{\psi}$. Consider the iterative routine on the $(k+1)^{\text{th}}$ step defined as,

E-step: Formulates an expectation $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$ of the log-likelihood function by dealing with the conditional expectation of the unobservable data, given both the observed data, and the current estimate for the parameters.

Compute $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$, where

$$Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) = \mathbb{E}_{\boldsymbol{\psi}^{(k)}}[\,\log L_c(\boldsymbol{\psi} \,|\, \boldsymbol{y}) \,|\, \boldsymbol{x}\,] \; .$$

M-step: Selects updated parameters that maximise the expected log-likelihood calculated on the previous E step.

Choose $\boldsymbol{\psi}^{(k+1)} \in \Omega$ such that $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$ is maximised, that is;

$$Q(\boldsymbol{\psi}^{(k+1)} \,|\, \boldsymbol{\psi}^{(k)}) \,\geq\, Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) \qquad \forall \;\; \boldsymbol{\psi} \in \Omega \; ,$$

i.e.

$$\boldsymbol{\psi}^{(k+1)} = \underset{\boldsymbol{\psi} \in \Omega}{\operatorname{argmax}} \; Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) \; .$$

Iteration is continued until convergence of the log-likelihood is achieved. Chapter 3 presents the EM algorithm in significantly more depth, motivating the EM equations with specific application to mixture models.

1.6 Goodness-of-fit Testing

Goodness-of-fit testing is the process of assessing the quality in which a statistical model describes a given dataset. Typically a measure of the discrepancy between the observed values and the values predicted by the candidate model is used to quantify the level of goodness-of-fit of a model. This thesis considers two goodness-of-fit frameworks, namely hypothesis based tests, and information criteria tests.

1.6.1 Hypothesis Based Tests

Hypothesis testing is the process of utilising test statistics to determine whether a null hypothesis H_0 can be rejected, given a set of observations $\boldsymbol{x} = \{x_1, x_2, ..., x_n\}$. This work is concerned with a null hypothesis H_0 that a given set of n observations $\boldsymbol{x} = \{x_1, x_2, ..., x_n\}$ arise from a population which is distributed according to a cumulative distribution function $F(\boldsymbol{x} \,|\, \hat{\boldsymbol{\psi}})$. A goodness-of-fit test defines a test statistic, a real scalar measure, which quantifies the level of agreement of observation and model, therefore ultimately determines whether one can reject the null-hypothesis H_0 given a set of observations. A smaller test statistic corresponds to a stronger level of agreement between the dataset $\boldsymbol{x}$ and $F(\boldsymbol{x} \,|\, \hat{\boldsymbol{\psi}})$, and hence the prediction of the null hypothesis. To determine whether or not to reject the null hypothesis, a comparison of the test statistic to a previously known *critical* *value* is made. **If the test statistic is smaller than the critical value, one cannot reject the null hypothesis**. Section 4.1.7 provides a more in-depth description of critical values, and how they can obtained. Hypothesis based goodness-of-fit tests can be easily implemented if the critical values are known, or easily calculable. If critical values exhibit a strong degree of sensitivity to the distribution, parameter vector, sample size, and/or confidence level, then hypothesis based tests can be non-trivial.

1.6.2 Information Criterion Tests

An alternate goodness-of-fit framework is *model selection* via *information criteria test.* This approach to goodness-of-fit testing compares the relative quality of candidate models via comparison of log-likelihood values, but imposes a penalty consistent with the complexity of each model. Information criteria were popularised by Hirotugu Akaike [14] who developed the Akaike Information Criterion, or AIC, which is an asymptotic approximation of the Kullback-Leibler information, which it uses as a basis for statistical model evaluation. Section 4.2.1 presents Kullback-Leibler information and provides a derivation of AIC,

$$\text{AIC} = -2\log L(\hat{\boldsymbol{\psi}}\,|\,\boldsymbol{x}) + 2k \ , \tag{1.16}$$

where k is the number of free parameters in the candidate model (or the dimensionality of the parameter vector $\boldsymbol{\psi}$).

Another popular information criterion is the Bayesian information criterion, or BIC, which was introduced by Schwarz [15], and involves a term that imposes a slightly stricter penalty on the number of free model parameters than that of AIC,

$$\text{BIC} = -2\log L(\hat{\boldsymbol{\psi}}\,|\,\boldsymbol{x}) + k\,\log n \ . \tag{1.17}$$

Model selection can be undertaken by selecting the candidate model with the smallest corresponding information criteria value.

It is important to note that information criteria tests do not assess a model in the same sense as a hypothesis test. They are merely a means for model selection, quantifying the quality of a model relative to others, with no information obtained on the absolute quality.

1.7 Thesis Outline and Application

This thesis aims to gain a thorough understanding of both mixture modelling and the EM algorithm, with Chapters 2 and 3 dedicated to presenting the general methodologies. Parameter estimation via the Expectation Maximisation (EM) algorithm is appropriately motivated as a strategy to deal with the incompleteness which is manifested by the mixtures.

Current research considers two goodness-of-fit frameworks, hypothesis testing and model selection via information criteria test. Chapter 4 outlines both of these frameworks with particular emphasis given to mixture models. It is important to note that very little literature has been dedicated to hypothesis testing of mixed distributions, therefore critical values are largely unknown. This study obtains critical values for Kolmogorov-Smirnov, Kuiper, Cramér-von Mises, and Anderson-Darling goodness-of-fit tests for two-component normal (Chapter 5) and Weibull (Chapter 6) distributions. Understanding the dependency of critical values on the parameters of the distribution, sample size, truncation level, and significance level, allows for accurate hypothesis testing results to be obtained.

Chapters [5 - 10] provide complete derivations of EM equations, and the corresponding numerical algorithms for normal, Weibull, exponential, gamma, loglogistic, and uniform component distributions. Finite normal mixtures were chosen to be studied because they are the most typical form of mixture model. In this case, EM equations exist in closed form, allowing parameter estimation to be undertaken with relative ease, and allowing such mixtures to be an appropriate test case for developing statistical methodology. However, contrary to intuition, finite normal mixtures possess many undesirable properties, including partial identifiability and an unbounded likelihood function [16], so careful treatment is required. Chapter 5 continues this discussion. The additional distributions were considered since they are central to the application of financial data which will be considered in this thesis.

When dealing with real data, an observed datum is often the rounded realisation of a continuous random variable. In order to tackle problems associated with rounding, a censored EM framework is presented in Chapter 11. The censored EM algorithm is a relatively new topic, therefore very few studies have been dedicated to it. We note that Chauveau [17] presented censored EM equations to deal with a two-component mixture, where the unknown parameter is the mixture proportion, and Verbelen et.al [18] derived censored EM equations for a mixture of Erlang distributions. Chapter 11 presents a complete derivation of censored EM equations for exponential and Weibull component distributions, noting no previous studies exist dealing with the Weibull distribution in this manner.

This thesis culminates with an analysis of financial data obtained from the London Stock Exchange (LSE) order book. The most significant motivation of this research is to apply mixture modelling techniques to describe stock exchange data related to high frequency trading.

1.7.1 Financial Application of the Thesis

The financial sector is a huge proponent of statistical modelling. Statistical finance, often referred to as econophysics, is an interdisciplinary research field concerned with the application of statistical concepts originally developed within mathematics and the natural sciences, to describe economic processes. The stock market is a highly complex, self-interacting, dynamical system with associated uncertainty. Understanding the stochastic nature of the stock market is of the utmost importance to traders who are both motivated by financial gain and a desire to assess the stability of the market. Statistical analysis typically involves the use of empirical studies to reveal statistical features of financial time-series data. The frequency with which buy or sell orders are submitted to a stock exchange is directly related to the rate of change of the stock price. An understanding of the temporal nature of orders placed on an exchange is therefore of great value to financial institutions. These statistical concepts are foundational to much of modern algorithmic trading. Financial institutions are now leveraging huge computing resources in order to partake in high-frequency trading, which involves the use of algorithmic decisions without human intervention.

An attempt will be made in this thesis to model the time separation of consecutive limit order arrivals, and mid-price waiting times. This research largely builds upon the existing work of Kizilersü et al. [19] and Guscott [20], who determined that a left-truncated Weibull distribution well describes the distribution of inter-arrival order times greater then around 10 milliseconds. Left-truncated distributions were used in order to ignore inter-arrival times which correspond to high-frequency trading, operating in the order of microseconds. Significant density (around 50% for limit orders) exists in the short time scale region of the distribution, hence a model including this region would be of great significance. We note that a remarkable degree of universality exists in the timing of orders placed on the stock exchange (for time separations greater than 10 milliseconds). Maximum entropy for the Weibull distribution corresponds to a shape parameter β which is consistent with the Euler-Mascheroni constant $\gamma \approx 0.577$ for inter-arrival time data, and the scale parameter α of the Weibull distribution is related to the inverse activity of the stock [19]. Unfortunately, time differences below 10 milliseconds display largely chaotic behaviour, and are non-trivial to model with a single distribution. This thesis explores the hypothesis that a mixture may appropriately describe the full distribution of inter-arrival and waiting times. The component distributions considered are given below, where $f(x \,|\, \boldsymbol{\psi})$ is the pdf, and $F(x \,|\, \boldsymbol{\psi})$ the cdf,

- **Weibull Distribution:**

$$f(x \,|\, \alpha, \beta) = \left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-(x/\alpha)^{\beta}} \ , \tag{1.18}$$

$$F(x \,|\, \alpha, \beta) = 1 - e^{-(x/\alpha)^{\beta}} \ , \qquad x \geq 0 \ , \tag{1.19}$$

where $\alpha \in (0, +\infty)$ is the scale parameter, $\beta \in (0, +\infty)$ is the shape parameter, and the distribution is defined on the domain $\mathbb{R}^+ \cup 0$ (i.e. $x \in [0, +\infty)$).

- **Exponential Distribution:**

$$f(x\,|\,\lambda) = \lambda\, e^{-\lambda\, x} \ , \tag{1.20}$$

$$F(x\,|\,\lambda) = 1 - e^{-\lambda\, x} \ , \qquad x \geq 0 \ , \tag{1.21}$$

where $\lambda \in (0, +\infty)$ is the rate, or the inverse scale parameter, and the distribution is defined on the domain $\mathbb{R}^+ \cup 0$ (i.e. $x \in [0, +\infty)$).

- **Gamma Distribution:**

$$f(x\,|\,K,\,\theta) = \frac{1}{\Gamma(K)\,\theta^K}\, x^{K-1}\, e^{-x/\theta} \ , \tag{1.22}$$

$$F(x\,|\,K,\,\theta) = \frac{1}{\Gamma(K)\,\theta^K}\, x^{K-1}\, e^{-x/\theta} \ , \qquad x \geq 0 \ , \tag{1.23}$$

where $\theta \in (0, +\infty)$ is the scale parameter, $K \in (0, +\infty)$ is the shape parameter, and the distribution is defined on the domain $\mathbb{R}^+ \cup 0$ (i.e. $x \in [0, +\infty)$).

- **Loglogistic Distribution:**

$$f(x\,|\,\alpha,\,\beta) = \left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1} \frac{1}{\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^2} \ , \tag{1.24}$$

$$F(x\,|\,\alpha,\,\beta) = \frac{1}{1+\left(\dfrac{x}{\alpha}\right)^{-\beta}} \ , \qquad x \geq 0 \ , \tag{1.25}$$

where $\alpha \in (0, +\infty)$ is the scale parameter, $\beta \in (0, +\infty)$ is the shape parameter, and the distribution is defined on the domain $\mathbb{R}^+ \cup 0$ (i.e. $x \in [0, +\infty)$).

- **Uniform Distribution:**

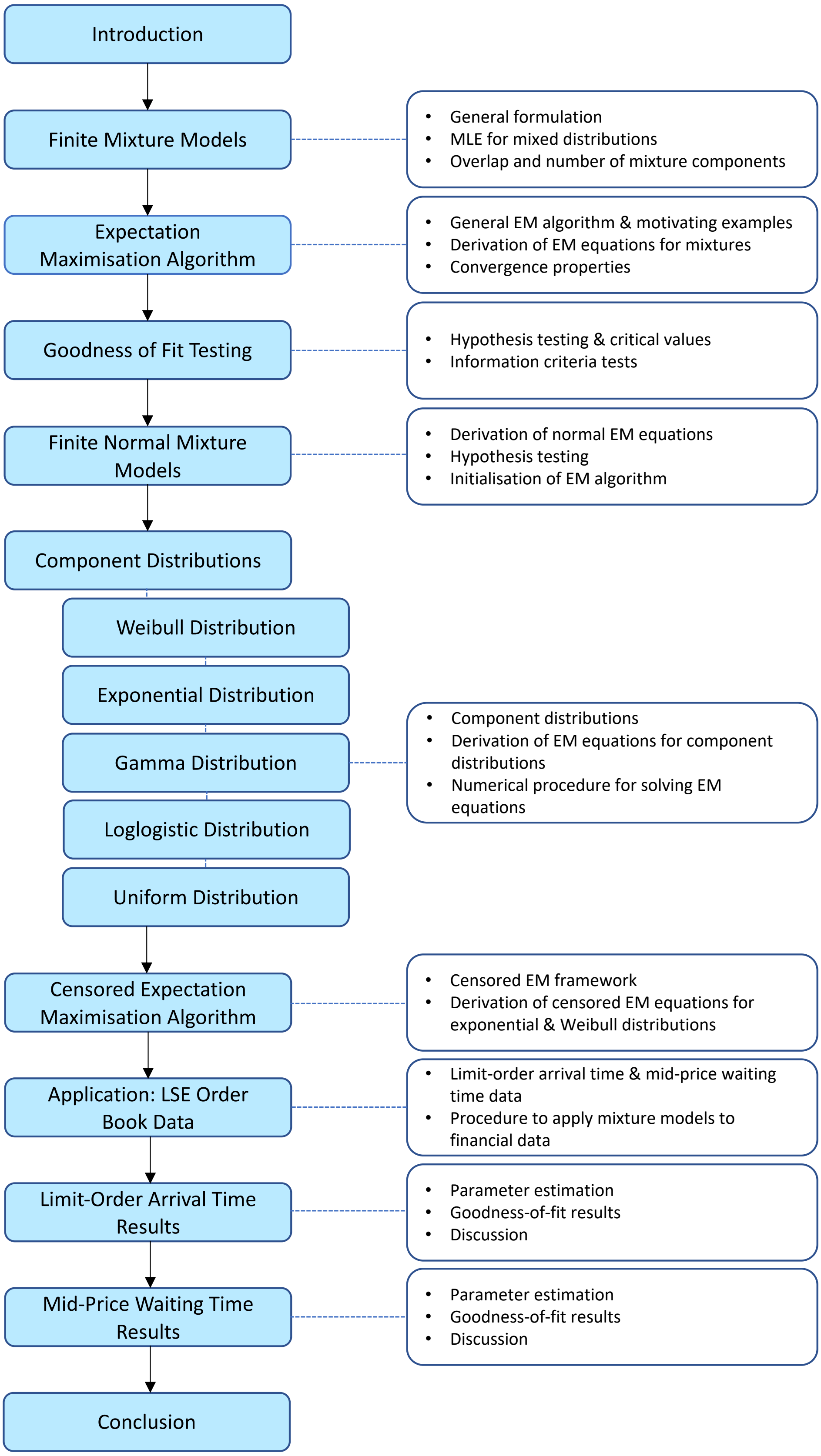
$$f(x\,|\,a,b) = \begin{cases} \dfrac{1}{b-a} \ , & \text{if } x \in [a,b] \ , \\ 0, & \text{otherwise.} \end{cases} \tag{1.26}$$

$$F(x\,|\,a,b) = \begin{cases} 0, & x < a, \\ \dfrac{x-a}{b-a} \ , & \text{if } x \in [a,b) \ , \\ 1, & x \geq b \ , \end{cases} \tag{1.27}$$

where a and b represent the corresponding minimum and maximum values of the region of support, $x \in [a,b]$, defined such that $-\infty < a < b < \infty$.

All of these distributions for given parameter combinations can present high density in the short time-scale region, falling off extremely quickly, such that little compromise to the intermediate and tail region of the distribution, which is well described by the Weibull distribution, is made. Goodness-of-fit testing will then be conducted on the resultant models.

1.7.2 Summary of Thesis Outline



Chapter 2

Finite Mixture Models

In practical situations a mathematical model can be used to describe certain behaviours of a system. Modelling enhances understanding of these behaviours so that informed predictions of the future state of the system can be made. Statistical (or stochastic) models present the example of interest in this work. What categorises a statistical model is the use of a probability distribution to describe data resulting from the system. This work considers probability distributions which can be defined as *mixed distributions*. A mixed distribution describes the probability distribution of a random variable which comprises a collection of additional random variables. In other words, the distribution, as the name suggests, describes a mixture of *component* probability distributions. The contribution of each component to the overall mixture is quantified by a set of *mixture proportions*. An important consideration is that there is no requirement that an observed dataset identifies to which mixture component an individual datum belongs. Mixture models are primarily used to model heterogeneous data which is characterised by the presence of subpopulations within an overall population. The heterogeneity of a population which exhibits physically identifiable subpopulations is well reflected by a mixture model. Component distributions in this case have a physical interpretation. An example of this was provided in Section 1.2 where a two-component Gaussian mixture model was used to describe 'population height' data. These component distributions can be physically interpreted as describing the distribution of height for both the male and female sub-population. Alternatively, mixture models can be used as a pure estimation technique. A mixture model is capable of describing (through appropriate choice of parametric components and proportions) a quite complex distribution of data, often in situations where a single parametric distribution is unable to provide a satisfactory result. It is this flexibility which enables mixture modelling to be a useful statistical tool and underpin techniques used in major areas of statistics including; discriminant analysis, latent class analysis, model based clustering, etc. Modern computing has largely dealt with the computational expense associated with mixture modelling, as such a plethora of mixture related research and applications have proliferated. Mixture models have appeared in studies in many fields including marketing, genetics, medicine, artificial intelligence, finance etc, with some specific applications including:

- **Survival Analysis:** Mixed distributions have been employed to deal with the heterogeneous structure of survival data. Angelis et al. [21] proposed an application of mixture models to the relative survival rate of cancer patients. Erisoglu et al. [22] applied mixture models to study the failure times for oral irrigators.
- **Epidemiology:** Poisson inference is the conventional approach for estimating risk in disease mapping. Militino et al. [23] utilised a discrete mixture of Poisson distributions to incorporate random effects in the model.
- **Economics:** Jump-diffusion modelling is common in option pricing, as well as popular in some credit risk and short-rate models. Merton [24] explains that a jump-diffusion model is a form of mixture model, where a jump-process and a diffusion-process constitute the mixture components.
- **Astronomy:** Mixture models are a natural statistical tool used in many situations in astronomy, such as cluster analysis in various data spaces, and surveys containing multiple types of objects. Modelling techniques including classification, semi-parametric density estimation, and clustering are commonplace in many astronomical applications [25].

- **Image Segmentation:** In computer vision, image segmentation is the process of partitioning an image into segments. Digital images are represented by arrays of pixels. A Gaussian mixture model can be used to partition the pixels into similar segments [26, 27].

Many additional examples can be found in Titterington et al. [28], Redner et al. [13], and McLachlan et al. [29]. These references also provide a thorough and general motivation for mixture modelling as a whole.

The work of Pearson [30] is widely regarded as the first explicit use of mixture modelling in the literature. Pearson used a two-component unimodal mixture of normal densities to model the ratio of crab forehead to body length. The mixture components were heteroscedastic normal densities, meaning the components exhibited unique variances σ_1^2 and σ_2^2 (heteroscedasticity), as well as unique means μ_1 and μ_2. A mixture proportion π_1 quantified the contribution of the first component to the overall mixture (and the second mixture proportion was given as $\pi_2 = 1 - \pi_1$). Figure 2.1 displays a histogram of binned ($n = 1000$) data, including an overlay of the two-component normal fit. Pearson showed that a mixture model could adequately explain the skewness in the data, and hence suggested that *two subspecies* were present within the overall crab population. The method of moments was used to obtain Pearson's estimates ($\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_2$, and $\hat{\pi}_1$), although a large level of agreement exists with more efficient, modern maximum likelihood based approaches.

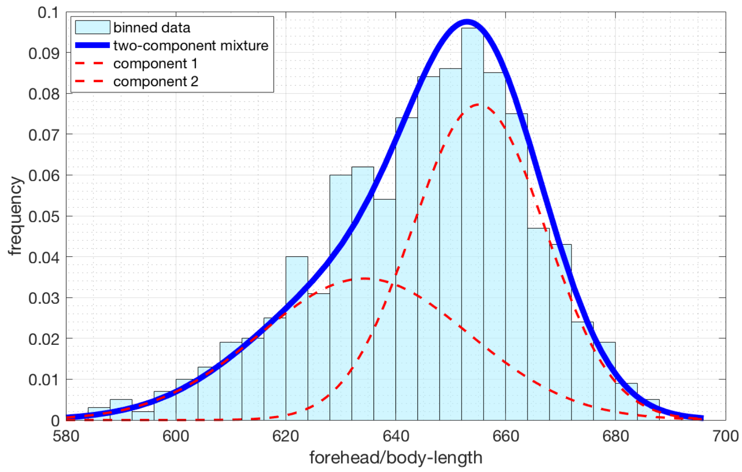


Figure 2.1: Pearson's crab data [30] ($n = 1000$): $\hat{\mu}_1 = 634.3$, $\hat{\mu}_2 = 655.1$, $\hat{\sigma}_1 = 19.1$, $\hat{\sigma}_2 = 12.1$, and $\hat{\pi}_1 = 0.4145$, $\hat{\pi}_2 = 0.5855$.

2.1 General Formulation

Consider a random variable X with realisation $\boldsymbol{x_t}$, and a corresponding random sample $\boldsymbol{x} = \{\boldsymbol{x_1}, ..., \boldsymbol{x_n}\}$ of size n. The general probability density function for a $g \in \mathbb{Z}^+$ component parametric mixture is

$$f(\boldsymbol{x_t} \,|\, \boldsymbol{\psi}) = \sum_{i=1}^{g} \pi_i f_i(\boldsymbol{x_t} \,|\, \boldsymbol{\theta}_i) \ . \tag{2.1}$$

The mixture model is parametrised by the vector $\boldsymbol{\psi} = (\boldsymbol{\pi}, \boldsymbol{\xi})$, where $\boldsymbol{\pi} = (\pi_1, ..., \pi_{g-1})$ is a vector containing (g - 1) mixture proportions (or weights), and $\boldsymbol{\xi} = (\boldsymbol{\theta_1}, ..., \boldsymbol{\theta_g})$ contains the parameter vectors for each component $f_i(\cdot \,|\, \boldsymbol{\theta}_i)$ $(i = 1, ..., g)$ of the mixture.

Each π_i represents the probability that a randomly selected $\boldsymbol{x_t} \in \boldsymbol{x}$ corresponds to the i^{th} subpopulation. Hence, only (g - 1) mixture proportions appear in the vector $\boldsymbol{\pi}$ because component densities are mixed such that proportions are constrained as follows

$$\sum_{i=1}^{g} \pi_i = 1 \quad \Rightarrow \quad \pi_g = 1 - \sum_{i=1}^{g-1} \pi_i \ , \tag{2.2}$$

and

$$0 \leq \pi_i \leq 1, \quad \text{for} \quad i = 1, ..., g \ .$$

It can be trivially shown that Eq. (2.1) in fact defines a probability density function.

$$\begin{aligned} 1 &= \sum_{i=1}^{g} \pi_i \\ &= \sum_{i=1}^{g} \pi_i \int_{\mathbb{R}^d} f_i(\boldsymbol{x_t} \,|\, \boldsymbol{\theta}_i) \ d\boldsymbol{x_t} \\ &= \int_{\mathbb{R}^d} \sum_{i=1}^{g} \pi_i \, f_i(\boldsymbol{x_t} \,|\, \boldsymbol{\theta}_i) \ d\boldsymbol{x_t} \\ &= \int_{\mathbb{R}^d} f(\boldsymbol{x_t} \,|\, \boldsymbol{\psi}) \ d\boldsymbol{x_t} \ . \end{aligned} \tag{2.3}$$

Finite mixture models can be extended to consider an infinite mixture of component densities as presented in [13],

$$f(x \,|\, \boldsymbol{\psi}) = \int_{\Lambda} f(x \,|\, \boldsymbol{\psi}(\lambda)) \, d\pi(\lambda) \ , \tag{2.4}$$

although this thesis doesn't consider mixtures of this type. Additionally, a major focus of this thesis will be on univariate finite mixtures. From this point onwards it will be assumed that all mixture models are univariate, hence $\boldsymbol{x_t} \ \Rightarrow \ x_t$ unless multi-variability is explicitly stated.

2.2 MLE Formulation of Mixture Models

There are a number of methods one can employ for parameter estimation of a statistical model for a given set of observations. Some examples are the method of moments, the Cramer-Rao estimator, the Bayes estimator, and the maximum likelihood estimator. This thesis will focus solely on maximum likelihood estimation because it has many desirable attributes [2–10]. Three of the most important properties are consistency, efficiency, and uniqueness which were proven by Lehmann and Casella [11]. An important result is that maximum likelihood parameter estimates tend towards true values for large sample sizes. Especially after the advent of modern computers, where estimation can be undertaken computationally, MLE has become the primary technique for estimating parameters in statistical models. The foundation of maximum likelihood estimation is based on the principle that parameters $\hat{\boldsymbol{\psi}}(\boldsymbol{x})$ which maximise the likelihood function are most likely to define the parameters by which the data is distributed.

The purpose of this section is formulating MLE for mixture models. Firstly, in order to find parameter estimates for a given mixed distribution, the density of the mixture is required to be identifiable. The parametric family of mixture densities is identifiable if distinct values of $\boldsymbol{\psi}$ determine distinct density values. A class of finite mixtures is identifiable for $\boldsymbol{\psi} \in \Omega$ if probability density functions for two members are

$$f(x \,|\, \boldsymbol{\psi}) = \sum_{i=1}^{g} \pi_i f_i(x \,|\, \boldsymbol{\theta}_i) \ , \tag{2.5}$$

and

$$f(x \,|\, \boldsymbol{\psi}') = \sum_{i=1}^{g'} \pi_i' f_i(x \,|\, \boldsymbol{\theta}_i') \ , \tag{2.6}$$

where $f(x \,|\, \boldsymbol{\psi}) \equiv f(x \,|\, \boldsymbol{\psi}')$ if and only if $g = g'$ and $f_i(x \,|\, \boldsymbol{\theta}_i) = f(x \,|\, \boldsymbol{\theta}_i')$.

Let $\boldsymbol{x} = (x_1, ..., x_n)$ be an observed random sample, a realisation of the random variable X, distributed by the probability density function of the general mixture model $f(x \,|\, \boldsymbol{\psi})$. The log-likelihood function for parameter vector $\boldsymbol{\psi}$ can be formed from the observed data $\boldsymbol{x}$ and is given as

$$\begin{aligned}\log L(\boldsymbol{\psi}\,|\,\boldsymbol{x}) &= \log \prod_{j=1}^{n} f(x_j\,|\,\boldsymbol{\psi}) \\ &= \sum_{j=1}^{n} \log f(x_j\,|\,\boldsymbol{\psi}) \\ &= \sum_{j=1}^{n} \log \left(\sum_{i=1}^{g} \pi_i f_i(\boldsymbol{x}\,|\,\boldsymbol{\theta}_i) \right) . \end{aligned} \tag{2.7}$$

Maximising the log-likelihood function requires setting the partial derivatives of Eq. (2.7) with respect to $\boldsymbol{\psi}$ to zero

$$\frac{\partial \log L(\boldsymbol{\psi}\,|\,\boldsymbol{x})}{\partial\, \boldsymbol{\psi}} = 0 \ . \tag{2.8}$$

The maximum likelihood parameter estimates are simultaneous solutions to this set of coupled non-linear equations. To illustrate, consider the case where parameters of specific mixture components are known *a priori* and only mixture proportions require estimation, i.e. $\boldsymbol{\psi} = (\pi_1, ..., \pi_{g-1})$, containing only $(g$ - $1)$ mixture proportions. Even in this case the following system of maximum likelihood equations doesn't yield an explicit solution for the mixture proportions

$$\hat{\boldsymbol{\psi}} = (\hat{\pi}_1, ..., \hat{\pi}_{g-1}) \ ,$$

$$\frac{\partial \log L(\boldsymbol{\psi}\,|\,\boldsymbol{x})}{\partial\, \boldsymbol{\psi}} = \sum_{j=1}^{n} \left\{ \frac{f_i(x_j)}{f(x_j\,|\,\hat{\boldsymbol{\psi}})} - \frac{f_g(x_j)}{f(x_j\,|\,\hat{\boldsymbol{\psi}})} \right\} = 0 \ . \tag{2.9}$$

Direct solutions of the maximum likelihood equations cannot be found for general mixture models, but the problem can be simplified by a reformulation as an incomplete-data problem. These problems typically involve missing or unobservable data. The incompleteness of mixture models originates from the fact that observed data samples are not labelled to identify which mixture component they are assigned to. The introduction of an unobserved (or missing) data vector $\boldsymbol{z} = (\boldsymbol{z}_1, ..., \boldsymbol{z}_{g-1})$ can deal with this. Here $\boldsymbol{z}_i$ represents a g-dimensional vector of zero-one indicator variables, i.e. $z_{ij} = (\mathbf{z}_i)_j$ is a one or zero according to whether x_j arose or did not arise from the i^{th} component of the mixture $(i = 1, ..., g, \quad j = 1, ..., n)$. The complete-data vector $\boldsymbol{y}$ is simply defined as

$$\boldsymbol{y} = (\boldsymbol{x}\,,\,\boldsymbol{z}) \ ,$$

taking into account the incomplete-observable data vector $\boldsymbol{x}$, and the unobservable data vector $\boldsymbol{z}$. The complete-data log-likelihood function $\log L_c(\boldsymbol{\psi}\,|\,\boldsymbol{y})$ for $\boldsymbol{\psi}$ now takes the form

$$\log L_c(\boldsymbol{\psi}\,|\,\boldsymbol{y}) = \sum_{i=1}^{g} \sum_{j=1}^{n} z_{ij} \log \big(\pi_i f(x_j\,|\,\boldsymbol{\theta}_i) \big) \tag{2.10}$$

$$= \sum_{i=1}^{g} \sum_{j=1}^{n} \big(z_{ij} \log \pi_i + z_{ij} \log f(x_j\,|\,\boldsymbol{\theta}_i) \big) \ . \tag{2.11}$$

Chapter 3 outlines the use an **Expectation Maximisation algorithm** as a tool to find maximum likelihood estimates of parameters in mixture models which have been formulated as incomplete. The algorithm handles the addition of the unobservable data by working with the current conditional expectation of the complete-data log-likelihood given the observed data.

2.3 Overlap of Mixture Components

The degree of overlap between mixture components often (inversely) categorises the ease in which subpopulations can be distinguished within an overall population. If mixture components display a low degree of overlap (or correspondingly high degree of separation) one can easily distinguish between mixture components, and hence easily discern the presence of subpopulations within an overall population. Estimation procedures encounter numerical difficulties estimating parameters of mixture components where overlap is high.

The following integral measure quantifies the overlap between components in a g-component univariate mixture [31],

$$OVL_g = 1 - \int_{\mathbb{R}} \max\left(\pi_1\, f_1(x\,|\,\boldsymbol{\theta}_1), ..., \pi_g\, f_g(x\,|\,\boldsymbol{\theta}_g)\right)(x)\; dx\;. \tag{2.12}$$

Simplification for the two-component ($g = 2$) mixture is given as follows

$$OVL_2 = \int_{\mathbb{R}} \min\left(\pi_1\, f_1(x\,|\,\boldsymbol{\theta}_1), \pi_2\, f_2(x\,|\,\boldsymbol{\theta}_2)\right)(x)\; dx\;. \tag{2.13}$$

Figures [2.2 - 2.4] display various univariate mixture models where red dotted curves represent component densities and blue curves are overall mixture densities. The coloured regions display component overlap which provides an indicative metric for categorising the ease in which subpopulations can be distinguished from an overall population.

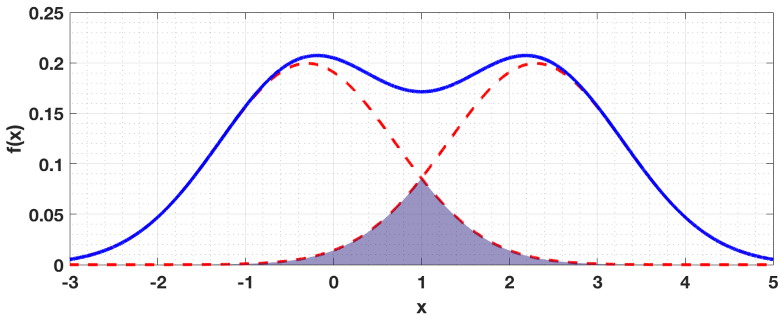


Figure 2.2: Two-component Gaussian mixture: $\boldsymbol{\mu} = (-0.3,\, 2.3)$, $\quad \boldsymbol{\sigma} = (1,\, 1)$, $\quad \pi_1 = 0.5$.

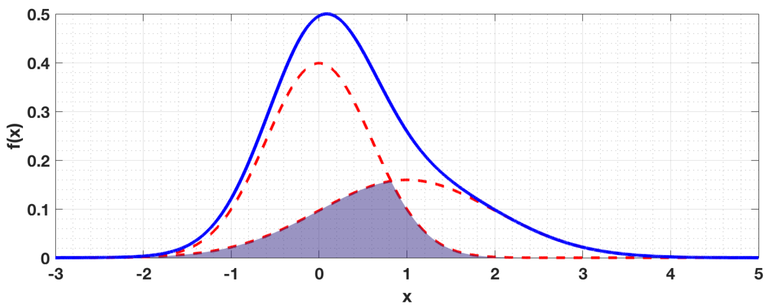


Figure 2.3: Two-component Gaussian mixture: $\boldsymbol{\mu} = (0,\, 1)$, $\quad \boldsymbol{\sigma} = (0.6,\, 1)$, $\quad \pi_1 = 0.6$.

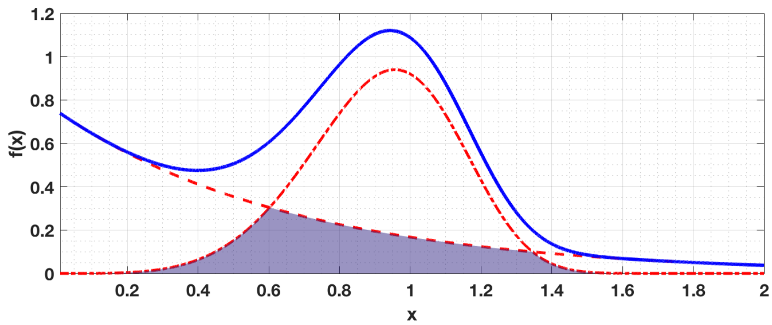


Figure 2.4: Exponential/Weibull mixture: $\lambda = 1.5$, $\quad \alpha = 1$, $\quad \beta = 5$, $\quad \pi_1 = 0.5$.

2.4 Determining the Number of Mixture Components

The number of components in a finite mixture is sometimes known or assumed to be known *a priori*. For example, this is the case when clear sub-populations (which can often be physically interpreted) exist within the data, or the statistician has constructed a numerical experiment such that this is true. If the number of mixture components is unknown, the task of determining it is often difficult but important. The obvious way of approaching this problem is to find the smallest value g (the number of components) which is consistent with the data, using the likelihood ratio test. Let Λ be the test statistic which is the ratio of likelihoods of competing models. McLachlan and Baseford [29] showed that the regularity conditions for mixture models do not hold for $-2\log\Lambda$ to have its usual asymptotic χ^2 distribution, with degrees of freedom equal to the difference in the number of parameters in the competing models.

This section first considers the connection between the number of components and the modality of a distribution, before presenting information criteria tests as a tool for undertaking this task.

2.4.1 Unimodality/Bimodality/Multimodality

Unimodality of a parametric distribution refers to the possession of one single mode in the distribution. This is characterised by a single local maximum (or peak) in the probability density function. Similarly if there are two modes the distribution is bimodal, or more generally multimodal. The term 'mode' is most often associated with discrete probability distributions. For a discrete random variable the mode represents the value which occurs most often in a data sample, corresponding to the realisation with greatest probability mass, meaning it has the largest associated probability of being sampled. For continuous probability distributions, the type of distributions of interest to this work, the mode describes the set of all local maxima in the probability density function. Unimodality is most often defined with reference to the cumulative distribution function. If the cdf is concave for $x < m$ and convex for $x > m$, the distribution is unimodal with mode m.

Although bimodal and more generally multimodal distributions exist (eg. beta distribution and arcsine distribution) the vast majority of popular continuous probability distributions are unimodal. For this reason, the majority of parametric components of mixture models are also unimodal. Many multimodal distributions are in fact finite mixtures of unimodal distributions. Although the modality of a model can often yield useful information on the number of sub-populations evident within an overall population, it can often be misleading. For example, the distribution of crab forehead to body length ratios which Pearson [30] described as a two-component mixture possessed one single mode.

Consider for reference the standard two-component Gaussian mixture model. Eisenberger [32] proved that a two-component Gaussian mixture model will be unimodal if the difference between component means μ_1 and μ_2 is sufficiently small, independent of the mixture proportions π_1 and π_2. If the difference between the means $d = |\mu_1 - \mu_2|$ exceeds a threshold value, then the unimodality is dependent on the mixture proportions π_1 and π_2. Lastly, there always exists values of π_1 and π_2 sufficiently close to zero and one such that the resultant distribution is unimodal. Many additional criteria for determining whether a distribution is unimodal exist [33].

The purpose of this discussion was to support the somewhat unintuitive result that the modality of a model gives little information about the number of components in the mixture. Because of this, alternate techniques are required.

2.4.2 Information Criteria Tests

Information criteria tests are a popular technique to determine the number of components in a mixture model. Section 4.2 provides a far more detailed explanation on these tests, including a description of their formulation from Kullback-Liebler information. This section will present two information criteria; Akaike Information Criterion [14] (AIC) and Bayesian Information Criterion [15] (BIC). The reader is encouraged to reference Section 4.2 for a more thorough treatment.

$$\mathrm{AIC} = -2\log L(\hat{\boldsymbol{\psi}}\,|\,\boldsymbol{x}) + 2k \ , \tag{2.14}$$

$$\text{BIC} = -2 \log L(\hat{\boldsymbol{\psi}} \,|\, \boldsymbol{x}) + k \, \log n \ , \tag{2.15}$$

where k is the number of free parameters in the candidate model (or the dimensionality of the parameter vector $\boldsymbol{\psi}$), proportional to the number of components. **An appropriate choice of g can be determined by selecting the candidate model which corresponds to the smallest information criteria value**. Information criteria tests balance the complexity of a model with the goodness-of-fit. In this context, the task is to determine the number of components in a mixture model which sufficiently describes the distribution of data, without overfitting. Recently alternate methods have been proposed including an EM-test for homogeneity [34].

2.5 Example 1: Unsupervised Learning/Clustering

Statistical techniques are the foundation of much of modern machine learning. A subset of machine learning is unsupervised learning which deals with data without labelling. Instead of evaluating the validity of a model by a feedback metric, unsupervised learning algorithms attempt to identify commonalities within the data and behave accordingly. For such algorithms, data classification is a central issue in data pre-processing, with mixture models often used to classify incomplete-data into parametric clusters. A cluster has no rigorous definition but is rather loosely defined as a collection of data objects which exhibit a commonality. Distribution based clustering involves the use of mixed distributions to model clustered data, typically via the method of moments or the Expectation Maximisation (EM) algorithm.[1] Figure 2.5 displays a bi-variate Gaussian mixture model. Data is grouped into three distinct clusters, each corresponding to a component distribution. This figure displays the final converged state into these distinct clusters after 25 EM iterations.

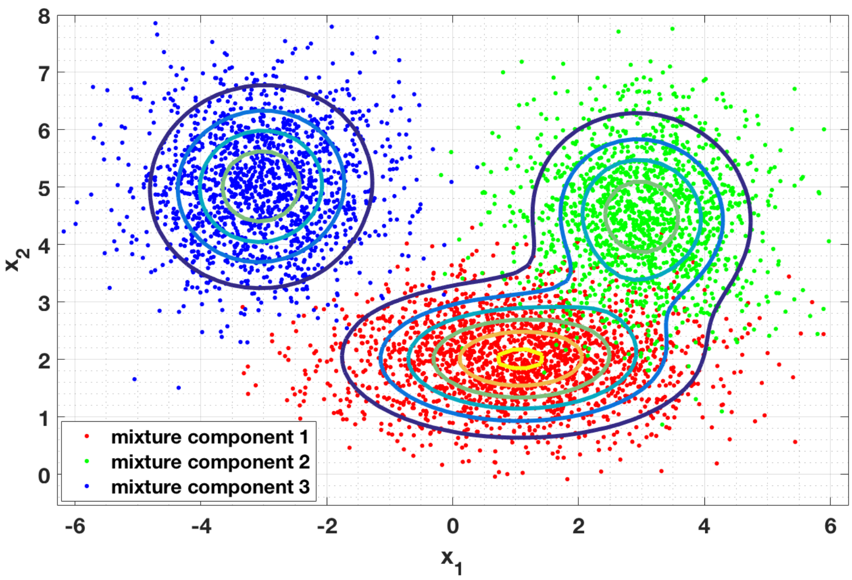


Figure 2.5: Bi-variate three-component Gaussian mixture model: $\mu_1 = [1, 2]$, $\mu_2 = [3, 4.5]$, $\mu_3 = [-3, 5]$, $\sigma_1 = [2, 0; 0, 0.5]$, $\sigma_2 = [1, 0; 0, 1]$, $\sigma_3 = [1, 0; 0, 1]$, $\pi_1 = 0.4$, $\pi_2 = 0.3$, $\pi_3 = 0.3$.

2.6 Example 2: Australian Climate Data

Climate data was obtained from the Australian Government Bureau of Meteorology [35]. Data for the daily maximum temperatures from the year 1995 to 2000 for the Adelaide Airport and Darwin Airport weather stations in the month of April was recorded. Consider the case where each individual datum isn't labelled with additional data indicating whether it corresponds to the Adelaide or Darwin weather station. The data can be seen as heterogeneous, with two clear sub-populations.

[1] Cluster analysis can also be undertaken without reference to statistical distributions. Typically a distance metric is defined, as standard in connectivity and centroid models.

A two-component Gaussian mixture model could be postulated to model the distribution of data, consequently differentiating the two sub-populations from the overall population and each other. Hence information about the mean and variance of the April max-temperatures in Adelaide and Darwin can be determined. (Refer to Chapter 3 for specific detail regarding how such problems can be formulated as incomplete, and estimation undertaken with the Expectation-Maximisation algorithm). The mixture model is shown in Fig. 2.6 as an overlay of the binned data. The first mixture component corresponds to a climate that is on average colder than the second, also with a larger variation between daily maximum temperatures. It is reasonable to assume that convergence of parameter estimates were in a manner in which the first component corresponds Adelaide, and the second Darwin. It is important to note the use of this example as merely a means to motivate mixture modelling. Due to the simplicity and usefulness of Gaussian mixture models they provide a good platform for introduction into modelling. Possibly a skew-normal is more appropriate for fitting climate data.

This thesis will not limit modelling to Gaussian components and deal with alternative parametric components as well.

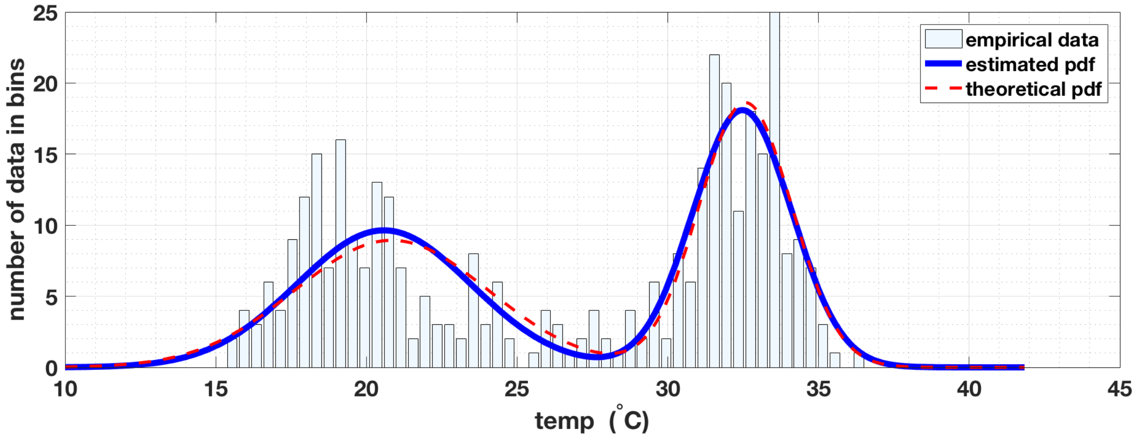


Figure 2.6: Adelaide and Darwin max-temperature data in April, 1995-2000 ($n = 360$): $\hat{\mu}_1 = 20.58$, $\hat{\mu}_2 = 32.48$, $\hat{\sigma}_1 = 2.89$, $\hat{\sigma}_2 = 1.63$, and $\hat{\pi}_1 = 0.49$, $\hat{\pi}_2 = 0.51$.

Chapter 3

Expectation Maximisation Algorithm

Maximum likelihood estimation is a method of parameter estimation for statistical models. Estimation is undertaken by finding parameters which maximise the likelihood function for a given set of observations. These parameters are solutions of a set of simultaneous nonlinear equations known as the *maximum likelihood equations* which are constructed by taking partial derivatives of the log-likelihood function with respect to elements of the parameter vector. For a general introduction to maximum likelihood estimation the reader is encouraged to reconsider Section 1.4.

The **Expectation Maximisation algorithm**, commonly known as the EM algorithm, is an iterative method for computing maximum likelihood estimates, typically used in *incomplete-data* problems. These problems most often involve missing or unobservable data. The EM algorithm is useful because maximum likelihood equations cannot be directly solved in these situations. The algorithm was first introduced by Dempster, Laird, and Rubin [36] in a widely renowned paper initially presented to the Royal Statistical Society in 1976 and subsequently published in 1977. As the name suggests, the algorithm alternates between an expectation, or E step, and a maximisation, or M step. The E step formulates an expectation of the log-likelihood function. This is calculated by dealing with the conditional expectation of the unobserved data given both the observed data and estimates of the parameters on the previous iteration. The M step is the more intuitive step because it is comparable to direct maximum likelihood estimation. It involves updating the iterative estimates by selecting the parameters which maximise the expected log-likelihood (which was calculated on the previous E step). What distinguishes the EM algorithm from more traditional maximum likelihood approaches is the expectation step. Although this thesis will focus on incompleteness which is manifested by mixtures, the EM algorithm can also be utilised to deal with incompleteness originating from truncation, censoring, and other non-traditional forms. Additionally, the incompleteness is often formulated by the statistician in a manner suitable for application of the algorithm. This is true for mixtures, and will be presented in this chapter.

The EM algorithm has become a standard tool for dealing with incomplete-data problems in a variety of fields. Some specific examples of modelling that use the algorithm in the field of medical imaging, machine learning, quantitative genetics, and epidemiology include:

- **Medical Imaging:** Shepp et al. [37] among others outline the maximum likelihood approach using the EM algorithm for image reconstruction in emission tomography. Blume et al. [38] outlines the iterative scheme explaining how the incompleteness originates from the lack of information regarding the location of emission events in-between detector pairings.

- **Quantitative Genetics:** Zhan et al. [39] and Ghosh et al. [40] among others outline the use of the EM algorithm for mapping quantitative resistance loci in genetic modelling.

- **Machine Learning:** The EM algorithm is largely used in data clustering and computer vision, so there exists a plethora of machine learning literature dedicated to it. Ng et al. [41] explains how the EM algorithm can be used to train multilayer perception and mixture of experts networks in application to multi-class classification.

- **AIDS epidemiology:** Becker [42] outlines how the EM algorithm is used in analysis of HIV/AIDS epidemiology data that is affected by the partial observance of infection processes.

McLachlan et al. [12] and Redner et al. [13] provide comprehensive discussions of the EM algorithm, including many additional applications and references. This chapter will first present a general formulation of the EM equation before specific consideration to mixture models is given.

3.1 General EM Algorithm

For simplicity, the EM algorithm will be generally formulated here for the univariate case, although multivariate extensions are also permissible. Consider an observed data vector $\boldsymbol{x} = (x_1, ..., x_n)$ which corresponds to a postulated probability density function $f(\boldsymbol{x} \,|\, \boldsymbol{\psi})$ where $\boldsymbol{\psi} = (\psi_1, \psi_2, ..., \psi_m)$ is a vector of unknown parameters which are restricted to the parameter space Ω. Parameter estimates $\hat{\psi}$ are driven using maximum likelihood estimation. The EM algorithm is an iterative procedure for computing maximum likelihood estimates, and would be elementary but for the absence of some additional data. In this framework, the data vector $\boldsymbol{x}$ is viewed as incomplete. (The notion of incompleteness can be made clearer by briefly digressing from the general formulation and considering mixture modelling. Incompleteness manifests in this case because the component an individual datum belongs to is unknown. The missing data in this context can be conceptualised as labels identifying for which component each datum corresponds). Let $\boldsymbol{y}$ denote the *complete* data vector, and $\boldsymbol{z}$ denote the additional unobserved (or missing) data, such that $\boldsymbol{y} = (\boldsymbol{x}, \boldsymbol{z})$. Let the probability density function for the complete data vector $\boldsymbol{y}$ be denoted $f_c(\boldsymbol{y} \,|\, \boldsymbol{\psi})$, and the corresponding log-likelihood function given as

$$\log L_c(\boldsymbol{\psi} \,|\, \boldsymbol{y}) = \sum_{j=1}^{n} \log f_c(y_j \,|\, \boldsymbol{\psi}) \ . \tag{3.1}$$

By iterating in terms of the complete log-likelihood function, the algorithm indirectly deals with the incomplete log-likelihood. As the missing data is unobservable, it is replaced by its conditional expectation given the observed data $\boldsymbol{x}$, in terms of the current estimate of the unknown parameters $\boldsymbol{\psi}$. The computed parameters on the previous iteration (or the initial values if it is the first iteration) is what is meant by the current estimate. The routine is categorised by two steps, the expectation step, and the maximisation step. Consider the general $(k{+}1)^{\text{th}}$ EM iteration,

E step: Compute $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$ where

$$Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) = \mathbb{E}_{\boldsymbol{\psi}^{(k)}}[\,\log L_c(\boldsymbol{\psi} \,|\, \boldsymbol{y}) \,|\, \boldsymbol{x}\,] \ .$$

M step: Choose $\boldsymbol{\psi}^{(k+1)} \in \Omega$ to maximise $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$, that is

$$Q(\boldsymbol{\psi}^{(k+1)} \,|\, \boldsymbol{\psi}^{(k)}) \ \geq \ Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) \qquad \forall \ \ \boldsymbol{\psi} \in \Omega \ .$$

i.e.

$$\boldsymbol{\psi}^{(k+1)} = \underset{\boldsymbol{\psi} \in \Omega}{\operatorname{argmax}} \ Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) \ .$$

On the first iteration ($k = 0$) the parameter vector $\boldsymbol{\psi}^{(0)}$ represents the initial values of the algorithm. Convergence of the EM algorithm is dependent on the choice of these values. Poor initialisation can lead to both slow convergence and convergence to undesired stationary points. Good initialisation of the algorithm is therefore desirable, although there seldom exists a perfect routine. Initialisation will be further discussed in subsequent chapters.

Iteration is continued by repeating both the E and M steps until convergence of the log-likelihood function is reached, i.e. until the difference of two consecutive log-likelihood values is smaller than a predetermined tolerance ϵ. Although the convergence properties of the EM algorithm haven't been explicitly presented, convergence of the EM algorithm is to a log-likelihood which is bounded above, increasing on each iteration, and is deemed when

$$\log L(\boldsymbol{\psi}^{(k+1)}) - \log L(\boldsymbol{\psi}^{(k)}) \ \leq \ \epsilon \ . \tag{3.2}$$

3.1.1 Motivating Examples

The following examples attempt to demonstrate the EM algorithm in a general sense.[1] The first example was contrived by Chuong and Serafim [43] where estimation of bias in a simple two-coin flipping experiment is formulated as incomplete and the EM algorithm is utilised. The second is the introductory example presented by Dempster, Laird, and Rubin [36] in their seminal paper. This example has been subsequently used many times in the literature and involves the estimation of a parameter in a multinomial distribution.

Coin Flipping Example:

An experiment considers two coins, A and B, of unknown respective biases, θ_A and θ_B. On any given flip, coin A will land on 'heads' with probability θ_A, and 'tails' with probability $1 - \theta_A$. The goal is to estimate $\boldsymbol{\theta} = (\theta_A, \theta_B)$, by the following procedure: **randomly choose (with equal probability) one of the two coins, and perform ten independent tosses with that coin. This procedure is repeated fives times, hence the total experiment involves the tossing of 50 coins**. Let $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5)$ be a vector indicating the number of 'heads' after each set of ten tosses, and $\boldsymbol{z} = (z_1, z_2, z_3, z_4, z_5)$ be a vector indicating which coin was chosen for each of the five repetitions, i.e. $z_i \in \{A, B\}$ for $i = 1, ..., 5$. If both $\boldsymbol{x}$ and $\boldsymbol{z}$ are observed, the estimation of parameter $\boldsymbol{\theta}$ is a complete-data problem, where the maximum likelihood estimation is simple and given by the intuitive result,

$$\hat{\theta}_A = \frac{\text{number of heads using coin A}}{\text{total number of flips using coin A}} \,, \tag{3.3}$$

$$\hat{\theta}_B = \frac{\text{number of heads using coin B}}{\text{total number of flips using coin B}} \,. \tag{3.4}$$

The incomplete-data problem arises when the vector $\boldsymbol{z}$ is now unobserved. In this scenario $\boldsymbol{x}$ the number of heads tossed during each repetition is known, but the coin used is unknown. The EM algorithm approaches the problem using the expectation of the missing data (the indicator vector $\boldsymbol{z}$) conditioned on the current parameter estimates. Now the incomplete-data problem can be handled with maximum likelihood estimation of a complete-data problem.

Assume the following experiment has been carried out (Table 3.1) according to the procedure outlined by the boldface text above:

1	**coin B**	**H T T T H H T H T H**
2	**coin A**	**H H H H T H H H H H**
3	**coin A**	**H T H H H H H T H H**
4	**coin B**	**H T H T T T H H T T**
5	**coin A**	**T H H H T H H H T H**

Table 3.1: Results of experiment: Select coin A or coin B at random and toss 10 times. Repeat 5 times.

First consider the complete-data problem, where it is known which coin was selected on each repetition of the procedure. Both vectors $\boldsymbol{x} = (5, 9, 8, 4, 7)$ and $\boldsymbol{z} = (B, A, A, B, A)$ are observable. For this case the parameter estimate is trivially given by $\hat{\theta}_A = \frac{24}{24+6} = 0.8$, and $\hat{\theta}_B = \frac{9}{9+11} = 0.45$.

[1]Noting that both are discrete examples intended to motivate the general concepts of the EM algorithm.

Now consider the incomplete-data problem where the indicator vector $\boldsymbol{z}$ is unobserved. The EM procedure is outlined below:

- The EM algorithm begins with an initial guess of the parameters, i.e. $\hat{\theta}_A^{(0)} = 0.6$ and $\hat{\theta}_B^{(0)} = 0.5$.

- The E-step, or expectation step, requires a probability distribution of possible completions to be built using the current parameter estimates. The binomial probability distribution gives the probability of K out of 10 tosses of coin $i \in \{A, B\}$ being 'heads',

$$P_i(K) = \binom{10}{K} \hat{\theta}_A^{(0)K} (1 - \hat{\theta}_A^{(0)})^{10-K} . \tag{3.5}$$

  Table 3.2 gives the expected number of heads and tails according to this probability distribution. The following procedure outlines an example calculation for the highlighted cells:

  eg.1: The normalised probability (or likelihood) that set 2 (9 heads, 1 tail) corresponds to coin A on the first iteration is:

$$\textbf{Coin A} \Rightarrow \quad \frac{\binom{10}{9} \hat{\theta}_A^{(0)9} (1 - \hat{\theta}_A^{(0)})^{10-9}}{\binom{10}{9} \hat{\theta}_A^{(0)9} (1 - \hat{\theta}_A^{(0)})^{10-9} + \binom{10}{9} \hat{\theta}_B^{(0)9} (1 - \hat{\theta}_B^{(0)})^{10-9}} = 0.805 .$$

  eg.2: Similarly the normalised probability that set 4 (4 heads, 6 tails) corresponds to coin B on the same iteration is:

$$\textbf{Coin B} \Rightarrow \quad \frac{\binom{10}{4} \hat{\theta}_B^{(0)4} (1 - \hat{\theta}_B^{(0)})^{10-4}}{\binom{10}{4} \hat{\theta}_A^{(0)4} (1 - \hat{\theta}_A^{(0)})^{10-4} + \binom{10}{4} \hat{\theta}_B^{(0)4} (1 - \hat{\theta}_B^{(0)})^{10-4}} = 0.648 .$$

Coin A		**Coin B**	
$0.45 \times x_1 \approx 2.2H$,	$0.45 \times (10 - x_1) \approx 2.2T$	$0.55 \times x_1 \approx 2.8H$,	$0.55 \times (10 - x_1) \approx 2.8T$
$0.81 \times x_2 \approx 7.2H$,	$0.81 \times (10 - x_2) \approx 0.8T$	$0.20 \times x_2 \approx 1.8H$,	$0.20 \times (10 - x_2) \approx 0.2T$
$0.73 \times x_3 \approx 5.9H$,	$0.73 \times (10 - x_3) \approx 1.5T$	$0.27 \times x_3 \approx 2.1H$,	$0.27 \times (10 - x_3) \approx 0.5T$
$0.35 \times x_4 \approx 1.4H$,	$0.35 \times (10 - x_4) \approx 2.1T$	$0.65 \times x_4 \approx 2.6H$,	$0.65 \times (10 - x_4) \approx 3.9T$
$0.65 \times x_5 \approx 4.5H$,	$0.65 \times (10 - x_5) \approx 1.9T$	$0.35 \times x_5 \approx 2.5H$,	$0.35 \times (10 - x_5) \approx 1.1T$
$\approx \mathbf{21.3H, 8.6T}$		$\approx \mathbf{11.7H, 8.4T}$	

Table 3.2: Expected number of heads/tails of each coin on each repetition according to the binomial distribution. A representative calculation of the highlighted cells was shown above.

- The M-step, or maximisation step on the first iteration can be undertaken using Eq. (3.3) and Eq. (3.4) now that an expectation of the missing data (the indicator vector $\boldsymbol{z}$) conditioned on the current parameter estimates has been determined

$$\hat{\theta}_A^{(1)} \approx \frac{21.3}{21.3 + 8.6} \approx 0.71 ,$$

$$\hat{\theta}_B^{(1)} \approx \frac{11.7}{11.7 + 8.4} \approx 0.58 .$$

This procedure is repeated iteratively until convergence is reached. $\hat{\theta}_A^{(1)}$ and $\hat{\theta}_B^{(1)}$ are used as initial values for the second iteration. Table 3.3 provides estimates for the coin biases on 10 EM iterations.

k	Coin A: $\hat{\theta}_A^{(k)}$	Coin B: $\hat{\theta}_B^{(k)}$
0	0.600	0.500
1	0.713	0.581
2	0.745	0.569
3	0.768	0.550
4	0.783	0.535
5	0.791	0.526
6	0.795	0.522
7	0.796	0.521
8	0.797	0.520
9	0.797	0.520
10	0.797	0.520

Table 3.3: Estimate for coin biases on each EM iteration.

Multinomial Distribution Example:

The probability mass function for a general multinomial distribution gives the probability of counts in k categories after n independent trials. Each category having a given fixed probability p of a count for a particular trial. The general probability mass function for a multinomial distribution is

$$g(\boldsymbol{x} \,|\, p_1, p_2, ..., p_k) = \frac{n!}{x_1!\, x_2! \dots x_k!}\, p_1^{x_1}\, p_2^{x_2} ... p_k^{x_k} \ , \tag{3.6}$$

where $x_i \in (0, 1, ..., n)$, and $\sum_{i=1}^{k} x_i = n$.

Now consider an observed data vector

$$\boldsymbol{x} = (x_1, x_2, x_3, x_4) \ ,$$

which corresponds to a multinomial distribution with $k = 4$. It is postulated that the multinomial distribution is parametrised by ψ, such that $p_1 = \frac{1}{2} + \frac{1}{4}\psi$, $p_2 = \frac{1}{4}(1-\psi)$, $p_3 = \frac{1}{4}(1-\psi)$ and $p_4 = \frac{1}{4}\psi$.

Hence the probability mass function $f(\boldsymbol{x} \,|\, \psi)$ is given by

$$f(\boldsymbol{x} \,|\, \psi) = \frac{n!}{x_1!\, x_2!\, x_3!\, x_4!} \left(\frac{1}{2} + \frac{1}{4}\psi\right)^{x_1} \left(\frac{1}{4}(1-\psi)\right)^{x_2} \left(\frac{1}{4}(1-\psi)\right)^{x_3} \left(\frac{1}{4}\psi\right)^{x_4} \ , \tag{3.7}$$

and the corresponding log-likelihood function in terms of ψ is

$$\log L(\psi \,|\, \boldsymbol{x}) = \log C(\boldsymbol{x})\, p_1^{x_1}\, p_2^{x_2}\, p_3^{x_3}\, p_4^{x_4} \tag{3.8}$$

$$= \log C(\boldsymbol{x}) + x_1 \log p_1 + x_2 \log p_2 + x_3 \log p_3 + x_4 \log p_4 \tag{3.9}$$

$$= \log C(\boldsymbol{x}) - n \log 4 + x_1 \log(2+\psi) + (x_2 + x_3)\log(1-\psi) + x_4 \log \psi, \tag{3.10}$$

where $C(\boldsymbol{x}) = \dfrac{n!}{x_1!\, x_2!\, x_3!\, x_4!}$.

Now consider a modification of the problem which involves unobservable data, and hence lends itself to use of the EM algorithm. Now the first of the original four multinomial cells with associated probability $p_1 = \frac{1}{2} + \frac{1}{4}\psi$ is split up into two sub-cells, x_{11} and x_{12}, having probabilities, $p_{11} = \frac{1}{2}$ and $p_{12} = \frac{1}{4}\psi$ respectively. It follows that $x_{11} + x_{12} = x_1$. Now the observable frequencies $\boldsymbol{x}$ are considered incomplete, and the complete vector of frequencies is given as

$$\boldsymbol{y} = (x_{11},\, x_{12},\, x_2,\, x_3,\, x_4)\ .$$

In this formulation x_{11} and x_{12} are regarded as unobservable and only their sum x_1 is observable. The probability mass function can now be written as

$$f(\boldsymbol{x}\,|\,\psi) = \sum_{\boldsymbol{y}} f_c(\boldsymbol{y}\,|\,\psi)\ , \tag{3.11}$$

where the summation considers all $\boldsymbol{y}$ such that $x_{11} + x_{12} = x_1$, and $f_c(\boldsymbol{y}\,|\,\psi)$ is the probability mass function for the complete data

$$f_c(\boldsymbol{y}\,|\,\psi) = \frac{n!}{x_{11}!\,x_{12}!\,x_2!\,x_3!\,x_4!}\left(\frac{1}{2}\right)^{x_{11}}\left(\frac{1}{4}\,\psi\right)^{x_{12}}\left(\frac{1}{4}\,(1-\psi)\right)^{x_2}\left(\frac{1}{4}\,(1-\psi)\right)^{x_3}\left(\frac{1}{4}\,\psi\right)^{x_4}. \tag{3.12}$$

The log-likelihood function in terms of ψ now takes the form,

$$\begin{aligned} \log L_c(\psi\,|\,\boldsymbol{y}) &= \log f_c(\boldsymbol{y}\,|\,\psi) && (3.13)\\ &= \log C(\boldsymbol{y}) - x_{11}\log 2 - (x_{12} + x_2 + x_3 + x_4)\log 4 \\ &\qquad + (x_{12} + x_4)\log\psi + (x_2 + x_3)\log(1-\psi) && (3.14) \end{aligned}$$

where

$$C(\boldsymbol{y}) = \frac{n!}{x_{11}!\,x_{12}!\,x_2!\,x_3!\,x_4!}\ .$$

Some of these terms are independent of ψ, so don't contribute on differentiation with respect to ψ.

$$\frac{\partial \log L_c(\psi\,|\,\boldsymbol{y})}{\partial\psi} = \frac{x_{12} + x_4}{\psi} - \frac{x_2 + x_3}{1-\psi}\ . \tag{3.15}$$

Equating $\dfrac{\partial \log L_c(\psi\,|\,\boldsymbol{y})}{\partial\psi} = 0$, and solving for ψ yields

$$\begin{aligned} \frac{x_{12} + x_4}{\psi} &= \frac{x_2 + x_3}{1-\psi}, && (3.16)\\ \Rightarrow\ \psi &= \frac{x_{12} + x_4}{x_{12} + x_2 + x_3 + x_4}\ . && (3.17) \end{aligned}$$

Since x_{12} is unobservable the maximum likelihood equation cannot be solved directly, so the EM algorithm is used. The algorithm uses a conditional expectation for x_{12} given the estimate for ψ on the previous iteration to update the parameter estimate. The E step is mathematically formulated as

$$Q(\psi\,|\,\psi^{(0)}) = \mathbb{E}_{\psi^{(0)}}\{\log L_c(\psi\,|\,\boldsymbol{y})\,|\,\boldsymbol{x}\}\ . \tag{3.18}$$

The two sub-cell frequencies x_{11} and x_{12} can be thought of as coming from two binomial distributions, given as, $X_{11} \sim \text{Bin}\left(x_1,\, \dfrac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{4}\psi^{(0)}}\right)$, and, $X_{12} \sim \text{Bin}\left(x_1,\, \dfrac{\frac{1}{4}\psi^{(0)}}{\frac{1}{2} + \frac{1}{4}\psi^{(0)}}\right)$. The expectation of these respective distributions is $\mathbb{E}[X_{11}] = \dfrac{\frac{1}{2}\,x_1}{\frac{1}{2} + \frac{1}{4}\psi^{(0)}}$, and $\mathbb{E}[X_{12}] = \dfrac{\frac{1}{4}\psi^{(0)}\,x_1}{\frac{1}{2} + \frac{1}{4}\psi^{(0)}}$.

Hence the initial conditional expectation of X_{11} given x_1 is

$$\mathbb{E}_{\psi^{(0)}}(X_{11}\,|\,x_1) = x_{11}^{(0)}, \qquad \text{where}\ \ x_{11}^{(0)} = \frac{\frac{1}{2}\,x_1}{\frac{1}{2} + \frac{1}{4}\psi^{(0)}}\ . \tag{3.19}$$

Similarly, the initial conditional expectation of X_{12} given x_1 is

$$\mathbb{E}_{\psi^{(0)}}(X_{12} \,|\, x_1) = x_{12}^{(0)}, \qquad \text{where} \;\; x_{12}^{(0)} = \frac{\frac{1}{4}\,\psi^{(0)}\,x_1}{\frac{1}{2}+\frac{1}{4}\psi^{(0)}} = x_1 - x_{11}^{(0)} \; . \tag{3.20}$$

These conditional expectations can be substituted into the Eq. (3.17) to obtain an estimate $\hat{\psi}^{(k)}$ yielding the following iterative scheme.

$$\begin{aligned} \hat{\psi}^{(k+1)} &= \frac{x_{12}^{(k)} + x_4}{x_{12}^{(k)} + x_2 + x_3 + x_4} \\ &= \frac{x_{12}^{(k)} + x_4}{n - x_{11}^{(k)}} \; , \end{aligned} \tag{3.21}$$

where

$$x_{11}^{(k)} = \frac{\frac{1}{2}x_1}{\frac{1}{2}+\frac{1}{4}\psi^{(k)}} \; , \tag{3.22}$$

and

$$x_{12}^{(k)} = x_1 - x_{11}^{(k)} \; . \tag{3.23}$$

3.2 Mixture Models with EM Algorithm

The usefulness of the Expectation Maximisation algorithm in dealing with incomplete-data problems has been demonstrated. This section focusses on incompleteness which is manifested by mixed distributions. The EM algorithm is a typical procedure for computing maximum likelihood estimates of such distributions.

3.2.1 General Formulation

The postulated probability density function of the finite mixture used to model the observed data vector $\boldsymbol{x} = (x_1, ..., x_n)$ is given as

$$f(x \,|\, \boldsymbol{\psi}) = \sum_{i=1}^{g} \pi_i \, f_i(x \,|\, \boldsymbol{\theta}_i) \; . \tag{3.24}$$

The model is parametrised by the vector $\boldsymbol{\psi} = (\boldsymbol{\pi}, \boldsymbol{\xi})$. As before, $\boldsymbol{\pi} = (\pi_1, ..., \pi_{g-1})$ is a vector containing the $(g$ - $1)$ mixture proportions. The vector $\boldsymbol{\xi} = (\boldsymbol{\theta_1}, ..., \boldsymbol{\theta_g})$ contains the parameter vectors for each component of the mixture. The problem is formulated as an incomplete-data problem by the introduction of an unobserved or missing data vector $\boldsymbol{z} = (\boldsymbol{z}_1, ..., \boldsymbol{z}_{g-1})$. Here $\boldsymbol{z}_i$ represents a g-dimensional vector of zero-one indicator variables, i.e. $z_{ij} = (\mathbf{z}_i)_j$ is a one or zero according to whether x_j arose or did not arise from the ith component of the mixture $(i = 1, ..., g$ and $j = 1, ..., n)$. The EM algorithm handles the addition of the unobservable data by working with the current conditional expectation of the complete-data log-likelihood, given the observed data and the current parameter estimates. As previously defined, the complete-data vector $\boldsymbol{y}$ is

$$\boldsymbol{y} = (\boldsymbol{x} \, , \, \boldsymbol{z}).$$

The complete-data log-likelihood for $\boldsymbol{\psi}$ takes the following form, where the first term is independent of $\boldsymbol{\xi}$ and the second term independent of $\boldsymbol{\pi}$.

$$\log L_c(\boldsymbol{\psi} \,|\, \boldsymbol{y}) = \sum_{i=1}^{g}\sum_{j=1}^{n} z_{ij} \log\big(\pi_i \, f_i(x_j \,|\, \boldsymbol{\theta}_i)\big) \tag{3.25}$$

$$= \sum_{i=1}^{g}\sum_{j=1}^{n} \big(z_{ij}\log \pi_i + z_{ij}\log f_i(x_j \,|\, \boldsymbol{\theta}_i)\big) \; . \tag{3.26}$$

E step:

Calculation of the current conditional expectation of Z_{ij}, given the observed data $\boldsymbol{x}$ is required. Z_{ij} is the random variable corresponding to the missing data z_{ij}.

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\psi}^{(k)}}(Z_{ij} \,|\, \boldsymbol{x}) &= \sum_{r=0}^{1} r \, P_{\boldsymbol{\psi}^{(k)}}(Z_{ij} = r \,|\, \boldsymbol{x}) \\ &= P_{\boldsymbol{\psi}^{(k)}}(Z_{ij} = 1 \,|\, \boldsymbol{x}) \\ &= z_{ij}^{(k)} . \end{aligned} \tag{3.27}$$

From Bayes' theorem the posterior probability that the jth member of the sample (observed value x_j) belongs to the ith component of the mixture is

$$\begin{aligned} z_{ij}^{(k)} &= P_{\boldsymbol{\psi}^{(k)}}(Z_{ij} = 1 \,|\, \boldsymbol{x}) \\ &= \frac{P_{\boldsymbol{\psi}^{(k)}}(\boldsymbol{x} \,|\, Z_{ij} = 1) \, P_{\boldsymbol{\psi}^{(k)}}(Z_{ij} = 1)}{P_{\boldsymbol{\psi}^{(k)}}(\boldsymbol{x})} \\ &= \frac{\pi_i^{(k)} \, f_i(x_j \,|\, \boldsymbol{\theta}_i^{(k)})}{f(x_j \,|\, \boldsymbol{\psi}^{(k)})} \\ &= \frac{\pi_i^{(k)} \, f_i(x_j \,|\, \boldsymbol{\theta}_i^{(k)})}{\sum_{m=1}^{g} \pi_m^{(k)} f_m(x_j \,|\, \boldsymbol{\theta}_m^{(k)})} , \qquad \text{for } i = 1, \ldots, g. \end{aligned} \tag{3.28}$$

Given an estimate $\boldsymbol{\psi}^{(k)}$, the expectation of the log-likelihood function Eq. (3.26) for a general mixed distribution is

$$Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) = \sum_{j=1}^{n} z_{ij}^{(k)} \log f(x_j \,|\, \boldsymbol{\psi}) \tag{3.29}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log \left[\pi_i \, f_i(x_j \,|\, \boldsymbol{\theta_i})\right] \tag{3.30}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log \pi_i + \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log f_i(x_j \,|\, \boldsymbol{\theta_i}) + \lambda \left(\sum_{i=1}^{g} \pi_i - 1 \right) . \tag{3.31}$$

We note the use of the Lagrange multiplier λ which deals with the constraint $\sum_{i=1}^{g} \pi_i = 1$, that all mixture proportions sum to one.

M step:

The EM algorithm requires finding $\boldsymbol{\psi}^{(k+1)}$ which maximises the expectation given by Eq. (3.31). Taking the derivative of this expression with respect to π_i, and equating it to zero, gives the value of $\boldsymbol{\pi}^{(k+1)}$ corresponding to the maximum value of $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$. This gives

$$\frac{1}{\pi_i} \sum_{j=1}^{n} z_{ij}^{(k)} + \lambda = 0 . \tag{3.32}$$

Noting that $\sum_{i=1}^{g} z_{ij}^{(k)} = 1$, one can solve for the lagrange multiplier λ.

$$\begin{aligned} \lambda \, \pi_i &= - \sum_{j=1}^{n} z_{ij}^{(k)} \\ \Rightarrow \lambda \sum_{i=1}^{g} \pi_i &= - \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \\ \Rightarrow \lambda &= -n . \end{aligned} \tag{3.33}$$

Hence, the iterative estimate of the mixture proportions is given as

$$\boxed{\pi_i^{(k+1)} = \sum_{j=1}^{n} \frac{z_{ij}^{(k)}}{n}}. \tag{3.34}$$

Similarly, the parameter vector $\boldsymbol{\xi}^{(k+1)}$ which maximises $Q(\boldsymbol{\psi}\,|\,\boldsymbol{\psi}^{(k)})$ is given as the appropriate root of the following expression

$$\frac{\partial\, Q(\boldsymbol{\psi}\,|\,\boldsymbol{\psi}^{(k)})}{\partial\, \boldsymbol{\xi}} = 0 \ , \tag{3.35}$$

$$\Rightarrow \quad \boxed{\sum_{j=1}^{n} z_{ij}^{(k)} \frac{\partial \log f_i(x_j\,|\,\boldsymbol{\theta_i})}{\partial\, \boldsymbol{\theta_i}} = 0}. \tag{3.36}$$

This procedure is iterated until convergence of the log-likelihood function is reached, i.e. until the difference of two consecutive log-likelihood values is smaller than a predetermined tolerance ϵ,

$$\log L(\boldsymbol{\psi}^{(k+1)}) - \log L(\boldsymbol{\psi}^{(k)}) \ \leq \ \epsilon \ . \tag{3.37}$$

The use of the term 'algorithm' in Expectation Maximisation algorithm is somewhat misleading. The algorithm is better described as a pseudo-algorithm, that is, for maximum likelihood estimation to be undertaken, specific EM equations are required to be derived for certain applications. No algorithm exists which can be applied in a general setting. Each parametric component in a mixed distribution corresponds to separate EM equations. Later chapters derive full EM equations for the parametric components of interest to this work.

3.2.2 Introductory Example: Gaussian Mixture Model

Consider a finite normal mixture where to begin with the parameters of each mixture component are known *a priori* and only mixture proportions require to be estimated. This section uses normal mixtures to motivate the use of mixture modelling with the EM algorithm. Chapter 5 provides a more thorough treatment of normal mixtures.

Consider a mixture of two univariate normal distributions. Mixture components have probability density functions $f_1(x_j)$ and $f_2(x_j)$ respectively, and cumulative distribution functions $F_1(x_j)$ and $F_2(x_j)$ respectively. It is assumed that parameters μ_1, σ_1, μ_2, and σ_2 are known *a priori*, hence only π_1 requires estimation.

- $$f_i(x_j\,|\,\mu_i\ \sigma_i^2) \quad = \quad \frac{1}{\sqrt{2\pi\sigma_i^2}}\, e^{-(x_j-\mu_i)^2/2\sigma_i^2} \ , \tag{3.38}$$
- $$F_i(x_j\,|\,\mu_i\ \sigma_i^2) \quad = \quad \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x_j-\mu_i}{\sigma_i\sqrt{2}}\right)\right] \ , \qquad i = 1, 2. \tag{3.39}$$

The indicator variable Eq. (3.28) on the k^{th} iteration of the EM algorithm becomes,

$$z_{1j}^{(k)} \quad = \quad \frac{\pi_1^{(k)} f_1(x_j\,|\,\mu_1, \sigma_1^2)}{\pi_1^{(k)} f_1(x_j\,|\,\mu_1, \sigma_1^2) + \pi_2^{(k)} f_2(x_j\,|\,\mu_2, \sigma_2^2)} \tag{3.40}$$

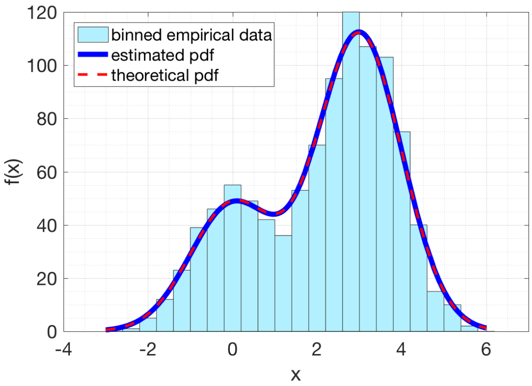
Hence estimation of mixture proportion π_1 on $(k+1)^{\text{th}}$ iteration is simply given by,

$$\pi_1^{(k+1)} = \sum_{j=1}^{n} \frac{z_{1j}^{(k)}}{n}. \tag{3.41}$$

Mixture proportion $\pi_2^{(k+1)}$ on each iteration follows by the normalisation condition as $1 - \pi_1^{(k+1)}$.

Consider an observed data vector $\boldsymbol{x} = (x_1, ..., x_{1000})$ which is sampled from a two-component normal mixture defined by parameters $\mu_1 = 0,\ \sigma_1 = 1,\ \mu_2 = 4,\ \sigma_2 = 1$, and $\pi_1 = 0.3$.
Figure 3.1 shows the probability density function of the mixture as an overlay of the empirical

data. The EM algorithm is used find an estimate $\hat{\pi}_1$ for the unknown mixture proportion. The parameters of the component distributions are assumed known *a priori*. Initialisation is defined by the mixture proportion $\pi_1^{(0)} = 0.7$, and iteration stopped under the convergence condition when $|\,\pi_1^{(k+1)} - \pi_1^{(k)}| < 10^{-5}$. In this case, because only one parameter is being estimated, convergence can simply be defined in terms of the parameter.



Iteration number	$\pi_1^{(k)}$
0	0.40000
1	0.33063
2	0.31705
3	0.31428
4	0.31371
5	0.31360
6	0.31357
7	0.31357

Figure 3.1: Estimated and theoretical probability density functions. Mixture proportion estimation on each iteration.

3.3 Convergence Properties of EM

This section highlights key convergence properties of the EM algorithm, although is by no means a detailed treatment. For a thorough review of convergence and rate of convergence properties including rigorous mathematical theory and proofs consult Wu [44], Redner et al. [13], or McLachlan et al. [12].

A fundamental property which characterises the behaviour of the EM algorithm is that the value of the likelihood function for the incomplete-data problem is not decreased after an EM iteration,

$$L(\boldsymbol{\psi}^{(k+1)}) \;\geq\; L(\boldsymbol{\psi}^{(k)}) \qquad\qquad \text{for } \; k = 0, 1, 2, \ldots \tag{3.42}$$

This was shown by Dempster et al. [36] in their seminal paper. The explanation provided within this section is largely based on the work of McLachlan et al. [12], with the main differences being notational. Let

$$k(\boldsymbol{y}\,|\,\boldsymbol{x},\,\boldsymbol{\psi}) = \frac{f_c(\boldsymbol{y}\,|\,\boldsymbol{\psi})}{f(\boldsymbol{x}\,|\,\boldsymbol{\psi})} \tag{3.43}$$

be the conditional density of random variable $\boldsymbol{Y}$ which is associated with the complete-data problem. Random variable $\boldsymbol{X}$ is associated with the incomplete-data problem. The observed data vector $\boldsymbol{x}$ is a realisation of random variable $\boldsymbol{X}$ such that $\boldsymbol{X} = \boldsymbol{x}$. Hence the log-likelihood can be expressed as

$$\begin{aligned} \log L(\boldsymbol{\psi}\,|\,\boldsymbol{x}) &= \log f(\boldsymbol{x}\,|\,\boldsymbol{\psi}) && (3.44)\\ &= \log f_c(\boldsymbol{y}\,|\,\boldsymbol{\psi}) - \log k\,(\boldsymbol{y}\,|\,\boldsymbol{x},\,\boldsymbol{\psi})\;. && (3.45) \end{aligned}$$

The E step of the EM algorithm requires an expectation of Eq. (3.45) to be taken conditioned on the random variable $\boldsymbol{X} = \boldsymbol{x}$, and the current parameter estimate $\boldsymbol{\psi}^{(k)}$.

$$\begin{aligned} \log L(\boldsymbol{\psi}\,|\,\boldsymbol{x}) &= \mathbb{E}_{\boldsymbol{\psi}^{(k)}}[\log L_c(\boldsymbol{\psi}\,|\,\boldsymbol{y})\,|\,\boldsymbol{x}\,] - \mathbb{E}_{\boldsymbol{\psi}^{(k)}}[\log k(\boldsymbol{Y}\,|\,\boldsymbol{x}\,,\boldsymbol{\psi}^{(k)})|\,\boldsymbol{x}\,] && (3.46)\\ &= Q(\boldsymbol{\psi}\,|\,\boldsymbol{\psi}^{(k)}) - H(\boldsymbol{\psi}\,|\,\boldsymbol{\psi}^{(k)}) && (3.47) \end{aligned}$$

where $H(\boldsymbol{\psi}\,|\,\boldsymbol{\psi}^{(k)}) = \mathbb{E}_{\boldsymbol{\psi}^{(k)}}[\log k(\boldsymbol{Y}\,|\,\boldsymbol{x}\,,\boldsymbol{\psi}^{(k)})|\,\boldsymbol{x}\,]$.

It follows from Eq. (3.47) that

$$\log L(\boldsymbol{\psi}^{(k+1)} \,|\, \boldsymbol{x}) - \log L(\boldsymbol{\psi}^{(k)} \,|\, \boldsymbol{x}) = \Big(Q(\boldsymbol{\psi}^{(k+1)} \,|\, \boldsymbol{\psi}^{(k)}) - Q(\boldsymbol{\psi}^{(k)} \,|\, \boldsymbol{\psi}^{(k)}) \Big) - \Big(H(\boldsymbol{\psi}^{(k+1)} \,|\, \boldsymbol{\psi}^{(k)}) - H(\boldsymbol{\psi}^{(k)} \,|\, \boldsymbol{\psi}^{(k)}) \Big) \,. \tag{3.48}$$

On the previous maximisation step $\boldsymbol{\psi}^{(k+1)}$ was chosen such that

$$Q(\boldsymbol{\psi}^{(k+1)} \,|\, \boldsymbol{\psi}^{(k)}) \,\geq\, Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) \qquad \forall\, \boldsymbol{\psi} \in \Omega \,. \tag{3.49}$$

Hence Eq. (3.42) has been shown if

$$H(\boldsymbol{\psi}^{(k+1)} \,|\, \boldsymbol{\psi}^{(k)}) \leq H(\boldsymbol{\psi}^{(k)} \,|\, \boldsymbol{\psi}^{(k)}) \,. \tag{3.50}$$

This can be shown for any $\boldsymbol{\psi}$

$$\begin{aligned} H(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) - H(\boldsymbol{\psi}^{(k)} \,|\, \boldsymbol{\psi}^{(k)}) &= \mathbb{E}_{\boldsymbol{\psi}^{(k)}} \left[\log \left(\frac{k(\boldsymbol{Y} \,|\, \boldsymbol{x}\,, \boldsymbol{\psi})}{k(\boldsymbol{Y} \,|\, \boldsymbol{x}\,, \boldsymbol{\psi}^{(k)})} \right) \middle| \, \boldsymbol{x} \right] \\ &\leq \log \left(\mathbb{E}_{\boldsymbol{\psi}^{(k)}} \left[\frac{k(\boldsymbol{Y} \,|\, \boldsymbol{x}\,, \boldsymbol{\psi})}{k(\boldsymbol{Y} \,|\, \boldsymbol{x}\,, \boldsymbol{\psi}^{(k)})} \,\middle|\, \boldsymbol{x} \right] \right) \\ &= \log \int_{\chi(\boldsymbol{X})} k(\boldsymbol{y} \,|\, \boldsymbol{x},\, \boldsymbol{\psi})\, d\boldsymbol{y} \\ &= 0 \,. \end{aligned} \tag{3.51}$$

where Jensen's inequality ($\phi(\mathbb{E}[X]) \leq \mathbb{E}[\phi(X)]$) and the concavity of the logarithmic function are used. This establishes Eq. (3.50). The value of the likelihood function will increase on each EM iteration if the inequality in Eq. (3.49) is strict. The key result can be stated as, **if a sequence of likelihood values $L(\boldsymbol{\psi}^{k})$ for $k = 0, 1, 2, \ldots$ is bounded from above then $L(\boldsymbol{\psi}^{k})$ will converge monotonically to a given value L^***. In almost all applications L^* is a stationary value of the likelihood function, meaning that $\partial L(\boldsymbol{\psi})/\partial \boldsymbol{\psi} = 0$. Almost surely the EM algorithm will converge to either a local or global maximum. Occasionally the algorithm may be caught in a saddle point, in which case a small perturbation of $\boldsymbol{\psi}$ will result in a divergence from the saddle point. Hence, whether the algorithm converges to one of these stationary points or more generally to a local or global maximum is dependent on the choice of initial values $\boldsymbol{\psi}^{(0)}$. A detailed treatment of this is provided in Wu [44].

3.4 Comparison to Alternate Methods

The EM algorithm is an iterative procedure for finding **maximum likelihood estimates** of parameters in a statistical model. What differentiates maximum likelihood estimation via the EM algorithm from many other methods is the requirement to specify initial values. Estimates are consequently dependent on the choice of these values. On the other hand, estimation via a direct solution of the maximum likelihood equations doesn't require initialisation so is only dependent on the data. Although the EM algorithm is the primary tool for parameter estimation in incomplete-data problems like mixtures, it would be remiss not to briefly discuss the main drawbacks of the algorithm and mention alternate methods.

Pearson [30] used the **method of moments** to find estimates of parameters for a two-component unimodal Gaussian mixture. The method of moments solves a system of equations equating sample moments to theoretical moments, moments which are a function of the parameters required to be estimated [45]. The method is a relatively simple, consistent estimator, calculable without large computing resources, but it is often biased. The ease of access to modern computing resources has allowed maximum likelihood estimation, which exhibits far less bias, to be a far superior method [13].

Other alternatives to maximum likelihood estimation include the **maximum a posteriori estimator, minimum χ^2 estimation, linear unbiased estimator, and bayes estimator, and various application specific graphical methods**.

If solutions of the maximum likelihood equations cannot be found directly, often an iterative maximum likelihood method, or a discretised direct search of the log-likelihood function is required. Common iterative procedures include, **Newton-Raphson**, various quasi-Newton procedures like **Fisher scoring**, **conjugate gradient methods**, and the **EM algorithm**. A summary of the brief properties, both pros and cons of the EM algorithm are discussed below with comparison to other iterative procedures.

Advantages:

- The value of the likelihood function is not decreased by an iteration of the EM algorithm. This fundamental convergence property ensures numerical stability of the algorithm.
- The EM algorithm has reliable convergence. Given an arbitrary initial value $\boldsymbol{\psi}^{(0)}$, convergence is nearly always to a local or global maximum of the likelihood function.
- The EM algorithm is fairly easy to implement, with EM equations often existing in closed form. The ease of implementation is due to the algorithm not requiring evaluation of the likelihood function, nor any of its derivatives on any iteration.[2]
- The EM algorithm easily deals with incompleteness. This can be from unobservable or missing data, truncation, censoring, mixtures, or other non-traditional forms.

Disadvantages:

- The main drawback of the EM algorithm is that it is often very slow to converge, even in seemingly rudimentary scenarios.
- If within the parameter space, multiple local maxima exist in the log-likelihood function, the EM algorithm is not guaranteed to converge to the global maximum (which is in general true of all optimisation procedures) with estimates dependent on choice of initial values $\boldsymbol{\psi}^{(0)}$.
- For multivariate mixtures, the EM algorithm provides no inbuilt procedure for producing an estimate of the covariance matrix of the parameter estimates, unlike Fisher scoring.

With this in mind, the EM algorithm is used exclusively for parameter estimation in this thesis.

[2]The likelihood function is often evaluated in order to define the convergence criteria.

Chapter 4

Goodness of Fit Testing

When dealing with real data, determining the adequacy of a statistical model is of the utmost importance. Goodness-of-fit testing is a method of assessing how well a set of observations fit a hypothesised statistical model. Many such tests exist, each quantifying goodness-of-fit in a slightly different manner. The majority of tests involve the use of a metric quantifying the level of discrepancy between observations and the values predicted by the statistical model in question. In the process of determining the validity of a statistical model, 'goodness-of-fit tests' assess how likely the observed data is, given a hypothesised distribution. This thesis will consider two goodness-of-fit frameworks, formal *hypothesis* based testing, and model selection via *information criteria* tests. These frameworks will be presented within this chapter, beginning with hypothesis testing, and emphasising goodness-of-fit testing of mixture models.

4.1 Hypothesis Based Tests

Hypothesis testing refers to the formal procedure of utilising *test statistics* to determine whether a null hypothesis H_0 can be rejected given a set of observations $\boldsymbol{x} = (x_1, x_2, ..., x_n)$. This work deals with null hypotheses H_0 that a given set of n independent identically distributed (i.i.d) observations $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ arise from a population that is distributed according to a hypothesised model, with a cumulative distribution function $F(\boldsymbol{x} \,|\, \hat{\boldsymbol{\psi}})$. In the context of this thesis, the cumulative distribution function $F(\boldsymbol{x} \,|\, \hat{\boldsymbol{\psi}})$ describes a mixture model, and maximum likelihood estimates of statistical parameters $\hat{\boldsymbol{\psi}}$ are found using the Expectation Maximisation algorithm. If the null hypothesis is rejected, it is rejected in favour of the alternative hypothesis H_1.

- **Null Hypothesis**, H_0:
  the observed data $\boldsymbol{x}$ originates from the proposed distribution $F(\boldsymbol{x} \,|\, \hat{\boldsymbol{\psi}})$.
- **Alternative Hypothesis**, H_1:
  the observed data doesn't originate from the proposed distribution.

Many goodness-of-fit tests exist, each defining a unique measure of fit and associated criteria for determining whether one can reject the null hypothesis H_0 for a given set of observations $\boldsymbol{x}$. Because of the prevalence of type I and type II errors, it is important to note that both the null and alternative hypothesis can never be proven. The quality of a goodness-of-fit test is determined by the extent of these errors.

- **Type I Error**: H_0 is true, but rejected

  The significance level α, is the measure of type I error. It is specified by the probability of rejecting the null hypothesis given that it is true,

$$\alpha = P\left(\text{reject } H_0 \,|\, H_0 \text{ is true}\right). \tag{4.1}$$

- **Type II Error**: H_0 is false, but not rejected

  The statistical power β, is the measure of type II error. The probability of not rejecting the null hypothesis given that it is false is,

$$1 - \beta = P\,(\text{do not reject } H_0 \,|\, H_0 \text{ is false})\,. \tag{4.2}$$

An accurate determination on the outcome of the null hypothesis can be made if both α and $1-\beta$ are small. In reality this is difficult because in an effort to decrease the probability of type I errors, an increase in the probability of type II errors often results, and vice versa. Goodness-of-fit tests usually produce a *test statistic* (real, scalar measure) which quantifies the level of agreement between sample data $\boldsymbol{x}$, and a selected statistical model (specified by the cumulative distribution function $F(\boldsymbol{x}\,|\,\hat{\boldsymbol{\psi}})$). A smaller test statistic corresponds to a stronger level of agreement between the dataset $\boldsymbol{x}$ and the hypothesised model $F(\boldsymbol{x}\,|\,\hat{\boldsymbol{\psi}})$, and hence the prediction of the null hypothesis. To determine whether or not to reject the null hypothesis, a comparison of the test statistic to a previously known *critical value* is made. **If the test statistic is smaller than the critical value, the null hypothesis cannot be rejected**.

The coming sections will present the test statistics of interest to this work, before Section 4.1.7 clarifies (by reference to the distribution of these test statistics) what a critical value represents, and how they can be determined. The formulation of test statistics is contingent on the empirical distribution function which will be presented first.

4.1.1 Empirical Distribution Function

The empirical distribution function (edf) is the cumulative distribution function of an empirical measure of a sample. In other words, it is the cumulative distribution function associated with the given set of observations $\boldsymbol{x} = (x_1, x_2, ..., x_n)$. The edf is a step function that increases by $1/n$ at each of the n data points within a sample. The value of the edf at an arbitrary point t is given by the proportion of observations in the sample $\boldsymbol{x}$ that are less than or equal to t. The edf can be conceptualised as an estimate of the cumulative distribution function describing the underlying distribution in which the data is sampled. According to the Glivenko-Cantelli theorem [46], as n increases, it will converge to the true cdf of the underlying distribution. The edf is denoted $F_n(t)$, and is defined as

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{x_i \le t} \quad \text{where} \quad \mathbb{I}_{x_i \le t} = \begin{cases} 1 & \text{if } x_i \le t \\ 0 & \text{if } x_i > t\,. \end{cases} \tag{4.3}$$

Figure 4.1 displays the cumulative distribution function for the standard normal distribution, and the empirical distribution function for $n = 25$ observations drawn from this same distribution. It can be observed that the edf estimates the cdf and is a step function, increasing the value of the cumulative probability by $1/25$ at each data point. In the context of hypothesis testing and edf based goodness-of-fit testing, the difference between edf, $F_n(t)$, and cdf, $F(t)$, is the basis for much of test statistic calculation.

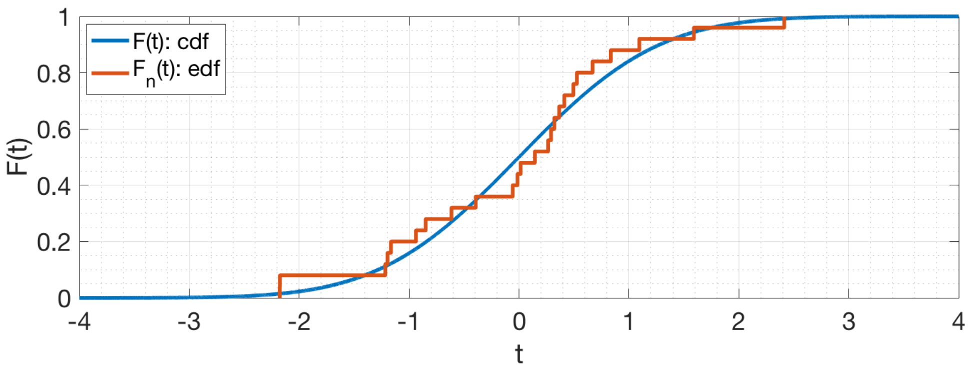


Figure 4.1: Comparison of standard normal distribution cdf and edf ($n = 25$).

4.1.2 Pearson's χ^2 Test

Pearson's χ^2 test is perhaps the most commonly used (hypothesis based) goodness-of-fit test used throughout history, and is the classical approach for goodness of fit problems [7, 47]. Like all goodness-of-fit tests discussed in this section, Pearson's χ^2 test allows a null hypothesis to be tested, determining whether a given observational dataset is consistent with a specified statistical distribution. For a given dataset $\boldsymbol{x} = (x_1, x_2, ..., x_n)$, the Pearson χ^2 test statistic is defined in terms of the edf as

$$\chi^2 = \sum_{i=1}^{n} \frac{(\frac{i}{n} - F(x_i))^2}{F(x_i)} \ . \tag{4.4}$$

Importantly Pearson [47] showed for the case where the null hypothesis is true, and n the number of data points is large enough, the χ^2 distribution describes the distribution of the χ^2 test statistics.[1] The cdf for the χ^2 distribution is given by

$$F(x \,|\, \nu) = \frac{1}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} \int_0^x t^{\frac{\nu}{2}-1} e^{-\frac{t}{2}} dt \ , \tag{4.5}$$

where $\nu = n - 1 - d$ is the number of degrees of freedom, d is the number of model parameters estimated from the data, and $\Gamma(x)$ is the Gamma function,

$$\Gamma(x) = \int_0^\infty z^{x-1} e^{-z} dz \ . \tag{4.6}$$

The test statistics for the Pearson χ^2 test are distributed according to the χ^2 distribution, which is fully specifiable, so it is simple to undertake hypothesis testing. Before Monte Carlo simulations were a viable way of producing critical values[2], Pearson's χ^2 test was one of the only tests in which the critical values could be determined when parameters were estimated from the data. While Pearson's χ^2 test is very popular within the literature, a number of studies have shown that under certain circumstances alternate edf based statistical tests are more powerful, with greater discriminatory power [7, 48–50].

4.1.3 Kolmogorov-Smirnov Test Statistics

The Kolmogorov-Smirnov (KS) test is a goodness-of-fit test proposed and named after mathematicians Andrey Kolmogorov and Nikolai Smirnov. Kolomogorov [51] developed a test which could be used to compare a set of observations to a theorised probability distribution (known as the one-sample KS-test). Later, Smirnov [52] developed the two-sample KS-test which could be used to make a comparison between two data samples. This work concerns itself with the one-sample KS-test. The Kolmogorov-Smirnov test statistic quantifies a maximum distance between the previously defined empirical distribution function $F_n(t)$, and the cumulative distribution function of the proposed reference distribution $F(t)$. KS-testing is restricted to continuous probability distributions, usually of univariate nature.

The Kolmogorov-Smirnov test statistic is a *supremum class* statistic meaning it relates to the supreme (or maximum) distance between the edf and cdf. D^+ denotes the maximum value of the cdf subtracted from the edf, correspondingly D^- denotes the maximum value of the edf subtracted from the cdf, which are given as

$$\begin{aligned} D^+ &= \sup_{\tau_l \le x \le \infty} [F_n(x) - F(x)] = \max_{1\le i\le n} \left[\frac{i}{n} - F(x_i)\right] \ , \\ D^- &= \sup_{\tau_l \le x \le \infty} [F(x) - F_n(x)] = \max_{1\le i\le n} \left[F(x_i) - \frac{i-1}{n}\right] \ , \end{aligned} \tag{4.7}$$

where observations $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ define the edf.

[1] Noting a distinction between the χ^2 test statistic, and the χ^2 distribution.

[2] Section 4.1.7 introduces a general formulation of critical values, with Section 4.1.8 providing a general Monte-Carlo procedure for their calculation.

The Kolmogorov-Smirnov test statistic is given as $\sqrt{n}D$, where D is the KS-distance, defined as

$$\begin{aligned} D &= \sup_{\tau_l \leq x \leq \infty} \left[|F_n(x) - F(x)|, |F(x) - F_n(x)| \right] \\ &= \max_{1 \leq i \leq n} \left[\left| \frac{i}{n} - F(x_i) \right|, \left| F(x_i) - \frac{i-1}{n} \right| \right] \\ &= \max \left[D^+, D^- \right]. \end{aligned} \tag{4.8}$$

The cdf and edf are both monotonically increasing functions. This means that generally the Kolmogorov-Smirnov test has a greater sensitivity to discrepancies near the median value of the sample than at the tails. The Kolmogorov-Smirnov test was one of the first and has become one of the most commonly used edf based goodness-of-fit tests. Critical values for the Kolmogorov-Smirnov test can be found if the full distribution of KS-test statistics can be determined under the assumption that the null-distribution is true. To the knowledge of the author, very few studies have been undertaken regarding the distribution of KS-test statistics for the case of mixed distributions (with corresponding parameters estimated from the data). Chapter 5 (normal distribution) and 6 (Weibull distribution) empirically justify a claim that the distribution of KS-test statistics are in fact dependent on the null distribution. The nature of this dependency will be further discussed.

4.1.4 Kuiper Test Statistics

Nicolaas Kuiper proposed a modification to the well known Kolomogorov-Smirnov goodness-of-fit test in 1960 [53]. Similar to the KS-test, the Kuiper test statistic involves the discrepancy statistics D^+ and D^- which represent the maximum positive and negative differences between the edf and cdf (of the reference distribution). The Kuiper test statistic uses the quantity $D^+ + D^-$, instead of $\max[D^+, D^-]$ which was seen in the KS-test. The test therefore is sensitive around the median of the distribution but also at the tails. The Kuiper test statistic is defined as $\sqrt{n}V$, where V is analogous to the KS-distance for the Kuiper test,

$$\begin{aligned} V &= \sup_{\tau_l \leq x \leq \infty} \Big[F_n(x) - F(x) \Big] + \sup_{\tau_l \leq x \leq \infty} \Big[F(x) - F_n(x) \Big] \\ &= \max_{1 \leq i \leq n} \left[\frac{i}{n} - F(x_i) \right] + \max_{1 \leq i \leq n} \left[F(x_i) - \frac{i-1}{n} \right] \\ &= D^+ + D^-. \end{aligned} \tag{4.9}$$

As for the Kolmogorov-Smirnov test, little is known regarding the distribution of Kuiper test statistics for mixed distributions.

4.1.5 Cramér-von Mises Test Statistics

The Cramér-von Mises test is a goodness-of-fit test named after Harold Cramér and Richard von Mises, who proposed the test in 1928 [54, 55]. The Cramér-von Mises test approaches the task of determining whether a given data sample was drawn from a proposed reference distribution in a manner that differs from previously discussed tests. The Cramér-von Mises test statistic is a *quadratic* edf class test statistic Q, which is defined in relation to the integral of the squared difference between the edf and cdf.

$$Q = n \int_{-\infty}^{\infty} \left[F_n(x) - F(x) \right]^2 \psi \left\{ F(x) \right\} dF(x), \tag{4.10}$$

where $\psi\{F(x)\}$ is a non-negative weight function. The Cramér-von Mises test statistic has the weight function set to one, i.e. $\psi\{F(x)\} = 1$, so the test statistic is simply the integral of the squared difference between the cdf and edf. The Cramér-von Mises test statistic W^2 is defined as follows

$$W^2 = n \int_{-\infty}^{\infty} \left[F_n(x) - F(x) \right]^2 dF(x) . \tag{4.11}$$

This integral can be written as a sum in terms of the vector $\boldsymbol{x}$, as shown by Anderson and Darling [56],

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{2i-1}{2n} - F(x_i) \right)^2 . \tag{4.12}$$

For mixed distributions, the distribution of Cramér-von Mises test statistics is unknown.

4.1.6 Anderson-Darling Test Statistics

The Anderson-Darling test was proposed in 1952 and named after mathematicians Theodore Anderson and Donald Darling [57]. It was proposed as an alternative to the Cramér-von Mises test, as it is even more sensitive in the tails of the distribution, and uses an alternate weight function,

$$\psi\left\{F(x)\right\} = \frac{1}{F(x)\left[1-F(x)\right]} \ . \tag{4.13}$$

The Anderson-Darling test statistic A^2 is then given by

$$A^2 = n\int_{-\infty}^{\infty} \frac{\left[F_n(x)-F(x)\right]^2}{F(x)\left[1-F(x)\right]} dF(x) \ , \tag{4.14}$$

which can be rewritten in terms of the following summation,

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}\left\{2i-1\right\}\left\{\log[F(x_i)] + \log[1-F(x_{n-i+1})]\right\} \ . \tag{4.15}$$

Like all goodness-of-fit tests discussed thus far, the distribution of Anderson-Darling test-statistics is unknown for mixed distributions. Empirical justification will be provided in this thesis that the distribution of test-statistics for mixed distributions is dependent on the null-distribution. Because of this, the critical values also display this dependency. The following section provides a more comprehensive approach to critical value representation, and calculation, with specific focus on mixed distributions.

4.1.7 Critical Values

In the context of hypothesis testing, a critical value is a pre-determined point on a distribution which is compared to a test statistic in order to determine whether a null hypothesis should be rejected. Fundamentally, **if the value of the test statistic is smaller than the critical value, the null hypothesis cannot be rejected**.

Recall that the significance level α is the probability of rejecting the null hypothesis given that it is true, i.e. $\alpha = P\left(\text{reject } H_0 \,|\, H_0 \text{ is true}\right)$. If the distribution of test-statistics is known, or can be determined under the assumption that the null-hypothesis is true, the critical value is determined such that $100(1-\alpha)\%$ of the test-statistics are less than or equal to it. For example, typically a statistical significance level of $\alpha = 0.05$ is selected. In this case, the critical value is determined such that 95% of test statistics fall below it (or equivalently 5% of the test statistics fall above it). Figure 4.2 illustrates the representation of critical values from the distribution of test-statistics.[3] The critical value is chosen such that the probability that a random test-statistic drawn from the distribution T is greater than the value $D_{cv\,0.95}$ is given as the confidence level $\alpha = 0.05$. i.e. $P(T > D_{cv\,0.95}) = 0.05$.

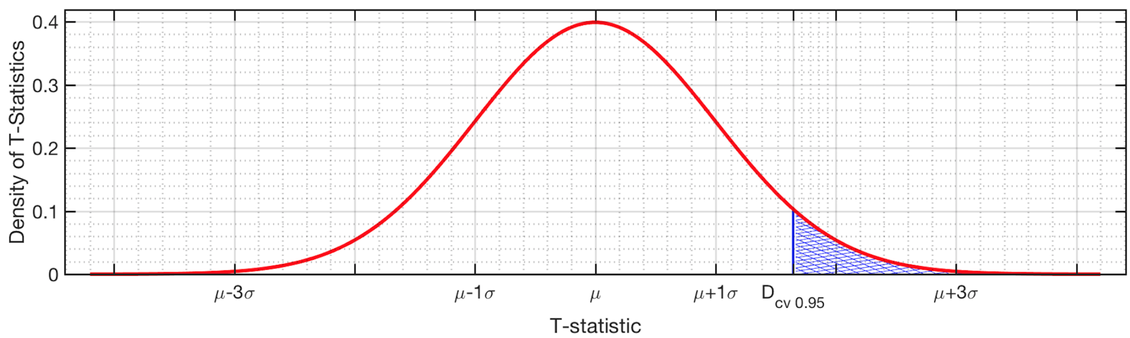


Figure 4.2: Distribution of test-statistics displaying the 95% ($\alpha = 0.05$) confidence level critical value.

[3] Note that this is just an illustrative example and the distribution of test-statistics is not necessarily Gaussian.

If a dataset $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ passes a goodness-of-fit test at the 95% ($\alpha = 0.05$) confidence level, the resulting test statistic is less than the highest $100\alpha\%$ of test statistics which resulted under the assumption of the null-hypothesis H_0 was true. A common misconception is that there is 95% confidence that the null-hypothesis is true. This false statement may be reworded as, "there is 95% chance that $\boldsymbol{x}$ was drawn from the distribution defining the null-hypothesis". What can actually be said is, for the given goodness-of-fit test, the sample $\boldsymbol{x}$ performed better than 5% of additional samples which were drawn from the distribution defining the null-hypothesis (when the null distribution is true). An increased confidence level α therefore corresponds to a less stringent test. A goodness-of-fit test at the 95% ($\alpha = 0.05$) significance level is more difficult to pass than those at 99% ($\alpha = 0.01$). Hence, if a null-hypothesis is not rejected at 95% significance level, we have more confidence that the null hypothesis is true than if it was not rejected at the 99% significance level. As the 99% critical value for a goodness-of-fit test is necessarily higher than the 95% critical value, if the null-hypothesis is not rejected at 95% significance level it will also not be rejected at 99%.

It should now be obvious that in order to determine specific critical values, it is necessary to understand the nature of the full distribution of test statistics, under the assumption of the null-hypothesis being true. For many early applications of goodness-of-fit tests on single distributions, especially when parameters weren't being estimated from the data, the distribution of test-statistics could be found analytically. The dependencies of critical values when testing a single distribution are summarised below [5, 6, 20],

- Critical values generally depend on sample size n, significance level α, and truncation level τ[4].
- When no parameter estimation is undertaken from the data, critical values are **distribution independent**.
- When parameter estimation is undertaken from the data, critical values are **distribution dependent**, but remain **parameter independent**. Kizilersü et al. [5] described a theoretical approach using pivotal functions to demonstrate this parameter independence.

When the distribution of test-statistics is unknown, or cannot be easily determined, a Monte-Carlo procedure can be utilised. The specific Monte-Carlo procedure developed for this work, for computation of critical values for mixed distributions, will be introduced in Section 4.1.8.

If parameters of the null-distribution are estimated from the given data vector $\boldsymbol{x}$, it is understood that the discrepancy between the hypothesised distribution and the empirical distribution function will be reduced. Mathematically, the edf will better approximate the hypothesised cdf specified by the null distribution. This will correspond to a reduction in the test-statistics, and hence a reduction of the critical values.

Critical Values for Mixture Models

This thesis considers null distributions which specify hypothesised g-component finite mixtures (of parametric distributions). Estimation of parameter vector $\boldsymbol{\psi} = (\boldsymbol{\pi}, \boldsymbol{\xi})$ is considered from the observed data $\boldsymbol{x}$. The individual parameter vectors of the component distributions are represented by $\boldsymbol{\xi} = (\boldsymbol{\theta_1}, ..., \boldsymbol{\theta_g})$ and $\boldsymbol{\pi} = (\pi_1, ..., \pi_{g-1})$ is the vector of $(g-1)$ mixture proportions. For mixed distributions, a significant outcome of this thesis was empirically determining the nature of critical value dependencies, which can be summarised as,

- The general dependence on sample size n, significance level α, and truncation level τ is retained.
- **Critical values now display parameter and distribution dependence**, even for the case when no estimation is undertaken, i.e model parameters assumed to be known *a priori*.

Section 5.2 and 6.2 provide empirical justification (and a significantly more thorough explanation) for normal and Weibull mixtures respectively. Because of these dependencies, critical values vary for different null distributions, and a Monte-Carlo procedure is required for their calculation.

[4] Although this thesis deals exclusively with untruncated distributions.

4.1.8 Monte-Carlo Procedure for Critical Value Computation

When critical values depend on parameter estimates (which come from a continuous solution space), one cannot have access to all required critical values, because there is an uncountably infinite extent of possible parameter estimates. A Monte-Carlo procedure can be used to numerically determine critical values for null-distributions in these cases.

Algorithm 1 outlines an example procedure for Kolmogorov-Smirnov critical value calculation of a two-component normal mixture. In this example parameter vectors $\boldsymbol{\xi} = (\boldsymbol{\theta_1},\, \boldsymbol{\theta_2})$ are assumed to be known *a priori*, with only the mixture proportion π_1 requiring estimation. To calculate the corresponding critical value, the distribution of test statistics under the assumption that the null-hypothesis is true, is required to be determined. This involves sampling data of a given sample size (eg. $n = 1000$) from the specified distribution of interest (eg. $\mu_1 = 0,\ \sigma_1 = 1,$ $\mu_2 = 3,\ \sigma_2 = 1$), estimating the required parameters (eg. π_1), and calculating the test statistic at a given significance (eg. $\alpha = 0.05$). This process is repeated for a given number of simulations (eg. numSims $= 10000$) to build the distribution of test statistics, and hence the corresponding critical value can be determined. An average over C repetitions (eg. $C = 100$) is then taken. This multisample method is similar to that proposed by Schafer [58].

Algorithm 1 Monte-Carlo procedure for critical value calculation.

$\mu_1 = 0, \quad \sigma_1 = 1, \quad \mu_2 = 3, \quad \sigma_2 = 1$
$\pi_1 = 0.3$
$n = 1000$ (number of data points)
numSims $= 10000$ (number of simulations)
$C = 100$ (number of repetitions)

for $i = 1$ to R **do**
  for $j = 1$ to numSims **do**
    • Sample n data points from the two component finite normal mixture defined by $\mu_1,\, \sigma_1,\, \mu_2,\, \sigma_2$, and π_1
    • Find estimate $\hat{\pi}_1$ using the EM algorithm
    • Calculate the KS-test statistic using Eq. (4.9) and store it as $D(i,\, j)$
  end for
  • Sort $D(i,\, :)$ into ascending order. At a significance level of $\alpha = 0.05$ the critical value is defined as, $D_{cv\,0.95} = \frac{1}{2}\left[D\left(i,\, \lfloor 0.95 * \text{numSims} \rfloor\right) + D\left(i,\, \lfloor 0.95 * \text{numSims} \rfloor + 1\right)\right]$
end for
• Calculate the mean $\bar{D}_{cv\,0.95}$, and variance $\sigma^2_{\bar{D}_{cv\,0.95}}$, from the R repetitions

4.1.9 Random Number Generation: Inverse Transform Method

The Monte-Carlo procedure presented in Section 4.1.8 requires generating observations according to a known probability distribution. Inverse transform sampling is a method for generating pseudo-random numbers, distributed according to an arbitrary probability distribution specified by the cdf $F(x \,|\, \boldsymbol{\psi})$. If the inverse cdf, denoted by $F^{-1}(u \,|\, \boldsymbol{\psi})$, where $u \in (0, 1)$, is evaluated at the point $F(x \,|\, \boldsymbol{\psi})$, the value x will be returned,

$$F^{-1}(F(x \,|\, \boldsymbol{\psi}) |\, \boldsymbol{\psi}) = x \ . \tag{4.16}$$

A cdf has range $(0, 1)$, hence, if a uniformly distributed random number $u_i \in (0, 1)$ is generated, and passed to the inverse cdf, a randomly generated data point x_i will result. This data point will be distributed according to the arbitrary probability distribution of interest, specified by the cdf $F(x \,|\, \boldsymbol{\psi})$.

$$F^{-1}(u_i \,|\, \boldsymbol{\psi}) = x_i \ . \tag{4.17}$$

Define the vector of uniform random variates $\boldsymbol{u}$,

$$\boldsymbol{u} = (u_1, u_2, ..., u_n) \quad \text{where} \quad u_i \in (0, 1) \quad \forall \quad i = (1, 2, ..., n) \ , \tag{4.18}$$

which correspond to the vector of observations, $\boldsymbol{x} = (x_1, x_2, ..., x_n)$.

The inverse transform sampling method can be better interpreted graphically, which can be seen in Figure 4.3. Consider selecting a random proportion of area under the curve of an arbitrary pdf $f(x \mid \boldsymbol{\psi})$. The random number generated according to this distribution corresponds to the value in the domain for which exactly this proportion of area exists to the left of the number. Namely, if X is a random variable distributed according to a given pdf, $P(X \leq x_i) = u_i$. Alternatively using the cdf, a uniformly distributed random number u_i maps from the vertical axis, to a number x_i on the horizontal axis.

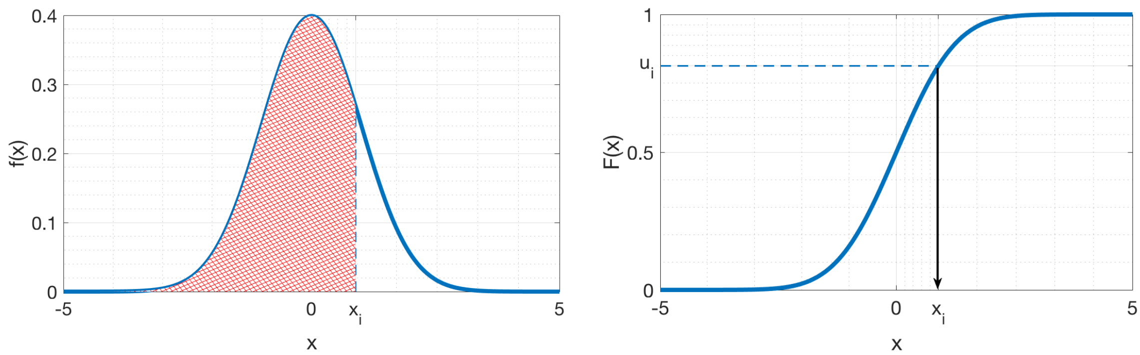


Figure 4.3: Inverse transform method graphical interpretation using the standard normal distribution.

4.2 Information Criteria Based Tests

All goodness-of-fit tests discussed thus far have been hypothesis based. Tests of this nature are straight forward to implement if the distribution of test-statistics, and hence critical values are known, or can be easily determined. Critical values for mixed distributions exhibit more complex parameter dependence, as such, hypothesis based goodness-of-fit tests are non-trivial. *Model selection* via *information criteria* provides an alternate goodness-of-fit framework which is considered more suitable for general application to mixed distributions [59,60]. This approach was popularised by Hirotugu Akaike [14], who used Kullback-Leibler information as a basis for statistical model evaluation. The Akaike information criterion (AIC) was formulated as an asymptotic approximation of the Kullback-Leibler information. Many other information criteria exist, all with subtle differences, most notably the Bayesian information criterion (BIC) proposed by Schwarz [15]. It is important to note that information criteria tests do not assess a model in the same sense as a hypothesis test. They are merely a means of model selection, quantifying the quality of a model relatively, with no information obtained on the absolute quality. This thesis is concerned with formulating information criteria tests in the context of model selection of mixture models. In order for this to be done, Kullback-Leibler information first needs to be considered.

4.2.1 Kullback-Leibler Information

Consider a set of n observations $\boldsymbol{x} = (x_1, ..., x_n)$, sampled from an unknown probability distribution which is defined by pdf $g(x_j)$. Consider a proposed statistical model (attempting to describe the distribution of the data), specified by a probability distribution with pdf $f(x_j \mid \hat{\boldsymbol{\psi}})$. In the context of this thesis $f(x_j \mid \hat{\boldsymbol{\psi}})$ describes a mixed distribution, although the following motivation holds generally for any distribution. The "goodness" (measure of model suitability) of the model $f(\cdot)$ can be broadly assessed as the closeness of the distribution to $g(\cdot)$, which defines the distribution of the data. Akaike proposed the use of Kullback-Leibler information (KL information) to quantify this closeness, defined as

$$I(g,f) = \mathbb{E}_G\left[\log\left(\frac{g(x)}{f(x\,|\,\hat{\boldsymbol{\psi}})}\right)\right] \tag{4.19}$$

$$= \mathbb{E}_G\Big[\log\, g(x)\Big] - \mathbb{E}_G\Big[\log\, f(x\,|\,\hat{\boldsymbol{\psi}})\Big] \tag{4.20}$$

$$= \int_{-\infty}^{\infty} \log\left(\frac{g(x)}{f(x\,|\,\hat{\boldsymbol{\psi}})}\right) g(x)\, dx\ , \tag{4.21}$$

where $\mathbb{E}_G$ represents the expectation with respect to $G(\cdot)$, the cdf of the distribution describing the data. From this definition, KL information has the property that $I(g,f) \geq 0$ and when $I(g,f) = 0$ it is implied that $g(\cdot) = f(\cdot)$. Hence a reduced KL information value corresponds to the proposed model distribution $f(\cdot)$ being closer to $g(\cdot)$. A proof of this and further motivation for the Kullback-Leibler information can be found in [60]. In statistical modelling the distribution $g(\cdot)$ is obviously unknown, so in practice the KL information cannot be directly calculated. Although this is the case, the first term in Eq. (4.20) $\mathbb{E}_g\Big[\log\, g(x)\Big]$ is constant, so model selection can be reduced to comparison of the second term $\mathbb{E}_G\Big[\log\, f(x\,|\,\hat{\boldsymbol{\psi}})\Big]$ for each competing model. This term is the expected log-likelihood of the proposed distribution $f(\cdot)$. A larger value of $\mathbb{E}_G\Big[\log\, f(x\,|\,\hat{\boldsymbol{\psi}})\Big]$ corresponds to a smaller KL information value, and hence a better model. The expectation is given as

$$\mathbb{E}_G\Big[\log\, f(x\,|\,\hat{\boldsymbol{\psi}})\Big] = \int_{-\infty}^{\infty} g(x)\, \log f(x\,|\,\hat{\boldsymbol{\psi}})\, dx\ , \tag{4.22}$$

and although still exhibits $g(\cdot)$ dependence, a good approximation of this expression can be used as a criterion for model selection. The empirical distribution function can be used for this approximation, with $g(x_j) = 1/n$, for $j = 1, ..., n$, yielding

$$\mathbb{E}_G\Big[\log\, f(x\,|\,\hat{\boldsymbol{\psi}})\Big] \approx \frac{1}{n}\sum_{j=1}^{n} \log f(x_j\,|\,\hat{\boldsymbol{\psi}})\ . \tag{4.23}$$

As the sample size tends to infinity, convergence is such that Eq. (4.23) becomes an equality. The expected log-likelihood can now be simply expressed in terms of the log-likelihood function. This is an important result because it allows the log-likelihood to be a criterion for model selection.

$$\mathbb{E}_G\Big[\log\, f(x\,|\,\hat{\boldsymbol{\psi}})\Big] = \frac{1}{n}\log L(\hat{\boldsymbol{\psi}}\,|\,\boldsymbol{x})\ . \tag{4.24}$$

The log-likelihood is an approximation of the Kullback Leibler information, although bias exists. For large sample sizes, and/or when the dimension of the candidate model parameter vector is small, this bias is also small. The bias manifests from the reuse of data for both parameter estimation and estimation of the expected log-likelihood. It is also important to note that the bias varies for different parameter vector dimensions. Information criteria can be generally conceptualised as the bias corrected log-likelihood, with bias defined as in [60] as

$$\text{bias}(G) = \mathbb{E}_{G(\boldsymbol{x})}\Big[\log\, f(\boldsymbol{x}\,|\,\hat{\boldsymbol{\psi}}) - n\,\mathbb{E}_{G(z)}\Big[\log\, f(z\,|\,\hat{\boldsymbol{\psi}})\Big]\Big]\ , \tag{4.25}$$

where $\mathbb{E}_{G(\boldsymbol{x})}$ denotes an expectation taken with respect to the joint distribution $G(\boldsymbol{x}) = \prod_{j=1}^{n} G(x_j)$ for the sample $\boldsymbol{x}$, and $\mathbb{E}_{G(z)}$ is the expectation with respect to the data sampling distribution $G(z)$. A general derivation of the bias is found in [60], and is as follows

$$\text{bias}(G) = \text{tr}(\hat{I}\hat{J}^{-1})\ , \tag{4.26}$$

where $I(\hat{\boldsymbol{\psi}})$, $J(\hat{\boldsymbol{\psi}})$ define matrices with elements,

$$I_{ik}(\hat{\boldsymbol{\psi}}) = \frac{1}{n}\sum_{j=1}^{n} \frac{\partial \log\, f(x_j\,|\,\boldsymbol{\psi})}{\partial \psi_i}\frac{\partial \log\, f(x_j\,|\,\boldsymbol{\psi})}{\partial \psi_k}\bigg|_{\hat{\boldsymbol{\psi}}} \tag{4.27}$$

$$J_{ik}(\hat{\boldsymbol{\psi}}) = -\frac{1}{n}\sum_{j=1}^{n} \frac{\partial^2 \log\, f(x_j\,|\,\boldsymbol{\psi})}{\partial \psi_i\, \partial \psi_k}\bigg|_{\hat{\boldsymbol{\psi}}}\ . \tag{4.28}$$

Now the general information criteria [60] can be defined,[5]

$$\mathrm{gIC}(\boldsymbol{x}, \hat{G}) = -2\log L(\hat{\boldsymbol{\psi}} \,|\, \boldsymbol{x}) + 2\,\mathrm{bias}(G)\ . \tag{4.29}$$

4.2.2 Akaike Information Criterion

The Akaike information criterion (AIC) is the most common information criterion, given as

$$\mathrm{AIC} = -2\log L(\hat{\boldsymbol{\psi}} \,|\, \boldsymbol{x}) + 2k\ , \tag{4.30}$$

where k is the number of free parameters in the candidate model (equal to the dimensionality of the parameter vector $\boldsymbol{\psi}$). The breakthrough Akaike made in [14] is that k, the number of free model parameters gives an estimate for the bias term. Although only strictly true if there exists an $\boldsymbol{\psi}_{\mathbf{0}}$ in the parameter space such that $g(\cdot) = f(\cdot \,|\, \boldsymbol{\psi}_{\mathbf{0}})$, the AIC is widely utilised in its seminal form, often even when the assumption does not hold. Further bias corrected variations of the AIC exist, including the AICc [61]. Model selection can be conceptualised by the test as an assessment of the relative quality of a model via the value of log-likelihood, but imposing an increasing penalty which is consistent with the number of free model parameters. The penalty is a measure of the complexity of the model, and is an attempt to mitigate overfitting. **Model selection can be undertaken by selecting the candidate model with the smallest corresponding AIC value.**

4.2.3 Bayesian Information Criterion

Another popular information criterion is the Bayesian information criterion (BIC). This information criterion was introduced by Schwarz [15] so is occasionally referred to as the Schwarz information criterion. A derivation formulated in the bayesian framework for BIC is presented in [60], and is given as

$$\mathrm{BIC} = -2\log L(\hat{\boldsymbol{\psi}} \,|\, \boldsymbol{x}) + k\,\log n\ . \tag{4.31}$$

The form of BIC is similar to that of AIC, but involves a bias correction term which imposes a slightly stricter penalty on the number of free model parameters than that of AIC. **Model selection is similarly undertaken by selecting the candidate model with the smallest corresponding BIC value.** A clear distinction exists between various information criteria, hence it is possible for a candidate model to be successful against one criterion, but not another.

This thesis will consider goodness-of-fit testing of mixed distributions via AIC and BIC model selection, in addition to hypothesis based testing with the Kolomogorov-Smirnov, Kuiper, Cramér-von Mises, and Anderson-Darling tests.

[5]Noting the multiplication of the bias corrected log-likelihood by a constant factor of -2.

Chapter 5

Finite Normal Mixture Models

Finite normal mixture models (commonly referred to as Gaussian mixture models) describe a class of mixtures whose components are parametric normal densities. Due to the commonality of the Gaussian distribution, finite mixtures of this type are undoubtedly one of the most typical, and easily motivated. Many natural phenomena can be described by a normal distribution, therefore populations with sub-populations of such processes will tend to have a probability density resembling a Gaussian mixture.

This thesis has already alluded to the existence and various applications of finite normal mixtures. For the sake of brevity, only a couple of these instances will be rementioned. Recall Pearson [30] used a two-component unimodal mixture of normal densities to model the ratio of female crab forehead to body length. Chapter 2 presented a multi-modal Gaussian distribution in the context of an unsupervised learning/clustering example. A finite normal mixture was also employed in Chapter 3 as the introductory example to EM equation derivation.

Finite normal mixtures are a powerful modelling tool with widespread uses, mainly because they can take shapes with varied modality, skewness, and kurtosis. The probability density function (pdf) of a univariate random variable X, distributed according to a single Gaussian distribution and making up a component of the finite normal mixture is

$$f_i(x \,|\, \mu_i, \sigma_i^2) = \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(x-\mu_i)^2/2\sigma_i^2} \qquad i = 1, \ldots, g \ , \tag{5.1}$$

where $\mu_i \in (-\infty, +\infty)$ is the mean parameter, $\sigma_i^2 \in (0, +\infty)$ is the variance parameter, and the distribution is defined on the domain $x \in \mathbb{R}$. The contribution of component $f_i(x \,|\, \mu_i, \sigma_i^2)$ to the overall mixture is quantified by mixture proportion π_i. The cumulative distribution function (cdf) of the normal component is

$$F_i(x \,|\, \mu_i, \sigma_i^2) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x - \mu_i}{\sigma_i\sqrt{2}}\right)\right] . \tag{5.2}$$

For observations drawn from a finite normal mixture with corresponding component distributions $f_i(x \,|\, \boldsymbol{\theta}_i)$, describing the distribution of data requires knowledge of parameter vector $\boldsymbol{\theta}_i$ estimated from the data.

5.1 EM Equations for Normal Distribution

Chapter 3 introduced the Expectation Maximisation algorithm as a tool for finding maximum likelihood estimates of parameters in mixture models. The specific EM equations required for parameter estimation of normal parametric components will now be derived.

Consider the postulated probability density function of the finite mixture used to model the observed data vector $\boldsymbol{x} = (x_1, \ldots, x_n)$, given as

$$f(x \,|\, \boldsymbol{\psi}) = \sum_{i=1}^{g} \pi_i f_i(x \,|\, \boldsymbol{\theta}_i) \ , \tag{5.3}$$

where parametric components $f_i(x \,|\, \boldsymbol{\theta}_i)$ are specified by normal densities. The model is parametrised by the vector $\boldsymbol{\psi} = (\boldsymbol{\pi}, \boldsymbol{\xi})$. As before, $\boldsymbol{\pi} = (\pi_1, ..., \pi_{g-1})$ is a vector of (g - 1) mixing proportions, and the vector $\boldsymbol{\xi} = (\boldsymbol{\theta_1}, ..., \boldsymbol{\theta_g})$ contains the parameter vectors for each component of the mixture, i.e. $\boldsymbol{\theta_i} = (\mu_i,\, \sigma_i^2)$ for $(i = 1, ..., g)$. Recall (from Section 3.2) that given a current estimate $\boldsymbol{\psi}^{(k)}$, the expectation of the log-likelihood function for a finite mixture is given as

$$Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) = \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log \pi_i + \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log f_i(x_j \,|\, \boldsymbol{\theta_i}) + \lambda \left(\sum_{i=1}^{g} \pi_i - 1 \right) , \tag{5.4}$$

noting the use of the Lagrange multiplier λ to deal with the constraint that all mixture proportions sum to one, and $z_{ij} = (\mathbf{z}_i)_j$ is a zero-one indicator variable specifying whether x_j arose or did not arise from the ith component of the mixture ($i = 1, ..., g$ and $j = 1, ..., n$). The EM equation for mixture proportions is constructed by taking the partial derivative of Eq. (5.4) with respect to π_i, and equating it to zero, i.e.

$$\frac{\partial\, Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})}{\partial\, \pi_i} = 0 \,. \tag{5.5}$$

The value of $\boldsymbol{\pi}^{(k+1)}$ (on the current iteration) is given as that which maximises $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$. A derivation of the following iterative estimate for the mixture proportions was provided in Section 3.2. The resulting expression is general for any parametric component, with all of the distribution dependence absorbed by the indicator variable $z_{ij}^{(k)}$.

$$\boxed{\pi_i^{(k+1)} = \sum_{j=1}^{n} \frac{z_{ij}^{(k)}}{n}} \,, \tag{5.6}$$

where

$$z_{ij}^{(k)} = \frac{\pi_i^{(k)}\, f_i(x_j \,|\, \mu_i, \sigma_i^2)}{\sum_{m=1}^{g} \pi_m^{(k)}\, f_m(x_j \,|\, \mu_m, \sigma_m^2)} \,, \qquad i = 1, ..., g \,. \tag{5.7}$$

The derivation of EM equations for normal component parameter estimation of $\boldsymbol{\theta_i} = (\mu_i,\, \sigma_i^2)$ will now be considered. Recall that the EM equations for iterative estimation of the parameter vector $\boldsymbol{\xi}^{(k+1)}$ can be constructed by taking partial derivatives of Eq. (5.4) with respect to $\boldsymbol{\theta_i}$. Selecting the parameter vector $\boldsymbol{\xi}^{(k+1)}$ which maximises $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$ on the current iteration is given as an appropriate solution of the expression

$$\sum_{j=1}^{n} z_{ij}^{(k)} \frac{\partial \log f_i(x_j \,|\, \boldsymbol{\theta_i})}{\partial\, \boldsymbol{\theta_i}} = 0 \,, \qquad i = 1, ..., g \,. \tag{5.8}$$

Taking the natural logarithm of the component pdf gives

$$\Rightarrow \; \log f_i(x_j \,|\, \mu_i, \sigma_i^2) \;=\; \log \frac{1}{\sqrt{2\pi\sigma_i^2}} - \frac{(x_j - \mu_i)^2}{2\sigma_i^2} \,. \tag{5.9}$$

Differentiation of Eq. (5.9) with respect to μ_i yields

$$\Rightarrow \; \frac{\partial \log f_i(x_j \,|\, \mu_i, \sigma_i^2)}{\partial \mu_i} \;=\; \frac{x_j - \mu_i}{\sigma_i^2} \,. \tag{5.10}$$

Substituting Eq. (5.10) into Eq. (5.8) gives the EM equation for μ_i,

$$\Rightarrow \; \frac{1}{\sigma_i^2} \sum_{j=1}^{n} z_{ij}^{(k)} (x_j - \mu_i) = 0 \,. \tag{5.11}$$

Rearrangement then yields the iterative estimate for μ_i:

$$\boxed{\mu_i^{(k+1)} = \sum_{j=1}^{n} z_{ij}^{(k)} x_j \Big/ \sum_{j=1}^{n} z_{ij}^{(k)}} \,. \tag{5.12}$$

Now consider the partial derivative of Eq. (5.9) with respect to σ_i,

$$\Rightarrow \frac{\partial \log f_i(x_j \,|\, \mu_i, \sigma_i^2)}{\partial \sigma_i} = -\frac{1}{\sigma_i} + \frac{(x_j - \mu_i)^2}{\sigma_i^3} \;. \tag{5.13}$$

A similar substitution of Eq. (5.13) into Eq. (5.8) gives the EM equation for σ_i,

$$\Rightarrow \; -\frac{1}{\sigma_i}\sum_{j=1}^{n} z_{ij}^{(k)}\left(1 - \frac{(x_j - \mu_i)^2}{\sigma_i^2}\right) = 0 \;. \tag{5.14}$$

Rearrangement yields the iterative estimate for σ_i:

$$\boxed{\sigma_i^{(k+1)} = \sqrt{\frac{\sum_{j=1}^{n} z_{ij}^{(k)}(x_j - \mu_i^{(k+1)})^2}{\sum_{j=1}^{n} z_{ij}^{(k)}}}} \;. \tag{5.15}$$

This section presented a derivation of EM equations for parameter estimation of normal components. In practice these equations are solved numerically.

5.1.1 Outline of Numerical Procedure

Mixtures of normal distributions present the unique case where all EM equations can be derived as closed form expressions. These situations offer numerical simplicity as no numerical root finding is required. Given a data vector $\boldsymbol{x}$, the following is an outline of the numerical procedure which can be utilised for parameter estimation of a g-component finite normal mixture with the EM algorithm.

- **Initialisation:** Choose vector $\boldsymbol{\psi}^{(0)}$, i.e. $\boldsymbol{\pi}^{(0)} = (\pi_1^{(0)}, ..., \pi_{g-1}^{(0)})$, and $\boldsymbol{\xi}^{(0)} = \Big((\mu_1^{(0)}, \sigma_1^{(0)}), ..., (\mu_g^{(0)}, \sigma_g^{(0)})\Big)$ (refer to Section 5.4 for more information on initialisation)

- **E step$^{(1)}$:** Compute indicator variables $\Big((z_{11}^{(0)}, z_{12}^{(0)}, ..., z_{1n}^{(0)}), (z_{21}^{(0)}, z_{22}^{(0)}, ..., z_{2n}^{(0)}), ..., (z_{g1}^{(0)}, z_{g1}^{(0)}, ..., z_{gn}^{(0)})\Big)$ using Eq. (5.7)

- **M step$^{(1)}$:** Compute vector $\boldsymbol{\psi}^{(1)}$, i.e. $\boldsymbol{\pi}^{(1)} = (\pi_1^{(1)}, ..., \pi_{g-1}^{(1)})$, and $\boldsymbol{\xi}^{(1)} = \Big((\mu_1^{(1)}, \sigma_1^{(1)}), ..., (\mu_g^{(1)}, \sigma_g^{(1)})\Big)$ using Eq. (5.6), Eq. (5.12), and Eq. (5.15)

- Compute the log-likelihood, $\log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x}) = \sum_{j=1}^{n} \log\big(\sum_{i=1}^{g} \pi_i^{(1)} f_i(\boldsymbol{x} \,|\, \mu_i^{(1)},\, \sigma_i^{(1)}\big)$

- **E step$^{(2)}$:** Recompute indicator variables, i.e. $\Big((z_{11}^{(1)}, z_{12}^{(1)}, ..., z_{1n}^{(1)}), (z_{21}^{(1)}, z_{22}^{(1)}, ..., z_{2n}^{(1)}), ..., (z_{g1}^{(1)}, z_{g1}^{(1)}, ..., z_{gn}^{(1)})\Big)$, with update $\boldsymbol{\psi}^{(1)}$

- **M step$^{(2)}$:** Compute updated parameter vector $\boldsymbol{\psi}^{(2)}$

- Compute updated log-likelihood, checking convergence condition $|\log L(\boldsymbol{\psi}^{(2)} \,|\, \boldsymbol{x}) - \log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x})\,| < \epsilon$, where ϵ defines a preset tolerance, i.e. $\epsilon = 10^{-5}$

- Continue iteration until convergence condition is satisfied

$$\vdots$$

- Maximum likelihood estimate obtained, $\hat{\boldsymbol{\psi}} = (\hat{\boldsymbol{\pi}}, (\hat{\boldsymbol{\mu}},\, \hat{\boldsymbol{\sigma}})\,)$

5.2 Hypothesis Testing for Normal Mixture Models

Hypothesis based goodness-of-fit testing was outlined in Section 4.1 for general mixture models. This section considers hypothesis testing of finite normal mixtures. Recall the null (H_0) and alternative (H_1) hypotheses,

> $\boldsymbol{H_0}$: the observed data $\boldsymbol{x}$ originates from the proposed distribution $F(\boldsymbol{x}\,|\,\hat{\boldsymbol{\psi}})$
>
> $\boldsymbol{H_1}$: the observed data doesn't originate from the proposed distribution

where the distribution $F(\boldsymbol{x}\,|\,\hat{\boldsymbol{\psi}})$ now defines a finite normal mixture model. The validity of the null hypothesis is assessed by comparison of a test-statistic to a pre-determined critical value. If the test statistic is smaller than the critical value, the null hypothesis cannot be rejected. Section 4.1.7 described how critical values can be determined from the distribution of test-statistics, highlighting the importance of understanding the resultant dependencies. Recall that for single distributions the critical values dependencies can be summarised as [5, 6, 20],

- Critical values generally depend on sample size n, significance level α, and truncation level τ.
- For the case when no parameter estimation is undertaken from the data, critical values are **distribution independent**.
- When parameter estimation is undertaken from the data, critical values are **distribution dependent**, but remain **parameter independent**.

This section computes critical values for Kolmogorov-Smirnov, Kuiper, Cramér-von Mises, and Anderson-Darling goodness of fit tests for a range of sample sizes $n = \{50, 100, 200, 500, 1000\}$, for a **two-component** Gaussian mixture model, at a 95% significance level ($\alpha = 0.05$). Few studies have been dedicated to goodness-of-fit testing of normal mixtures in this manner. Neus et al. [62] considered Kolmogorov-Smirnov testing for two-component mixtures with equal variance. For this study mixture proportion estimation was considered with a metric minimisation algorithm. Chen [16] used a hypothesis testing framework to test for homogeneity.

5.2.1 Random Number Generation: Normal Distribution

In order to determine critical values (for a given goodness-of-fit) the full distribution of test-statistics is required to be specified under the assumption that the null hypothesis is true. A Monte-Carlo procedure was presented in Section 4.1.8 which can be utilised to numerically approach this task. The procedure requires randomly sampling data from a known distribution. Inverse transform sampling can be used for this. Recall that one can generate a general pseudo-random number x_i, defined as

$$x_i = F^{-1}(u_i\,|\,\boldsymbol{\psi}) \ , \tag{5.16}$$

where u_i is a uniform variate, and $F^{-1}(\cdot\,|\,\boldsymbol{\psi})$ is the inverse cdf of the normal distribution, commonly referred to as the quantile function. This function does not have a closed form expression, but can be expressed in terms of the inverse error function.

$$u_i = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x_i - \mu}{\sigma\sqrt{2}}\right)\right] \tag{5.17}$$

$$\Rightarrow\ 2u_i - 1 = \operatorname{erf}\left(\frac{x_i - \mu}{\sigma\sqrt{2}}\right) \tag{5.18}$$

$$\Rightarrow\ \sigma\sqrt{2}\,\operatorname{erf}^{-1}(2u_i - 1) + \mu = x_i = F^{-1}(u_i\,|\,\mu,\,\sigma) \ . \tag{5.19}$$

The inverse error function is widely available in most numerical software packages, if not, it can also be formulated oneself with a finite sum.

5.2.2 Two-component Normal Mixture

Consider a mixture of two univariate normal distributions. The mixture components have probability density functions $f_1(x_j)$ and $f_2(x_j)$ respectively, and cumulative distribution functions $F_1(x_j)$

and $F_2(x_j)$ respectively. Parameter vector $\boldsymbol{\psi} = (\boldsymbol{\pi}, \boldsymbol{\xi})$ for this case is simply given as $\boldsymbol{\psi} = (\pi_1,\, \mu_1,\, \sigma_1,\, \mu_2,\, \sigma_2)$.

$$\bullet \; f_i(x_j \,|\, \mu_i \; \sigma_i^2) \;=\; \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(x_j-\mu_i)^2/2\sigma_i^2} \; , \tag{5.20}$$

$$\bullet \; F_i(x_j \,|\, \mu_i \; \sigma_i^2) \;=\; \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x_j - \mu_i}{\sigma_i\sqrt{2}}\right)\right] \; , \qquad i = 1, 2. \tag{5.21}$$

Two-component normal mixtures can be defined as either *homoscedastic* or *heteroscedastic*, defined as,

- **homoscedastic**: $f_1(x_j)$ and $f_2(x_j)$ share common variances, i.e. $\sigma_1 = \sigma_2$
- **heteroscedastic**: $f_1(x_j)$ and $f_2(x_j)$ have unique variances, i.e. $\sigma_1 \neq \sigma_2$

The Monte-Carlo procedure outlined in Section 4.1.8 was used to numerically compute critical values for both homosecdastic and heteroscedastic two-component normal mixtures. Simulations were undertaken via the proposed multisample method with $C = 100$ (number of repetitions) and numSims $= 10000$ (number of simulations), for the parameter values, mixture proportions, sample sizes, and goodness-of-fit tests listed below. The parameter values were chosen systematically such that the corresponding mixture components displayed varied overlap.

- homoscedastic: $\boldsymbol{\mu} = \Big\{0,\ [0.5, 1, 1.5, 3, 5]\Big\}, \quad \boldsymbol{\sigma} = \Big\{1, 1\Big\}$
- heteroscedastic: $\boldsymbol{\mu} = \Big\{0,\ 3\Big\}, \quad \boldsymbol{\sigma} = \Big\{1,\ [1, 3, 5, 7, 9]\Big\}$

The mixture proportions and sample sizes considered were

$$\pi_1 = \Big\{0,\ 0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9,\ 1\Big\}, \tag{5.22}$$

$$n = \Big\{50,\ 100,\ 200,\ 500,\ 1000\Big\}, \tag{5.23}$$

for Kolmogorov-Smirnov, Kuiper, Cramér-von Mises, and Anderson-Darling goodness-of-fit tests.

Before the results are presented, for completeness consider an assessment of the modality of these two-component mixtures. The following assessment accounts for the size of the mixture proportions, but also the degree of overlap between the mixture components. Holzmann et al. [33] proposed the following criterion to determine the modality of a two-component normal mixture, noting other criteria also exist including [32, 63].

A two-component normal mixture is unimodal if and only if:

- $d \leq 1$

  OR

- $|\log \pi_1 - \log \pi_2| \;\geq 2\log\big(d - \sqrt{d^2-1}\big) + 2d\sqrt{d^2-1}$

where

$$d = \frac{|\,\mu_1 - \mu_2\,|}{2\sqrt{\sigma_1\,\sigma_2}} = \frac{\Delta}{2\sqrt{\sigma_1\,\sigma_2}} \; .$$

An assessment on the modality of the proposed two-component normal mixtures is provided in Table 5.1 and Table 5.2, where the following notation has been introduced to represent the inequality in the criterion

$$A = |\,\log \pi_1 - \log \pi_2\,| \; ,$$

and

$$B = 2\log\big(d - \sqrt{d^2-1}\big) + 2d\sqrt{d^2-1} \; .$$

	$\boldsymbol{\Delta = 0.5}$	$\boldsymbol{\Delta = 1}$	$\boldsymbol{\Delta = 1.5}$	$\boldsymbol{\Delta = 3}$	$\boldsymbol{\Delta = 5}$
$\boldsymbol{\pi_1 = 0}$	✓ $d = 0.25$ $A = \infty$ $B = -$	✓ $d = 0.5$ $A = \infty$ $B = -$	✓ $d = 0.75$ $A = \infty$ $B = -$	✓ $d = 1.5$ $A = \infty$ $B = 1.429$	✓ $d = 2.5$ $A = \infty$ $B = 8.323$
$\boldsymbol{\pi_1 = 0.1}$	✓ $d = 0.25$ $A = 2.197$ $B = -$	✓ $d = 0.5$ $A = 2.197$ $B = -$	✓ $d = 0.75$ $A = 2.197$ $B = -$	✓ $d = 1.5$ $A = 2.197$ $B = 1.429$	$\times$ $d = 2.5$ $A = 2.197$ $B = 8.323$
$\boldsymbol{\pi_1 = 0.3}$	✓ $d = 0.25$ $A = 0.847$ $B = -$	✓ $d = 0.5$ $A = 0.847$ $B = -$	✓ $d = 0.75$ $A = 0.847$ $B = -$	$\times$ $d = 1.5$ $A = 0.847$ $B = 1.429$	$\times$ $d = 2.5$ $A = 0.847$ $B = 8.323$
$\boldsymbol{\pi_1 = 0.5}$	✓ $d = 0.25$ $A = 0$ $B = -$	✓ $d = 0.5$ $A = 0$ $B = -$	✓ $d = 0.75$ $A = 0$ $B = -$	$\times$ $d = 1.5$ $A = 0$ $B = 1.429$	$\times$ $d = 2.5$ $A = 0$ $B = 8.323$
$\boldsymbol{\pi_1 = 0.7}$	✓ $d = 0.25$ $A = 0.847$ $B = -$	✓ $d = 0.5$ $A = 0.847$ $B = -$	✓ $d = 0.75$ $A = 0.847$ $B = -$	$\times$ $d = 1.5$ $A = 0.847$ $B = 1.429$	$\times$ $d = 2.5$ $A = 0.847$ $B = 8.323$
$\boldsymbol{\pi_1 = 0.9}$	✓ $d = 0.25$ $A = 2.197$ $B = -$	✓ $d = 0.5$ $A = 2.197$ $B = -$	✓ $d = 0.75$ $A = 2.197$ $B = -$	✓ $d = 1.5$ $A = 2.197$ $B = 1.429$	$\times$ $d = 2.5$ $A = 2.197$ $B = 8.323$
$\boldsymbol{\pi_1 = 1.0}$	✓ $d = 0.25$ $A = \infty$ $B = -$	✓ $d = 0.5$ $A = \infty$ $B = -$	✓ $d = 0.75$ $A = \infty$ $B = -$	✓ $d = 1.5$ $A = \infty$ $B = 1.429$	✓ $d = 2.5$ $A = \infty$ $B = 8.323$

Table 5.1: Ticks represent component mixtures which are classified as unimodal (homoscedastic case).

	$\sigma_2 = 1$	$\sigma_2 = 3$	$\sigma_2 = 5$	$\sigma_2 = 7$	$\sigma_2 = 9$
$\pi_1 = 0$	✓	✓	✓	✓	✓
	$d = 1.5$	$d = 0.866$	$d = 0.671$	$d = 0.567$	$d = 0.5$
	$A = \infty$	$A = \infty$	$A = \infty$	$A = \infty$	$A = \infty$
	$B = 1.429$	$B = -$	$B = -$	$B = -$	$B = -$
$\pi_1 = 0.1$	✓	✓	✓	✓	✓
	$d = 1.5$	$d = 0.866$	$d = 0.671$	$d = 0.567$	$d = 0.5$
	$A = 2.197$	$A = 2.197$	$A = 2.197$	$A = 2.197$	$A = 2.197$
	$B = 1.429$	$B = -$	$B = -$	$B = -$	$B = -$
$\pi_1 = 0.3$	×	✓	✓	✓	✓
	$d = 1.5$	$d = 0.866$	$d = 0.671$	$d = 0.567$	$d = 0.5$
	$A = 0.847$	$A = 0.847$	$A = 0.847$	$A = 0.847$	$A = 0.847$
	$B = 1.429$	$B = -$	$B = -$	$B = -$	$B = -$
$\pi_1 = 0.5$	×	✓	✓	✓	✓
	$d = 1.5$	$d = 0.866$	$d = 0.671$	$d = 0.567$	$d = 0.5$
	$A = 0$	$A = 0$	$A = 0$	$A = 0$	$A = 0$
	$B = 1.429$	$B = -$	$B = -$	$B = -$	$B = -$
$\pi_1 = 0.7$	×	✓	✓	✓	✓
	$d = 1.5$	$d = 0.866$	$d = 0.671$	$d = 0.567$	$d = 0.5$
	$A = 0.847$	$A = 0.847$	$A = 0.847$	$A = 0.847$	$A = 0.847$
	$B = 1.429$	$B = -$	$B = -$	$B = -$	$B = -$
$\pi_1 = 0.9$	✓	✓	✓	✓	✓
	$d = 1.5$	$d = 0.866$	$d = 0.671$	$d = 0.567$	$d = 0.5$
	$A = 2.197$	$A = 2.197$	$A = 2.197$	$A = 2.197$	$A = 2.197$
	$B = 1.429$	$B = -$	$B = -$	$B = -$	$B = -$
$\pi_1 = 1.0$	✓	✓	✓	✓	✓
	$d = 1.5$	$d = 0.866$	$d = 0.671$	$d = 0.567$	$d = 0.5$
	$A = \infty$	$A = \infty$	$A = \infty$	$A = \infty$	$A = \infty$
	$B = 1.429$	$B = -$	$B = -$	$B = -$	$B = -$

Table 5.2: Ticks represent component mixtures which are classified as unimodal (heteroscedastic case).

The parameter values to be studied in this analysis were systematically chosen such that the resultant mixture components displayed varied overlap. Table 5.1 and Table 5.2 indicated that many of these mixtures can in fact also be classified as unimodal. This allows the behaviour of critical values for different forms of mixture models to be further analysed. It is also important to note that, for a given data vector $\boldsymbol{x}$ sampled according to a unimodal two-component mixture, it may be possible that a single distribution more appropriately describes the data. Recall Section 2.4.2 proposed an information criteria to test for homogeneity.

5.3 Results: Critical Values for Two-component Normal Mixtures

5.3.1 Parameters/Mixture Proportions Known *a Priori*

To begin with consider the simplest case where the full parameter vector $\boldsymbol{\psi}$ is known *a priori*. For this case no estimation is required for either the component parameter vectors, or the mixture proportions. Consider the null hypothesis that the observed data $\boldsymbol{x}$ originates from a mixed distribution specified by $F(\boldsymbol{x} \mid \boldsymbol{\psi})$, where parameter vector $\boldsymbol{\psi}$ is fully specified. The corresponding case for a single distribution yields critical values which are distribution independent. This section will provide critical values for two-component normal mixtures. Due to the cumulative size of the full set of the critical values only a representative sample is provided here for Kolomogorov-Smirnov goodness-of-fit testing and sample sizes $n = \{50, 1000\}$. Tables 5.3 and 5.4 (and Fig. 5.1) provide the critical values for the homoscedastic mixtures, whilst Tables 5.5 and 5.6 (and Fig. 5.2) the heteroscedastic. The remainder of the critical values are provided in Appendix A.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.3329(17)	1.3243(16)	1.3159(15)	1.3135(17)	1.3164(16)	1.3255(15)	1.3313(15)
$\Delta = 1.0$	1.3326(17)	1.3083(16)	1.2730(15)	1.2595(13)	1.2734(15)	1.3082(15)	1.3321(17)
$\Delta = 1.5$	1.3321(14)	1.2918(15)	1.2151(13)	1.1864(14)	1.2168(15)	1.2903(15)	1.3328(16)
$\Delta = 3.0$	1.3320(14)	1.2633(16)	1.1155(11)	1.0260(10)	1.1159(13)	1.2654(17)	1.3309(15)
$\Delta = 5.0$	1.3318(14)	1.2611(16)	1.1120(12)	1.0158(10)	1.1101(14)	1.2615(13)	1.3320(15)

Table 5.3: **homoscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5,\ 1,\ 1.5,\ 3,\ 5]$, $\sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.3521(18)	1.3471(18)	1.3353(16)	1.3336(15)	1.3376(15)	1.3458(15)	1.3531(17)
$\Delta = 1.0$	1.3525(17)	1.3296(15)	1.2928(16)	1.2813(14)	1.2955(15)	1.3297(16)	1.3531(15)
$\Delta = 1.5$	1.3519(16)	1.3110(16)	1.2371(15)	1.2072(12)	1.2366(13)	1.3085(14)	1.3518(16)
$\Delta = 3.0$	1.3525(14)	1.2837(15)	1.1371(15)	1.0489(10)	1.1366(13)	1.2840(16)	1.3520(15)
$\Delta = 5.0$	1.3533(17)	1.2833(15)	1.1318(14)	1.0399(11)	1.1324(15)	1.2824(16)	1.3537(17)

Table 5.4: **homoscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5,\ 1,\ 1.5,\ 3,\ 5]$, $\sigma_2 = 1$.

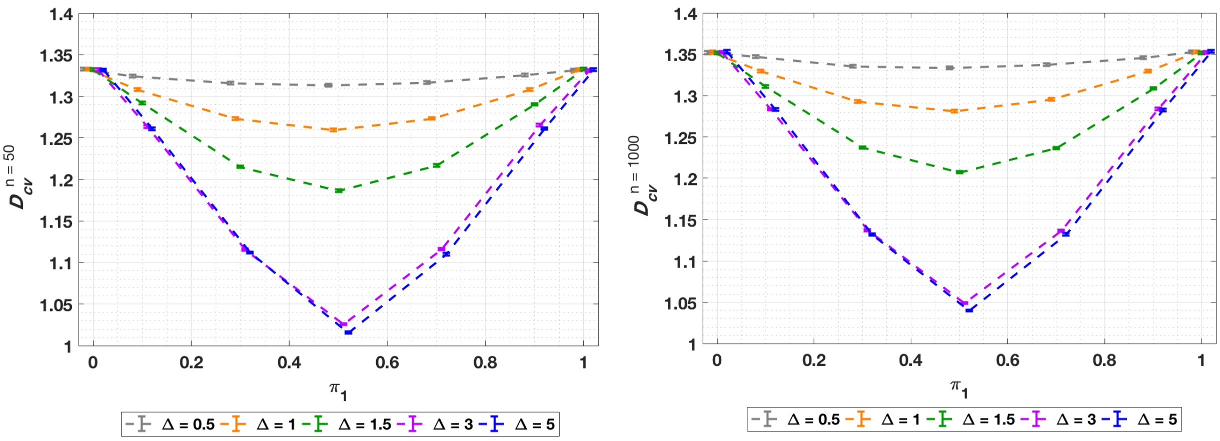


Figure 5.1: **homoscedastic** KS critical values: $n = \{50,\ 1000\}$. Case I.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3320(14)	1.2633(16)	1.1155(11)	1.0260(10)	1.1159(13)	1.2654(17)	1.3309(15)
$\sigma = 3$	1.3325(15)	1.2796(14)	1.2026(14)	1.1898(14)	1.2360(16)	1.2997(15)	1.3312(16)
$\sigma = 5$	1.3307(17)	1.2934(16)	1.2477(15)	1.2466(15)	1.2762(18)	1.3150(15)	1.3317(17)
$\sigma = 7$	1.3322(14)	1.2981(16)	1.2612(16)	1.2646(15)	1.2893(14)	1.3199(15)	1.3312(16)
$\sigma = 9$	1.3329(17)	1.2990(17)	1.2663(16)	1.2726(16)	1.2960(18)	1.3206(16)	1.3318(15)

Table 5.5: **heteroscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3$, $\sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3525(14)	1.2837(15)	1.1371(15)	1.0489(10)	1.1366(13)	1.2840(16)	1.3520(15)
$\sigma = 3$	1.3520(17)	1.2998(14)	1.2248(12)	1.2102(14)	1.2553(16)	1.3189(15)	1.3522(15)
$\sigma = 5$	1.3534(15)	1.3151(14)	1.2698(15)	1.2674(17)	1.2966(17)	1.3358(17)	1.3518(14)
$\sigma = 7$	1.3521(18)	1.3200(16)	1.2824(15)	1.2855(15)	1.3120(16)	1.3398(16)	1.3512(14)
$\sigma = 9$	1.3530(14)	1.3188(17)	1.2880(13)	1.2919(18)	1.3174(17)	1.3408(14)	1.3528(15)

Table 5.6: **heteroscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3$, $\sigma_2 = [1, 3, 5, 7, 9]$.

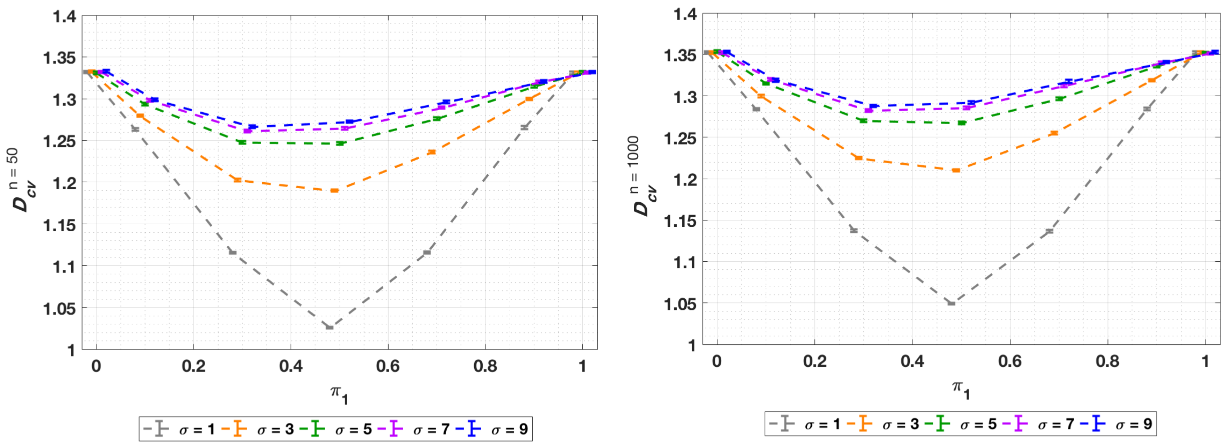


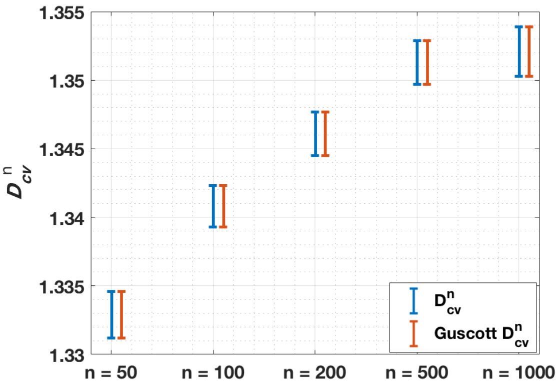
Figure 5.2: **heteroscedastic** KS critical values: $n = \{50, 1000\}$. Case I.

Even for the case when no parameter estimation is undertaken from the data, critical values $D^n_{cv\,0.95}$ for two-component normal mixtures have evident dependencies:

- Critical values $D^n_{cv\,0.95}$ are **parameter dependent**, i.e. they depend on both $\boldsymbol{\theta}$ the parameter vector of each each component distribution, and $\boldsymbol{\pi}$ the mixture proportions.
- Dependence on the sample size n is also evident. An increase in n corresponds to an increase in the critical values $D^n_{cv\,0.95}$.

The intuitive result, that for homoscedastic two-component normal mixtures critical values are symmetric about even mixture proportions (i.e. $\pi_1 = 0.5$) has been numerically justified. For heteroscedastic mixtures, if critical values are shown as a function of mixture proportion π_1 (as in Fig. 5.2), they exhibit a positive skew if $\sigma_1 < \sigma_2$ and a negative skew otherwise. The disparity of critical values for a varying mixture proportion is heightened if the component distributions exhibit a greater level of separation. A comforting result is that at the mixture proportion boundaries (i.e.

$\pi_1 = 0$ or 1) the single distribution independent critical values are recovered exactly. Figure 5.3 compares the Kolmogorov-Smirnov critical values obtained at a 95% significance level in this work to those obtained by Guscott [20].



	$D^n_{cv\,0.95}$	Guscott [20] $D^n_{cv\,0.95}$
$n = 50$	1.3329(17)	1.3332(14)
$n = 100$	1.3408(15)	1.3405(16)
$n = 200$	1.3461(16)	1.3446(15)
$n = 500$	1.3513(16)	1.3500(16)
$n = 1000$	1.3521(18)	1.3520(13)

Figure 5.3: Comparison of boundary value critical values to single distribution independent critical values of Guscott [20]. Tabulated $D^n_{cv\,0.95}$ corresponds to a homoscedastic mixture with $\Delta = 0.5$.

Strong agreement is evident in both the value of the estimates and the error margins. Both works adopted the same method for determining uncertainty for multisampled data which was proposed by Schafer [20, 58, 64].

5.3.2 Component Parameters Known *a Priori*, Mixture Proportions Requiring Estimation

Now consider the case where the specific parameter vectors $\boldsymbol{\theta_i}$ of the component distributions are known *a priori*, but the mixture proportions require estimation. This section provides the same representative sample of Kolomogorov-Smirnov critical values for $n = \{50, 1000\}$. Tables 5.7 and 5.8 (and Fig. 5.4) correspond to homoscedastic mixtures, whilst Tables 5.9 and 5.10 (and Fig. 5.4) heteroscedastic. The remainder of the critical values are similarly provided in Appendix A.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.2094(16)	1.1079(14)	0.9921(10)	0.9614(10)	0.9921(10)	1.1092(16)	1.2082(16)
$\Delta = 1.0$	1.2361(15)	1.1119(15)	0.9827(10)	0.9248(10)	0.9826(12)	1.1123(14)	1.2366(15)
$\Delta = 1.5$	1.2725(16)	1.1630(13)	0.9997(11)	0.9047(08)	1.0003(11)	1.1633(13)	1.2729(14)
$\Delta = 3.0$	1.3250(15)	1.2406(14)	1.0722(13)	0.9661(09)	1.0723(12)	1.2401(14)	1.3259(15)
$\Delta = 5.0$	1.3327(16)	1.2594(15)	1.1078(12)	1.0115(09)	1.1070(12)	1.2609(15)	1.3315(15)

Table 5.7: **homoscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 50$, $\mu_1 = 0,\ \sigma_1 = 1,\ \mu_2 = [0.5,\ 1,\ 1.5,\ 3,\ 5],\ \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.2401(18)	1.0278(12)	0.9648(09)	0.9440(09)	0.9664(10)	1.0264(10)	1.2388(16)
$\Delta = 1.0$	1.2776(13)	1.1155(14)	0.9814(10)	0.9134(10)	0.9812(12)	1.1143(13)	1.2771(15)
$\Delta = 1.5$	1.3165(15)	1.1784(12)	1.0075(11)	0.9036(07)	1.0083(11)	1.1788(15)	1.3164(15)
$\Delta = 3.0$	1.3499(16)	1.2603(16)	1.0912(13)	0.9855(10)	1.0912(13)	1.2610(15)	1.3519(15)
$\Delta = 5.0$	1.3524(14)	1.2805(16)	1.1283(14)	1.0340(10)	1.1278(12)	1.2824(14)	1.3529(17)

Table 5.8: **homoscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 1000$, $\mu_1 = 0,\ \sigma_1 = 1,\ \mu_2 = [0.5,\ 1,\ 1.5,\ 3,\ 5],\ \sigma_2 = 1$.

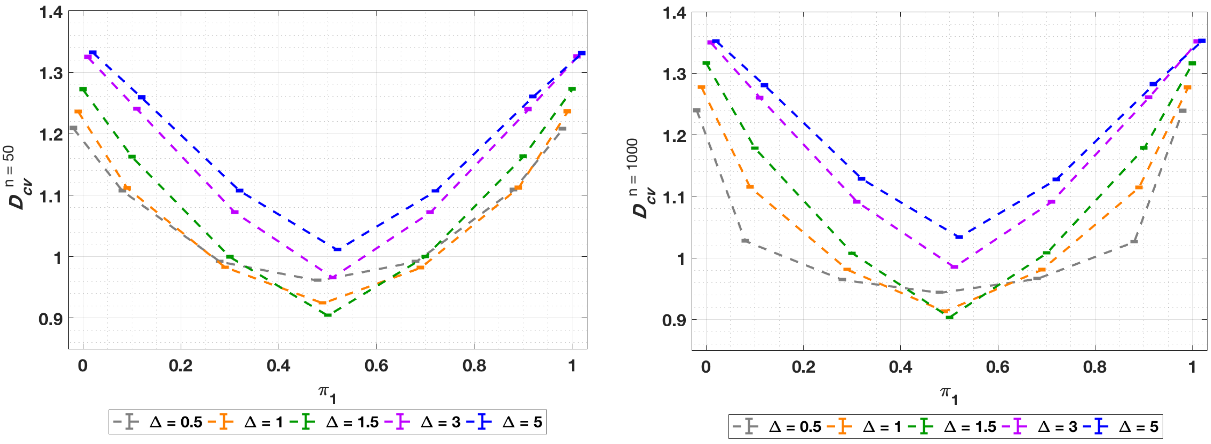


Figure 5.4: **homoscedastic** KS critical values: $n = \{50, 1000\}$. Case II.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3250(15)	1.2406(14)	1.0722(13)	0.9661(09)	1.0723(12)	1.2401(14)	1.3259(15)
$\sigma = 3$	1.2405(15)	1.0929(12)	1.0278(11)	1.0696(12)	1.1702(16)	1.2736(13)	1.3291(16)
$\sigma = 5$	1.2736(15)	1.1807(13)	1.1577(15)	1.1924(15)	1.2493(14)	1.3042(15)	1.3317(15)
$\sigma = 7$	1.2949(16)	1.2167(13)	1.2040(14)	1.2327(15)	1.2751(15)	1.3127(17)	1.3313(17)
$\sigma = 9$	1.3042(17)	1.2342(15)	1.2219(14)	1.2486(16)	1.2841(15)	1.3163(16)	1.3309(15)

Table 5.9: **heteroscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 50$, $\mu_1 = 0,\ \ \sigma_1 = 1,\ \ \mu_2 = 3,\ \ \sigma_2 = [1,\ 3,\ 5,\ 7,\ 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3499(16)	1.2603(16)	1.0912(13)	0.9855(10)	1.0912(13)	1.2610(15)	1.3519(15)
$\sigma = 3$	1.2705(14)	1.1015(12)	1.0422(10)	1.0866(12)	1.1893(14)	1.2941(15)	1.3513(15)
$\sigma = 5$	1.2931(15)	1.1933(13)	1.1766(16)	1.2144(15)	1.2692(14)	1.3225(16)	1.3518(15)
$\sigma = 7$	1.3115(16)	1.2347(14)	1.2228(16)	1.2523(14)	1.2930(16)	1.3336(16)	1.3519(18)
$\sigma = 9$	1.3203(15)	1.2527(15)	1.2400(16)	1.2687(17)	1.3046(18)	1.3374(15)	1.3519(17)

Table 5.10: **heteroscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 1000$, $\mu_1 = 0,\ \ \sigma_1 = 1,\ \ \mu_2 = 3,\ \ \sigma_2 = [1,\ 3,\ 5,\ 7,\ 9]$.

The critical values for the case when parameters of component distributions are known *a priori*, but mixture proportions estimated, exacerbate the same dependencies as the first case when no estimation was undertaken. Critical values at a 95% significance level are once again dependent on the sample size n and the parameter vector $\boldsymbol{\psi}$. A decrease in the critical values is also evident now that parameters are estimated from the data. This is because the discrepancy between the hypothesised cdf and edf is reduced, corresponding to a reduction in the test-statistics and hence a reduction in the critical values. Symmetry is retained about even mixture proportions (i.e. $\pi_1 = 0.5$) for homoscedastic mixtures, and the skewness evident in the critical values for heteroscedastic mixtures is much more significant. For this case, single distribution independent critical values **are not** generally recovered at the mixture proportion boundaries (i.e. $\pi_1 = 0$ or 1). At the boundaries the resulting distribution is inherently single, although the EM algorithm has flexibility to include a second component within the model. The algorithm tends to exploit the additional component where possible to model spurious data even if sampling was purely from a single distribution. Including additional components into the model ensures a better fit but this is consistent with the classical notion of overfitting. This also leads to a reduction in test-statistics, hence a reduction in the critical values.

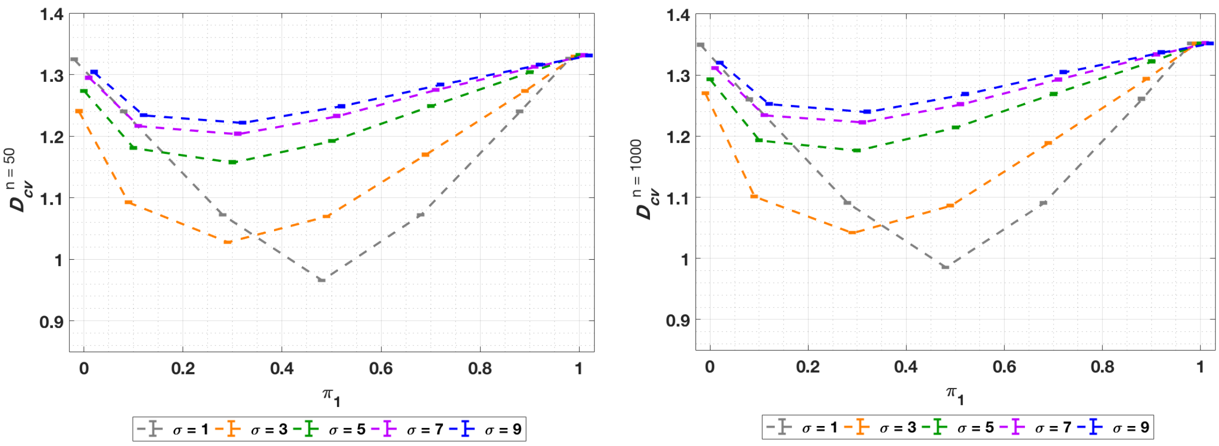


Figure 5.5: **heteroscedastic** KS critical values: $n = \{50,\, 1000\}$. Case II.

For homoscedastic mixtures there is disparity between the critical values at both boundaries for the various mixtures. Critical values are smaller for mixtures which exhibit a higher degree of overlap. For mixtures with high separation, no small additional contribution of one component can assist describing the distribution of data from the second, and as a result the single distribution independent critical values are once again recovered at the boundaries.

Similar behaviour is evident for heteroscedastic mixtures although only at one boundary. Generally the disparity in the critical values occurs at the left-hand mixture proportion boundary (i.e. $\pi_1 = 0$) if $\sigma_1 < \sigma_2$, and the right-hand boundary (i.e. $\pi_1 = 1$) if $\sigma_1 > \sigma_2$. This is because small contributions of the component with smaller variance can assist modelling data which is exclusively sampled from the second, although the counter is not true. As the two-component mixtures were arbitrarily specified such that $\sigma_1 < \sigma_2$ the disparity is exclusively evident at the left-hand boundary.

It is important to note that the behaviours described above are not exclusive to boundary values. Even if the boundaries were excluded, which is logical for analysis of mixed distributions, the critical values from the first case when no estimation was undertaken are still not recovered. In addition, further disparity between critical values would be evident for mixtures of greater than two components.

As discussed, parameter dependent critical values are largely inadequate for use in goodness-of-fit testing, unless a simple numerically tractable interpolant can be found to deal with the parameter dependence. To recap, critical values $D^n_{cv\,\alpha}$ exhibit a dependency on the parameter vector $\boldsymbol{\psi}$ (both the component parameter vectors $\boldsymbol{\theta_i}$ and the mixture proportions $\boldsymbol{\pi}$), the sample size n, and although no empirical justification was provided within this section, the significance level α. For a two-component normal mixture, the dimensionality of the component parameter vector in which the critical values are dependent can be reduced by considering a shift/scaling procedure. At first it can be recognised that critical values are not dependent on both μ_1 and μ_2 independently, but in fact $\Delta = |\mu_1 - \mu_2|$ which is the difference between the means. Mathematically this corresponds to a simple translation of the data, or a redefinition of the origin. Furthermore if mixture components have differing means, a scaling can also be applied such that the parameter dependence is restricted to only two variance parameters. The procedure is outlined as:

Consider $\boldsymbol{\mu} = (a,\, b), \quad \boldsymbol{\sigma} = (c,\, d), \quad$ assuming $a \neq b$.

- Translation: $\boldsymbol{\mu} = (0,\, b - a), \quad \boldsymbol{\sigma} = (c,\, d)$
- Scaling: $\boldsymbol{\mu} = (0,\, 1), \quad \boldsymbol{\sigma} = (\frac{c}{b-a},\, \frac{d}{b-a})$

Numerical justification of this procedure is provided by considering Kolomogorov-Smirnov critical values for $n = 1000$, $\alpha = 0.05$, $\boldsymbol{\mu} = (4,\ 8)$, $\boldsymbol{\sigma} = (2,\ 2)$, and $\boldsymbol{\mu'} = (0,\ 1)$, $\boldsymbol{\sigma'} = (1/2,\ 1/2)$. Noting that these critical values correspond to the case where the full parameter vector is known *a priori*. Figure 5.6 presents the critical values.

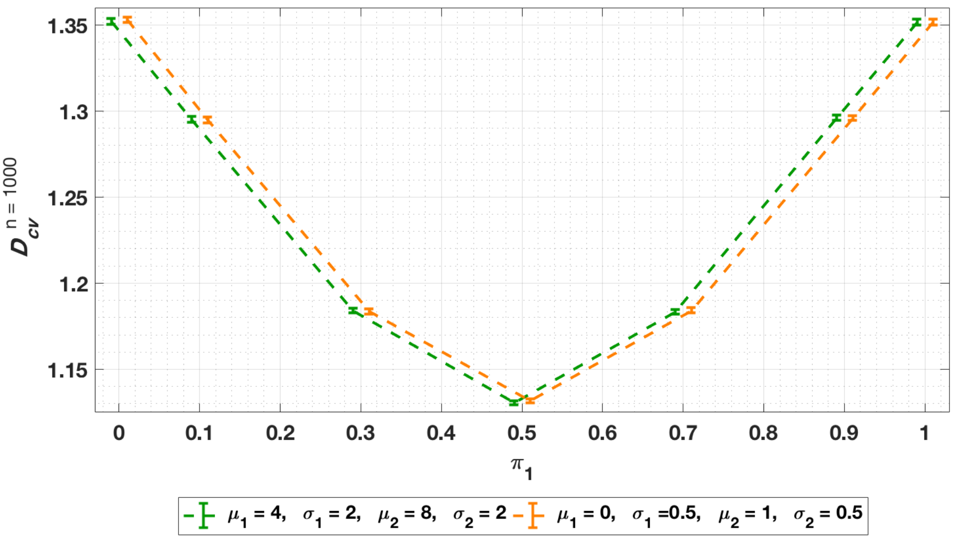


Figure 5.6: Comparison of critical values under the shift/scale procedure.

Interpolation of critical values between pre-calculated grid points is required because of the continuous nature of the parameter space. Using the shift/scale procedure outlined above, the parameter space can be restricted for the case of two-component normal mixtures to two variance parameters σ_1 and σ_2, and one mixture proportion π_1. Figure 5.7 displays surface plots of the critical value interpolant when considering a parameter space grid of $\sigma_1 \in \{0.1, 0.2, ..., 2.0\}$ and $\sigma_2 \in \{0.1, 0.2, ..., 2.0\}$. The displayed critical values correspond to Kolmogorov-Smirnov goodness-of-fit testing for $n = 1000$ and $\alpha = 0.05$.

5.3.3 Estimation of Full Parameter Vector

Estimating the component parameter vector in addition to the mixture proportions would further decrease the critical values. It has been demonstrated that critical values for mixtures of normal densities display complex parameter dependence. The complexity will further increase when considering estimation of the full parameter vector $\boldsymbol{\psi}$. As more parameters are estimated from the data, the log-likelihood function grows in dimensionality, and greater attention is required for appropriate initialisation of the EM algorithm. Initialisation is important such that convergence is to the stationary values (of the log-likelihood function) which are of interest.

It has become clear that information criteria based goodness-of-fit tests are more suitable for mixed distribution analysis because parameter dependent critical values are largely inadequate for hypothesis based goodness-of-fit testing. For this reason, critical values will not be considered for the case of full parameter vector estimation. For completeness, the critical values will be considered in Chapter 6 for the Weibull distribution for the cases specified within this section.

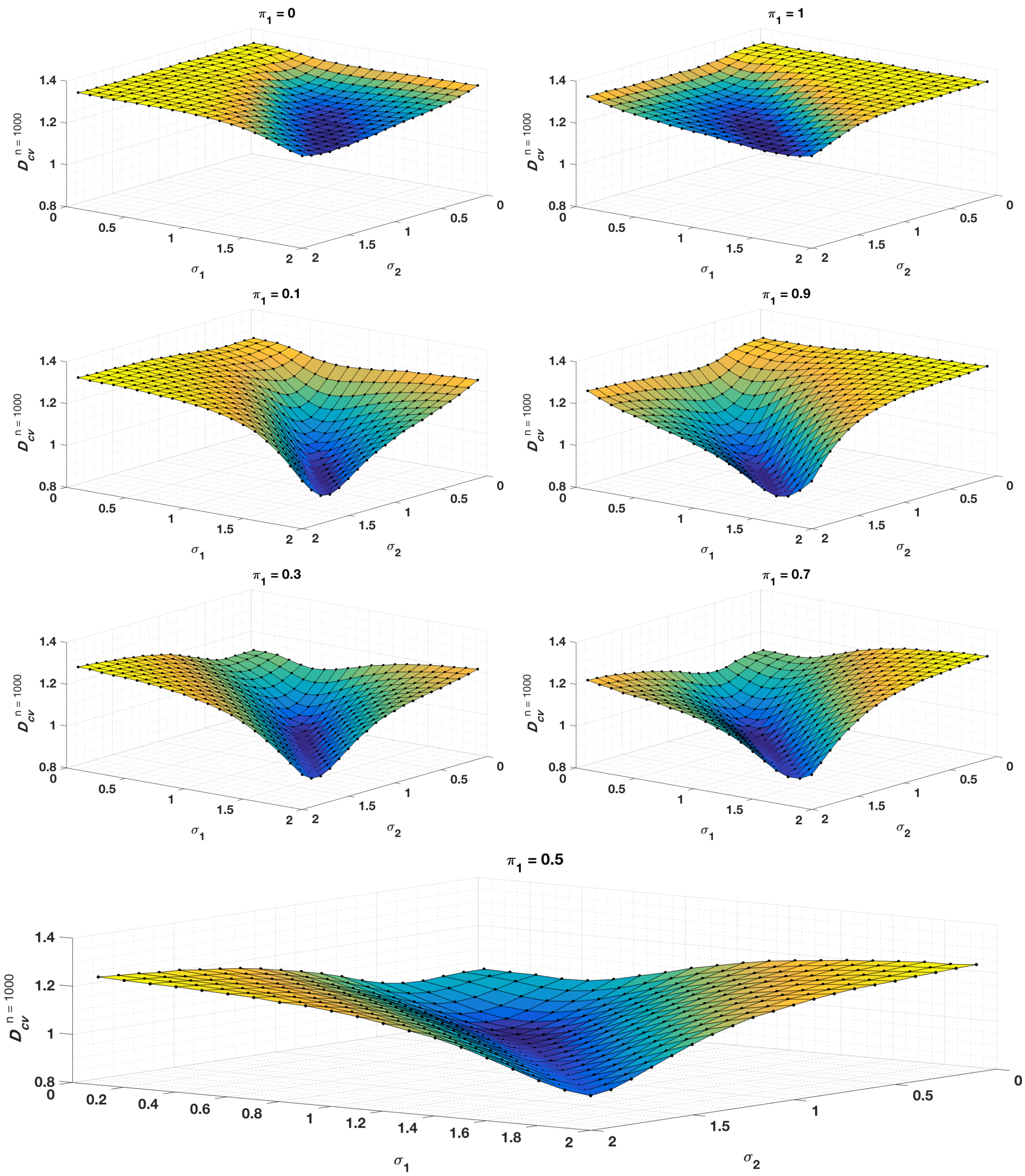


Figure 5.7: Interpolated Kolmogorov-Smirnov critical values, $n = 1000$.

5.4 Initialisation of EM Algorithm

Convergence of the EM algorithm is dependent on both the stopping criterion, and the initialisation of parameter vector $\boldsymbol{\psi}$. The issue of selecting an appropriate stopping criteria has largely been dealt with in the literature. Seidel et al. [65] investigated the effect of a weak criteria where iteration is stopped and convergence deemed to have occurred early. The majority of stopping criteria are either based on the relative change of parameter estimates, or the relative change of log-likelihood values. A negligible change in the log-likelihood indicates a lack of progress rather than genuine convergence. Chapter 3 introduced the EM algorithm, deeming convergence when the absolute difference between the log-likelihood value of two consecutive iterations is smaller than a predetermined tolerance $\epsilon = 10^{-5}$.

The issue of initialisation is by far a more complicated one. Ideal convergence of the EM algorithm is to a global maximum of the log-likelihood function. Selecting poor initial values can exacerbate slow convergence times, or lead to convergence to undesired stationary points. For example, if the algorithm is stuck in a region of the parameter space with relatively flat log-likelihood, it would be desirable to restart iteration elsewhere in the parameter space, rather than continue iteration indefinitely in the problematic region with little reward. Ideally parameter convergence is studied from multiple initial values spread across the parameter space. Here the full nature of the log-likelihood function can be understood, with increased confidence that convergence is to a true global maximum, but initialisation of this nature is often limited by computational expense. A further difficulty is that the log-likelihood can take a maximum value at more than one location in the parameter space (i.e. more than one choice of $\hat{\boldsymbol{\psi}}$). This can be easily observed with a rather trivial example of a two-component mixture where both mixture components are specified by the same parametric family. The value of the log-likelihood won't change if the parameter estimates are interchanged, i.e. $\boldsymbol{\theta_1} \to \boldsymbol{\theta_2}$ and $\pi_1 \to \pi_2$, and vice versa. This is in effect just switching the labels of the two parametric components, where the labelling convention is purely determined by the nature of the initial values. This effect is termed partial identifiability.

Many procedures have been proposed for initialisation of the EM algorithm. Unfortunately it is an impossible task to determine the most superior method as different procedures outperform each other for various data samples $\boldsymbol{x}$, and mixture models. Some common initialisation procedures include:

- **True values** $\boldsymbol{\psi}^{(\mathbf{0})} = \boldsymbol{\psi}_{\text{true}}$: Initialisation by true values is a typical procedure utilised to test an algorithm, or benchmark the performance of the algorithm on an artificial dataset where the exact distribution of which the data is sampled is known.
- **Perturbed true values** $\boldsymbol{\psi}^{(\mathbf{0})} = \boldsymbol{\psi}_{\text{true}} + \boldsymbol{\delta}$: Initialisation by perturbed values is similar to initialisation by true values, but includes a random perturbation specified by the vector $\boldsymbol{\delta}$. Whilst also being used to test an algorithm on an artificial dataset, the sensitivity of convergence to the true values can be determined by varying the size of the perturbation. Typically a uniform perturbation is used.
- **Random** : Random initialisation is the simplest procedure which can be used on a dataset where no information is known about the underlying distribution. Usually the parameter space can be constrained in some manner, and the initial values typically selected at random, uniformly distributed across the space.
- **Grid** : An initialisation procedure exists where the parameter space is discretised, and initial values selected at various locations. A small number of EM iterations are considered from each initial value, and iteration continued from the value which corresponds to the greatest log-likelihood. Laird [66] proposed a similar grid search.
- **k-means** : An iterative clustering procedure where n observations are clustered into g groups (each observation belonging to the cluster with the nearest mean) exists [67]. Once all observations are assigned a cluster, the means are re-calculated and the process repeated. If component distributions exhibit a small degree of overlap, data clusters can easily be determined. Initial values can be specified by single distribution maximum likelihood estimation of each mixture component for the clustered data.
- **Finch et al. [68]** : A method was proposed for initialisation of a two-component Gaussian mixture. For this case only a mixture proportion π_1 requires initialisation, as the remainder of initial values can be determined from the data. The data sample $\boldsymbol{x}$ is divided into two sections, one containing the first $n\pi_1^{(0)}$ observations (integer value) in the dataset, and the second containing the remainder. Initialisation of the mean and variance of the two-components is given by the determining the mean and variance of each of the two sections of data. This procedure can be undertaken by first ordering the dataset $\boldsymbol{x}$, or not.
- **Extension of Finch et al. [68]** : An extension of the method described above exists where initialisation of the component weight π_1 is also not required. A vector of weights, eg. $\boldsymbol{\pi} = (0.1, 0.2, ..., 0.9)$ can be considered, and initialisation undertaken with the method described above. From here, a small number of EM iterations can be considered, and iteration continued from the case which corresponds to the greatest log-likelihood value.

- **Karlis et al. [69] new method** : An initialisation procedure exists which distributes observations to specific mixture components. This procedure assigns the initial values to the latent indicator vector $\boldsymbol{z_i} = (z_{1j}, ..., z_{gj})$, where $z_{ij} = 1$ if the j-th observation originates from the i-th component of the mixture, and zero otherwise. Initially the percentiles which split the dataset into equal parts are considered as the initial values for the means. Each datum is assigned the component which has the closest mean in absolute distance. The $\boldsymbol{z_i}$ vector can then be populated, and the initial values of the parameters can be selected by using this latent variable and the M-step of the EM algorithm.

This is not an exhaustive list of procedures used for initialisation of the EM algorithm, but rather intended to emphasise the importance of initialisation of the algorithm. Initialisation was discussed with reference to normal mixtures, it is also important to note that much of the above methodology generalises to other parametric components also.

Chapter 6

Weibull Component Distribution

The Weibull distribution is a continuous probability distribution which was described and subsequently named after Swedish mathematician Waloddi Weibull in 1951 [70]. Because of its nature and versatility, the distribution is frequently used within survival analysis, failure analysis, and reliability modelling, with the following examples of particular interest to this thesis. Kizilersü et al. [19] described the intermediate and tail region of the distribution of limit order inter-arrival times (of data from the London Stock Exchange) using a left-truncated Weibull distribution. Guscott [20] reconciled the time differences between market orders with the same distribution. Kreer et al. [6] successfully used a Weibull distribution to describe the time separation between consecutive earthquakes in California, and Kizilersü et al. [5] used the distribution to model the duration of failed marriages in the US.

The probability density function (pdf) of a random variable X, distributed according to a single Weibull distribution and making up a component of a mixture model is

$$f_i(x \,|\, \alpha_i, \beta_i) = \left(\frac{\beta_i}{\alpha_i}\right)\left(\frac{x}{\alpha}\right)^{\beta_i - 1} e^{-(x/\alpha_i)^{\beta_i}} \quad , \quad x \geq 0 \; , \tag{6.1}$$

where $\alpha_i \in (0, +\infty)$ is the scale parameter, $\beta_i \in (0, +\infty)$ is the shape parameter, and the distribution is defined on the domain $\mathbb{R}^+ \cup 0$ (i.e. $x \in [0, +\infty)$). Figure 6.1 describes the distribution of probability density for various scale and shape parameters, demonstrating the shape the Weibull distribution takes for different parameter values. When the shape parameter $\beta = 1$ the distribution is equivalent to the exponential distribution, similarly when $\beta = 2$ it is identical to the Rayleigh distribution.

The cumulative distribution function (cdf) of the Weibull component is

$$F_i(x \,|\, \alpha_i, \beta_i) = 1 - e^{-(x/\alpha_i)^{\beta_i}} \quad , \quad x \geq 0 \; . \tag{6.2}$$

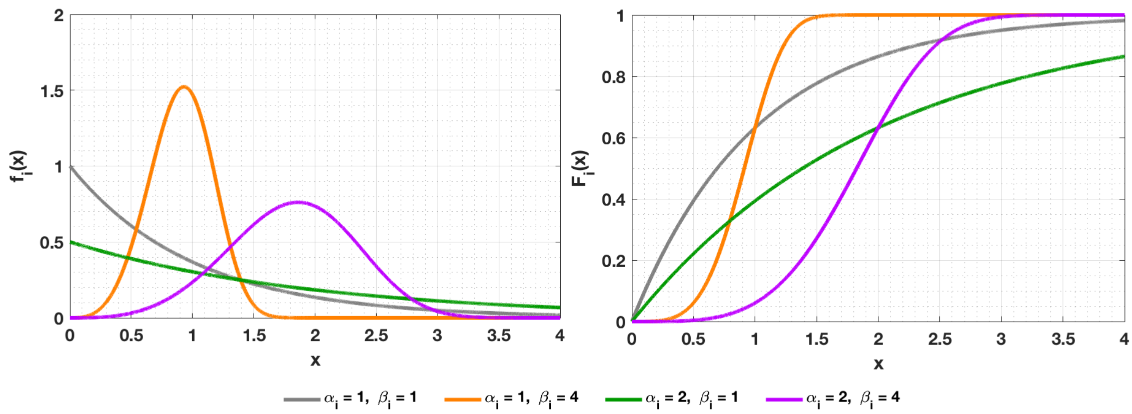


Figure 6.1: pdf and cdf of Weibull distribution for various scale and shape parameters.

In order to describe observations drawn from a mixture model, knowledge of the parameter vector $\boldsymbol{\theta}_i$ estimated from the data, is required. The following section considers estimation for a Weibull component distribution.

6.1 EM Equations for Weibull Distribution

Chapter 3 introduced the Expectation Maximisation algorithm and presented the EM framework for maximum likelihood estimation of parameters in mixed distributions. The specific EM equations required for parameter estimation of Weibull parametric components will now be derived.

Consider the probability density function of the finite mixture, postulated to model the observed data vector $\boldsymbol{x} = (x_1, ..., x_n)$, given as

$$f(x \,|\, \boldsymbol{\psi}) = \sum_{i=1}^{g} \pi_i f_i(x \,|\, \boldsymbol{\theta}_i) \ , \tag{6.3}$$

where at least one mixture component is specified by a Weibull distribution. For the convenience of this section consider the i^{th} parametric component $f_i(x \,|\, \boldsymbol{\theta}_i)$ to be Weibull. The model is parametrised by the vector $\boldsymbol{\psi} = (\boldsymbol{\pi}, \boldsymbol{\xi})$. As before, $\boldsymbol{\pi} = (\pi_1, ..., \pi_{g-1})$ is a vector containing the (g - 1) mixing proportions, and the vector $\boldsymbol{\xi} = (\boldsymbol{\theta_1}, ..., \boldsymbol{\theta_g})$ contains the parameter vectors for each component of the mixture. For the Weibull component $f_i(x \,|\, \boldsymbol{\theta}_i)$, the parameter vector is given as $\boldsymbol{\theta_i} = (\alpha_i, \beta_i)$, encapsulating both the shape and scale parameters of the Weibull distribution. Recall (from Section 3.2) that given a current estimate $\boldsymbol{\psi}^{(k)}$, the expectation of the log-likelihood function for a finite mixture is given as

$$Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) = \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log \pi_i + \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log f_i(x_j \,|\, \boldsymbol{\theta_i}) + \lambda \left(\sum_{i=1}^{g} \pi_i - 1 \right) \ , \tag{6.4}$$

recalling the use of the Lagrange multiplier λ to deal with the constraint that all mixture proportions sum to one, and $z_{ij} = (\mathbf{z}_i)_j$ is a zero-one indicator variable specifying whether x_j arose or did not arise from the ith component of the mixture ($i = 1, ..., g$ and $j = 1, ..., n$). The EM equation for mixture proportions is then constructed by taking the partial derivative of Eq. (6.4) with respect to π_i, and equating to zero, i.e.

$$\frac{\partial \, Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})}{\partial \, \pi_i} = 0 \ . \tag{6.5}$$

The value of $\boldsymbol{\pi}^{(k+1)}$ (on the current iteration) is that which maximises $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$. A derivation of the following iterative estimate for the mixture proportions was provided in Section 3.2 and will not be repeated here, because the resulting expression is general for any parametric component. All of the distribution dependence is absorbed by the indicator variable $z_{ij}^{(k)}$.

$$\boxed{\pi_i^{(k+1)} = \sum_{j=1}^{n} \frac{z_{ij}^{(k)}}{n}} \ , \tag{6.6}$$

where

$$z_{ij}^{(k)} = \frac{\pi_i^{(k)} \, f_i(x_j \,|\, \boldsymbol{\theta_i})}{\sum_{m=1}^{g} \pi_m^{(k)} f_m(x_j \,|\, \boldsymbol{\theta_m})} \ , \qquad i = 1, \, ..., \, g \ . \tag{6.7}$$

The derivation of EM equations for Weibull component parameter estimation of $\boldsymbol{\theta_i} = (\alpha_i, \, \beta_i)$ will now be considered. Recall that EM equations for iterative estimation of the parameter vector $\boldsymbol{\xi}^{(k+1)}$ can be constructed by taking partial derivatives of Eq. (6.4) with respect to $\boldsymbol{\theta_i}$. Selecting the parameter vector $\boldsymbol{\xi}^{(k+1)}$ which maximises $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$ on the current iteration is given as an appropriate solution of the expression,

$$\sum_{j=1}^{n} z_{ij}^{(k)} \frac{\partial \log f_i(x_j \,|\, \boldsymbol{\theta_i})}{\partial \, \boldsymbol{\theta_i}} = 0 \ , \qquad i = 1, \, ..., \, g \ . \tag{6.8}$$

Taking the natural logarithm of the component pdf gives

$$\Rightarrow \log f_i(x_j \,|\, \alpha_i, \beta_i) \;=\; \log\left(\frac{\beta_i}{\alpha_i}\right) + (\beta_i - 1)\log\left(\frac{x_j}{\alpha_i}\right) - \left(\frac{x_j}{\alpha_i}\right)^{\beta_i} \tag{6.9}$$

$$\begin{aligned} &= \log\beta_i - \log\alpha_i + \beta_i \log x_j - \beta_i \log\alpha_i \\ &\quad - \log x_j + \log\alpha_i - \left(\frac{x_j}{\alpha_i}\right)^{\beta_i} . \end{aligned} \tag{6.10}$$

Differentiation of Eq. (6.10) with respect to the scale parameter α_i yields

$$\Rightarrow \frac{\partial \log f_i(x_j \,|\, \alpha_i, \beta_i)}{\partial \alpha_i} \;=\; -\frac{1}{\alpha_i} - \frac{\beta_i}{\alpha_i} + \frac{1}{\alpha_i} + \left(\frac{\beta_i}{\alpha_i}\right)\left(\frac{x_j}{\alpha_i}\right)^{\beta_i} \tag{6.11}$$

$$= \left(\frac{\beta_i}{\alpha_i}\right)\left(-1 + \left(\frac{x_j}{\alpha_i}\right)^{\beta_i}\right) . \tag{6.12}$$

Substituting Eq. (6.12) into Eq. (6.8) gives the EM equation for α_i,

$$\Rightarrow \sum_{j=1}^{n} z_{ij}^{(k)} \left(\frac{\beta_i}{\alpha_i}\right)\left(-1 + \left(\frac{x_j}{\alpha_i}\right)^{\beta_i}\right) = 0 , \tag{6.13}$$

$$\Rightarrow \sum_{j=1}^{n} z_{ij}^{(k)} = \sum_{j=1}^{n} z_{ij}^{(k)} \left(\frac{x_j}{\alpha_i}\right)^{\beta_i} , \tag{6.14}$$

$$\Rightarrow \boxed{\alpha_i = \left(\frac{\sum_{j=1}^{n} z_{ij}^{(k)}\, x_j}{\sum_{j=1}^{n} z_{ij}^{(k)}}\right)^{1/\beta_i}} , \tag{6.15}$$

noting the coupling with shape parameter β_i.

Now consider the partial derivative of Eq. (6.10) with respect to the shape parameter β_i,

$$\Rightarrow \frac{\partial \log f_i(x_j \,|\, \alpha_i, \beta_i)}{\partial \beta_i} \;=\; \frac{1}{\beta_i} + \log x_j - \log\alpha_i - \left(\frac{x_j}{\alpha_i}\right)^{\beta_i} \log\left(\frac{x_j}{\alpha_i}\right) . \tag{6.16}$$

A similar substitution of Eq. (6.16) into Eq. (6.8) gives the EM equation for β_i,

$$\Rightarrow \sum_{j=1}^{n} z_{ij}^{(k)} \left(\frac{1}{\beta_i} + \log x_j - \log\alpha_i - \left(\frac{x_j}{\alpha_i}\right)^{\beta_i} \log\left(\frac{x_j}{\alpha_i}\right)\right) = 0 , \tag{6.17}$$

$$\begin{aligned} \Rightarrow \frac{1}{\beta_i}\sum_{j=1}^{n} z_{ij}^{(k)} + \sum_{j=1}^{n} z_{ij}^{(k)} \log x_j - \log\alpha_i \sum_{j=1}^{n} z_{ij}^{(k)} - \frac{1}{\alpha_i^{\beta_i}} \sum_{j=1}^{n} z_{ij}^{(k)} x_j^{\beta_i} \log x_j \\ + \frac{\log\alpha_i}{\alpha_i^{\beta_i}} \sum_{j=1}^{n} z_{ij}^{(k)} x_j^{\beta_i} \;=\; 0 , \end{aligned} \tag{6.18}$$

$$\begin{aligned} \Rightarrow \frac{1}{\beta_i}\sum_{j=1}^{n} z_{ij}^{(k)} + \sum_{j=1}^{n} z_{ij}^{(k)} \log x_j - \log\alpha_i \sum_{j=1}^{n} z_{ij}^{(k)} - \frac{\sum_{j=1}^{n} z_{ij}^{(k)} x_j^{\beta_i} \log x_j \sum_{j=1}^{n} z_{ij}^{(k)}}{\alpha_i^{\beta_i} \sum_{j=1}^{n} z_{ij}^{(k)}} \\ + \frac{\log\alpha_i}{\alpha_i^{\beta_i}} \sum_{j=1}^{n} z_{ij}^{(k)} x_j^{\beta_i} \;=\; 0 . \end{aligned} \tag{6.19}$$

Using $\alpha_i^{\beta_i} \sum_{j=1}^{n} z_{ij}^{(k)} = \sum_{j=1}^{n} z_{ij}^{(k)} x_j^{\beta_i}$ from Eq. (6.14) yields

$$\Rightarrow \frac{1}{\beta_i}\sum_{j=1}^{n} z_{ij}^{(k)} + \sum_{j=1}^{n} z_{ij}^{(k)} \log x_j - \frac{\sum_{j=1}^{n} z_{ij}^{(k)} x_j^{\beta_i} \log x_j \sum_{j=1}^{n} z_{ij}^{(k)}}{\sum_{j=1}^{n} z_{ij}^{(k)} x_j^{\beta_i}} = 0 . \tag{6.20}$$

A solution of the following expression in terms of β_i is required,

$$\Rightarrow \boxed{g(\beta_i) = \frac{1}{\beta_i} + \frac{\sum_{j=1}^{n} z_{ij}^{(k)} \log x_j}{\sum_{j=1}^{n} z_{ij}^{(k)}} - \frac{\sum_{j=1}^{n} z_{ij}^{(k)} \log x_j \, x_j^{\beta_i}}{\sum_{j=1}^{n} z_{ij}^{(k)} \, x_j^{\beta_i}} = 0} \,. \tag{6.21}$$

Hence finding maximum likelihood estimates $\hat{\alpha}_i$ and $\hat{\beta}_i$ for the parameters of a Weibull distribution (as a parametric component of a finite mixture model) using the EM algorithm reduces to numerically solving the following coupled equations on the $(k+1)^{\text{th}}$ iteration.

$$\boxed{g(\beta_i^{(k+1)}) = \frac{1}{\beta_i^{(k+1)}} + \frac{\sum_{j=1}^{n} z_{ij}^{(k)} \log x_j}{\sum_{j=1}^{n} z_{ij}^{(k)}} - \frac{\sum_{j=1}^{n} z_{ij}^{(k)} \log x_j \, x_j^{\beta_i^{(k+1)}}}{\sum_{j=1}^{n} z_{ij}^{(k)} \, x_j^{\beta_i^{(k+1)}}} = 0} \,, \tag{6.22}$$

$$\boxed{\alpha_i^{(k+1)} = \left[\frac{\sum_{j=1}^{n} z_{ij}^{(k)} \, x_j^{\beta_i^{(k+1)}}}{\sum_{j=1}^{n} z_{ij}^{(k)}} \right]^{1/\beta_i^{(k+1)}}} \,. \tag{6.23}$$

6.1.1 Outline of Numerical Procedure

Given a data vector $\boldsymbol{x}$, the following is an outline of the numerical procedure for estimation of Weibull parameters for a given component of a mixed distribution.

- **Initialisation:** Choose Weibull parameters $\pi_i^{(0)}$ and $\boldsymbol{\theta}_i^{(0)} = (\alpha_i^{(0)}, \, \beta_i^{(0)})$, noting that the full parameter vector $\boldsymbol{\psi}^{(0)}$ also requires initialisation (which could be made up of further Weibull components, or other parametric distributions defined on $\mathbb{R}^+$)
- **E step$^{(1)}$:** Compute indicator variables $(z_{i1}^{(0)}, z_{i2}^{(0)}, ..., z_{in}^{(0)})$ using Eq. (6.7)
- **M step$^{(1)}$:**
  - Compute iterative estimate of mixing proportion $\pi_i^{(1)}$ using Eq. (6.6)
  - Compute iterative estimate of shape parameter $\beta_i^{(1)}$ using Eq. (6.22). This work used Newton-Raphson method to find a solution where $\beta_i^{(k+1)} = \beta_i^{(k)} - g(\beta_i^{(k)})/g'(\beta_i^{(k)})$
  - Compute iterative estimate of scale parameter $\alpha_i^{(1)}$ using Eq.(6.23), and current estimate for parameter $\beta_i^{(1)}$
- Compute the log-likelihood, $\log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x}) = \sum_{j=1}^{n} \log \left(\sum_{i=1}^{g} \pi_i^{(1)} f_i(x_j \,|\, \boldsymbol{\theta}_i^{(1)} \right)$, which is dependent on all of the component distributions
- **E step$^{(2)}$:** Recompute indicator variables $(z_{i1}^{(1)}, z_{i2}^{(1)}, ..., z_{in}^{(1)})$, with update $\boldsymbol{\psi}^{(1)}$
- **M step$^{(2)}$:** Compute updated Weibull parameters, i.e $\pi_i^{(2)}$ and $\boldsymbol{\theta}_i^{(2)} = (\alpha_i^{(2)}, \, \beta_i^{(2)})$ using methodology specified above
- Compute updated log-likelihood, checking convergence condition $|\log L(\boldsymbol{\psi}^{(2)} \,|\, \boldsymbol{x}) - \log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x})\,| < \epsilon$, where ϵ defines a preset tolerance, i.e. $\epsilon = 10^{-5}$
- Continue iteration until convergence condition is satisfied

$$\vdots$$

- Maximum likelihood estimate obtained for Weibull parameters $\hat{\pi}_i$ and $\hat{\boldsymbol{\theta}}_i = (\hat{\alpha}_i, \, \hat{\beta}_i)$

6.2 Hypothesis Testing for Weibull Mixture Models

Hypothesis based goodness-of-fit testing was presented in a general setting in Section 4.1, and considered more specifically for finite normal mixtures in Section 5.2. This section will consider hypothesis testing for Weibull mixture models. Recall the null (H_0) and alternative (H_1) hypotheses,

> $\boldsymbol{H_0}$: the observed data $\boldsymbol{x}$ originates from the proposed distribution $F(\boldsymbol{x} \,|\, \hat{\boldsymbol{\psi}})$
>
> $\boldsymbol{H_1}$: the observed data doesn't originate from the proposed distribution

where the distribution $F(\boldsymbol{x} \,|\, \hat{\boldsymbol{\psi}})$ now defines a finite mixture of Weibull components. An assessment of the null hypothesis can be made by comparison of test-statistics to previously determined critical values. If the test statistic is smaller than the critical value, the null hypothesis cannot be rejected. This section computes critical values for Kolmogorov-Smirnov, Kuiper, Cramér-von Mises, and Anderson-Darling goodness of fit tests for a range of sample sizes $n = \{50, 100, 200, 500, 1000\}$, for a **two-component** Weibull mixture model, at a 95% significance level ($\alpha = 0.05$). Section 5.2 justified empirically that critical values for finite normal mixtures are dependent on the parameters of the component distributions, the mixture proportions, and the sample size. It is hypothesised that critical values for finite Weibull mixtures will exhibit similar dependencies.

6.2.1 Random Number Generation: Weibull Distribution

In order to determine critical values for a given null hypothesis and goodness-of-fit test, the full distribution of test-statistics is required to be known under the assumption that the null hypothesis is true. The Monte-Carlo procedure outlined in Section 4.1.8 can be used to approach this task. The inverse transform method can be used to obtain observations sampled from a known distribution. Recall that one can generate a general pseudo-random number x_i, defined as

$$x_i \;=\; F^{-1}(u_i \,|\, \boldsymbol{\psi}) \;, \tag{6.24}$$

where u_i is a uniform variate. The inverse of the cumulative distribution function, commonly referred to as the quantile function, is required to be established for the Weibull distribution.

$$\begin{aligned} u_i &= 1 - e^{-\left(\frac{x_i}{\alpha}\right)^{\beta}} && (6.25)\\ \Rightarrow\; \log(1 - u_i) &= -\left(\frac{x_i}{\alpha}\right)^{\beta} && (6.26)\\ \Rightarrow\; \alpha\,[\,-\log(1 - u_i)\,]^{1/\beta} &= x_i \;=\; F^{-1}(u_i \,|\, \alpha,\, \beta)\;. && (6.27) \end{aligned}$$

6.2.2 Two-component Weibull mixture

Consider a mixture of two univariate Weibull distributions. The mixture components have probability density functions $f_1(x_j)$ and $f_2(x_j)$ respectively, and cumulative distribution functions $F_1(x_j)$ and $F_2(x_j)$ respectively, defined on the support $x \in [0, \infty)$. Parameter vector $\boldsymbol{\psi} = (\boldsymbol{\pi}, \boldsymbol{\xi})$ for this case is simply $\boldsymbol{\psi} = (\pi_1,\, \alpha_1,\, \beta_1,\, \alpha_2,\, \beta_2)$, which encapsulates the shape and scale parameters of both Weibull distributions and one mixture proportion

- $$f_i(x_j \,|\, \alpha_i\;\beta_i) \;=\; \left(\frac{\beta_i}{\alpha_i}\right)\left(\frac{x_j}{\alpha_i}\right)^{\beta_i - 1} e^{-(x_j/\alpha_i)^{\beta_i}}, \tag{6.28}$$

- $$F_i(x_j \,|\, \alpha_i\;\beta_i) \;=\; 1 - e^{-(x_j/\alpha_i)^{\beta_i}}, \qquad i = 1, 2. \tag{6.29}$$

The Monte-Carlo procedure outlined in Section 4.1.8 was used to numerically compute critical values for two-component Weibull mixtures. Simulations were undertaken via the proposed multisample method with $C = 100$ (number of repetitions) and numSims $= 10000$ (number of simulations), for the parameter values, mixture proportions, sample sizes, and goodness-of-fit tests listed below. The parameter values chosen to assess the critical values were

$$\boldsymbol{\alpha} = \{1,\ 1\}, \quad \boldsymbol{\beta} = \{\ [0.5,\ 0.5,\ 0.5,\ 1,\ 1,\ 1.5],\ [1,\ 1.5,\ 5,\ 1.5,\ 5,\ 5]\ \}\ . \tag{6.30}$$

The mixture proportions and sample sizes considered were

$$\pi_1 = \{0, 0.1, 0.3, 0.5, 0.7, 0.9, 1\}, \tag{6.31}$$

$$n = \{50, 100, 200, 500, 1000\}, \tag{6.32}$$

for Kolmogorov-Smirnov, Kuiper, Cramér-von Mises, and Anderson-Darling goodness-of-fit tests.

6.3 Results: Critical Values for Two-component Weibull Mixtures

6.3.1 Parameters/Mixture Proportions Known *a Priori*

Consistent with the methodology in Section 5.2, first consider the case where the full parameter vector $\boldsymbol{\psi}$ is known *a priori*. This case considers no estimation for either the component parameter vectors or the mixture proportions. A representative sample of the critical values is provided in Tables 6.1 and 6.2 (and Fig. 6.2 and 6.3) for Kolomogorov-Smirnov goodness-of-fit testing and sample sizes $n = \big\{50, 1000\big\}$, the remainder are provided in Appendix B.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3324(17)	1.3288(16)	1.3249(16)	1.3216(17)	1.3230(18)	1.3284(15)	1.3314(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3326(16)	1.3271(14)	1.3180(14)	1.3114(17)	1.3123(15)	1.3214(18)	1.3326(16)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3314(16)	1.3216(15)	1.2969(14)	1.2770(15)	1.2711(14)	1.3002(13)	1.3315(15)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3303(16)	1.3308(15)	1.3304(15)	1.3299(17)	1.3287(17)	1.3320(17)	1.3315(16)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3313(15)	1.3235(15)	1.3091(16)	1.2949(15)	1.2934(15)	1.3123(15)	1.3326(17)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3317(14)	1.3274(17)	1.3151(17)	1.3080(17)	1.3079(14)	1.3206(17)	1.3329(16)

Table 6.1: Kolomogorov-Smirnov critical values, Case I: $n = 50$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3521(15)	1.3517(15)	1.3463(17)	1.3428(16)	1.3436(16)	1.3485(15)	1.3530(16)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3530(12)	1.3462(17)	1.3385(15)	1.3315(18)	1.3337(17)	1.3419(15)	1.3540(16)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3534(14)	1.3424(17)	1.3195(15)	1.2973(16)	1.2941(15)	1.3207(16)	1.3521(15)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3524(17)	1.3524(16)	1.3504(15)	1.3514(15)	1.3490(15)	1.3517(16)	1.3526(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3519(14)	1.3453(17)	1.3277(17)	1.3145(16)	1.3111(16)	1.3322(18)	1.3537(16)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3527(16)	1.3469(15)	1.3352(13)	1.3283(16)	1.3281(16)	1.3404(16)	1.3526(13)

Table 6.2: Kolomogorov-Smirnov critical values, Case I: $n = 1000$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

The analogous dependencies are evident for mixtures of Weibull components as were observed for normal densities. Even for the case when no parameter estimation is undertaken from the data, critical values $D^n_{cv\,0.95}$ for two-component Weibull mixtures have evident dependencies:

- Critical values $D^n_{cv\,0.95}$ are **parameter dependent**, both on $\boldsymbol{\theta}$ the parameter vector of each component distribution, and $\boldsymbol{\pi}$ the mixture proportions.

- Dependence on the sample size n is also evident. An increase in n corresponds to an increase in the critical values $D^n_{cv\,0.95}$.

For component distributions which exhibit close shape parameters, the two-component mixture reduces to a single distribution and critical values are consistent and independent of $\boldsymbol{\pi}$ as expected. This is best illustrated by Fig. 6.2 and 6.3 for the case of $\beta_1 = 1$ and $\beta_2 = 1.5$. Additionally, at the mixture proportion boundaries (i.e. $\pi_1 = 0$ or 1) the single distribution independent critical values are once again recovered exactly.

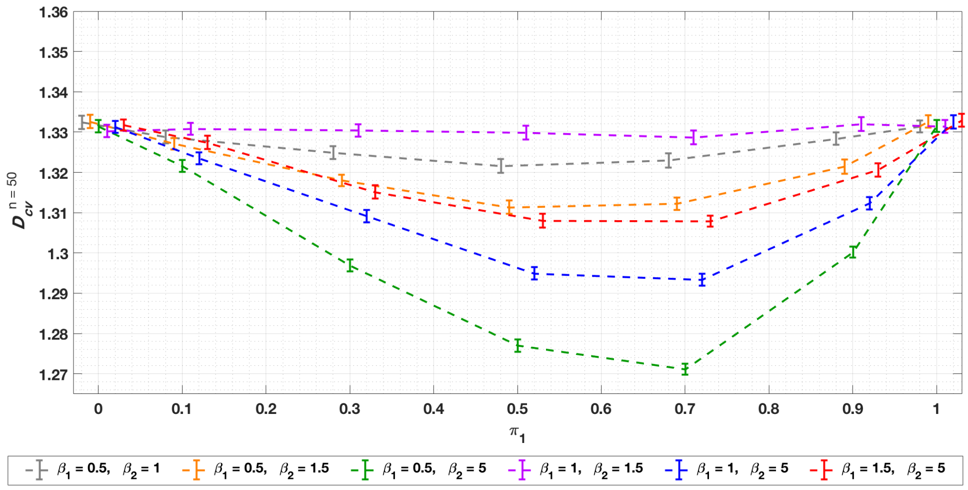


Figure 6.2: KS critical values: $n = 50$. Case I.

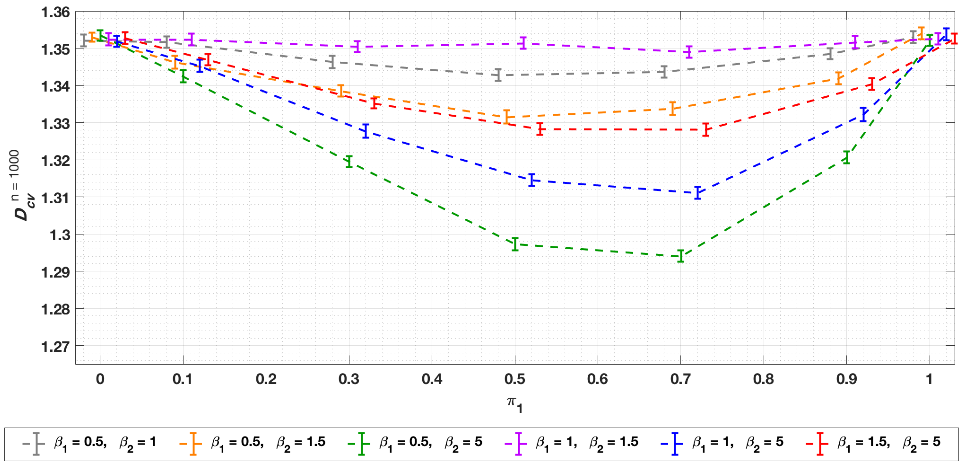


Figure 6.3: KS critical values: $n = 1000$. Case I.

6.3.2 Component Parameters Known *a Priori*, Mixture Proportions Requiring Estimation

Now consider the case where the specific parameter vectors $\boldsymbol{\theta_i}$ of the component distributions are known *a priori*, but the mixture proportions require estimation. This section provides the same representative sample of Kolomogorov-Smirnov critical values for $n = \{50, 1000\}$ in Tables 6.3 and 6.4 (and Fig. 6.4 and 6.5). The remainder of the critical values are similarly provided in Appendix B.

Similar exacerbation of the same critical value dependencies is evident here as was the case for the analysis of normal mixtures. The single distribution independent critical values are somewhat recovered at the mixture proportion boundary $\pi_1 = 0$. This is due to the labelling convention $\beta_1 < \beta_2$ adopted for this case. If $\beta_1 > \beta_2$ this behaviour would have occurred at the other mixture

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3203(15)	1.3126(17)	1.2936(15)	1.2734(16)	1.2600(16)	1.2735(14)	1.3082(16)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3268(16)	1.3157(19)	1.2915(14)	1.2675(17)	1.2430(16)	1.2472(15)	1.3031(17)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3328(18)	1.3173(16)	1.2870(16)	1.2522(15)	1.2277(15)	1.2398(17)	1.3069(16)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3083(15)	1.3035(15)	1.2941(15)	1.2843(15)	1.2818(17)	1.2967(16)	1.3161(18)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3291(17)	1.3164(19)	1.2872(17)	1.2586(15)	1.2310(17)	1.2318(17)	1.2992(17)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3259(15)	1.3156(15)	1.2906(18)	1.2637(15)	1.2408(16)	1.2428(14)	1.3002(17)

Table 6.3: Kolomogorov-Smirnov critical values, Case II: $n = 50$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3491(15)	1.3346(14)	1.3131(16)	1.2945(16)	1.2771(16)	1.2630(16)	1.3136(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3508(13)	1.3364(15)	1.3112(16)	1.2875(14)	1.2627(17)	1.2456(16)	1.3091(13)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3525(15)	1.3388(17)	1.3065(15)	1.2713(14)	1.2476(17)	1.2578(16)	1.3222(15)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3437(17)	1.3325(18)	1.3158(17)	1.3047(15)	1.2914(16)	1.2834(16)	1.3226(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3514(16)	1.3373(18)	1.3080(16)	1.2794(17)	1.2506(16)	1.2426(13)	1.3121(16)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3514(16)	1.3364(19)	1.3111(15)	1.2835(16)	1.2609(15)	1.2432(15)	1.3082(14)

Table 6.4: Kolomogorov-Smirnov critical values, Case II: $n = 1000$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

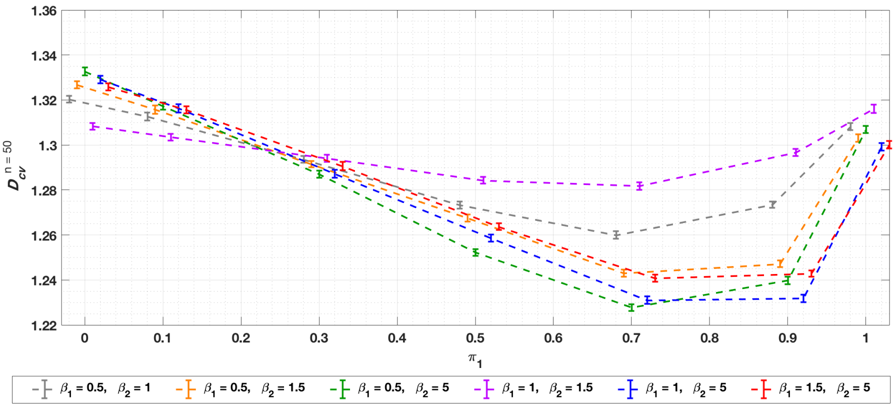


Figure 6.4: KS critical values: $n = 50$. Case II.

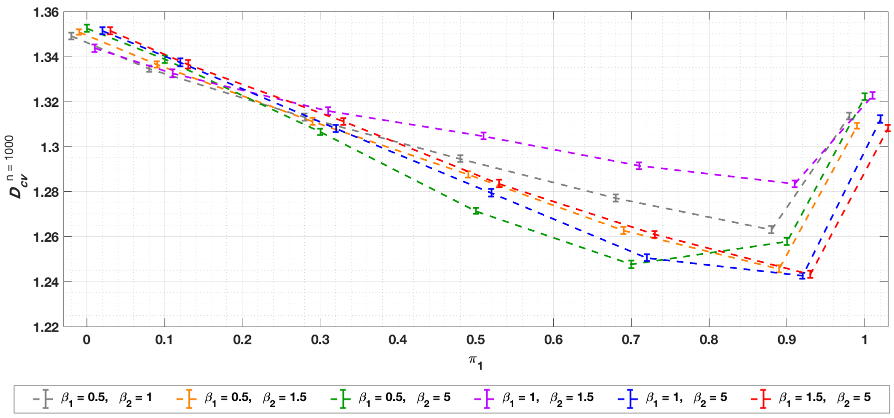


Figure 6.5: KS critical values: $n = 1000$. Case II.

proportion boundary $\pi_1 = 1$. If critical values are represented as a function of the mixture proportion π_1 (as in Fig. 6.2 and 6.3) the values are negatively skewed if $\beta_1 < \beta_2$ and positively skewed if $\beta_1 < \beta_2$. This is consistent with the behaviour which was observed for the variance parameters in two-component normal mixtures. Section 5.2 provided rationale for much of this behaviour.

For any non-trivial examples, parameter dependent critical values are largely inadequate for use in goodness-of-fit testing. A study of the dependencies of critical values has now been undertaken for two-component normal and Weibull mixtures. **The remainder of this thesis will employ information criteria based goodness-of-fit tests, which have been demonstrated to be more appropriate for use in mixture modelling.**

Chapter 7

Exponential Component Distribution

The exponential distribution is a popular continuous probability distribution found in numerous contexts within probability theory and statistics. One key property of the exponential distribution is that it is memoryless, eg. $P(X > m+n \mid P > m) = P(X > n)$. The distribution therefore appears naturally when describing the time separation of events in a Poisson process, typical examples include the time taken for a radioactive isotope to decay, or the waiting times in queuing theory.

The probability density function (pdf) of a random variable X, distributed according to a single exponential distribution and making up a component of a mixture model is

$$f_i(x \mid \lambda_i) = \lambda_i \, e^{-\lambda_i \, x}, \quad x \geq 0 \;, \tag{7.1}$$

where $\lambda_i \in (0, +\infty)$ is the rate, or the inverse scale parameter, and the distribution is defined on the domain $\mathbb{R}^+ \cup 0$ (i.e. $x \in [0, +\infty)$). The cumulative distribution function (cdf) of the exponential component distribution is given as

$$F_i(x \mid \lambda) = 1 - e^{-\lambda_i \, x}, \quad x \geq 0 \;. \tag{7.2}$$

Figure 7.1 describes the distribution of probability density, as well as the cumulative distribution function, for various rate parameters.

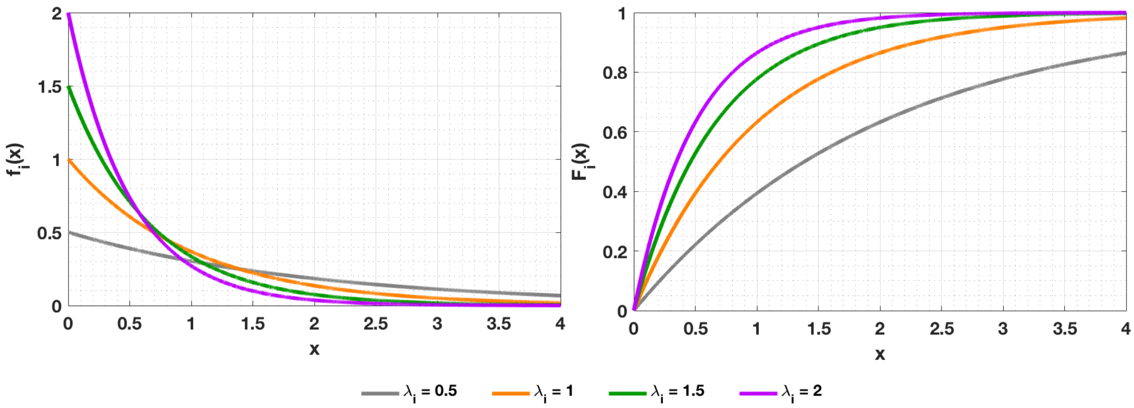


Figure 7.1: pdf and cdf of exponential distribution for various rate parameters.

In order to describe observations drawn from a mixture model, knowledge of the parameter vector $\boldsymbol{\theta}_i$ estimated from the data, is required. The following section considers estimation for an exponential component distribution.

7.1 EM Equations for Exponential Distribution

Chapter 3 introduced the Expectation Maximisation algorithm and presented the EM framework for maximum likelihood estimation of parameters in mixed distributions. The specific EM equations required for parameter estimation of exponential parametric components will now be derived.

Recall the postulated probability density function of the finite mixture used to model the observed data vector $\boldsymbol{x} = (x_1, ..., x_n)$, given as

$$f(x \,|\, \boldsymbol{\psi}) = \sum_{i=1}^{g} \pi_i f_i(x \,|\, \boldsymbol{\theta}_i) \ , \tag{7.3}$$

where at least one mixture component is now specified by an exponential distribution. Consider the i^{th} parametric component $f_i(x \,|\, \boldsymbol{\theta}_i)$ to be exponential. The model is parametrised by the vector $\boldsymbol{\psi} = (\boldsymbol{\pi}, \boldsymbol{\xi})$. As before, $\boldsymbol{\pi} = (\pi_1, ..., \pi_{g-1})$ is a vector containing the $(g$ - $1)$ mixture proportions, and the vector $\boldsymbol{\xi} = (\boldsymbol{\theta_1}, ..., \boldsymbol{\theta_g})$ contains the parameter vectors for each component of the mixture. For the exponential component $f_i(x \,|\, \boldsymbol{\theta}_i)$ the parameter vector is given as $\boldsymbol{\theta_i} = \lambda_i$ which only contains the rate parameter that singly parametrises the exponential distribution. Recall (from Section 3.2) that given a current estimate $\boldsymbol{\psi}^{(k)}$ the expectation of the log-likelihood function for a finite mixture is given as

$$Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) = \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log \pi_i + \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log f_i(x_j \,|\, \boldsymbol{\theta_i}) + \lambda \left(\sum_{i=1}^{g} \pi_i - 1 \right) \ , \tag{7.4}$$

noting the use of the Lagrange multiplier λ to deal with the constraint that all mixture proportions sum to one, and $z_{ij} = (\mathbf{z}_i)_j$ is a zero-one indicator variable specifying whether x_j arose or did not arise from the ith component of the mixture ($i = 1, ..., g$ and $j = 1, ..., n$). The EM equation for mixture proportions is constructed by taking the partial derivative of Eq. (7.4) with respect to π_i, and equating it to zero, i.e.

$$\frac{\partial \, Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})}{\partial \, \pi_i} = 0 \ . \tag{7.5}$$

The value of $\boldsymbol{\pi}^{(k+1)}$ (on the current iteration) is given as that which maximises $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$. As previously discussed, a derivation of the following iterative estimate for the mixture proportions was provided in Section 3.2 and won't be repeated here because the resulting expression is general for any parametric component. All of the distribution dependence is absorbed by the indicator variable $z_{ij}^{(k)}$.

$$\boxed{\pi_i^{(k+1)} = \sum_{j=1}^{n} \frac{z_{ij}^{(k)}}{n}} \ , \tag{7.6}$$

where

$$z_{ij}^{(k)} = \frac{\pi_i^{(k)} \, f_i(x_j \,|\, \boldsymbol{\theta_i})}{\sum_{m=1}^{g} \pi_m^{(k)} \, f_m(x_j \,|\, \boldsymbol{\theta_m})} \ , \qquad i = 1, \, ..., \, g \ . \tag{7.7}$$

The derivation of EM equations for exponential component parameter estimation of $\boldsymbol{\theta_i} = \lambda_i$ will now be considered. Recall that EM equations for iterative estimation of parameter vector $\boldsymbol{\xi}^{(k+1)}$ can be constructed by taking partial derivatives of Eq. (6.4) with respect to $\boldsymbol{\theta_i}$. Selecting parameter vector $\boldsymbol{\xi}^{(k+1)}$ which maximises $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$ on the current iteration is given as an appropriate solution of the expression,

$$\sum_{j=1}^{n} z_{ij}^{(k)} \frac{\partial \log f_i(x_j \,|\, \boldsymbol{\theta_i})}{\partial \, \boldsymbol{\theta_i}} = 0, \qquad i = 1, \, ..., \, g. \tag{7.8}$$

Taking the natural logarithm of the component pdf gives

$$\Rightarrow \ \log f_i(x_j \,|\, \lambda_i) = \log \lambda_i - \ \lambda_i \, x_j \ , \tag{7.9}$$

Differentiation of Eq. (7.9) with respect to the rate parameter λ_i yields

$$\Rightarrow \frac{\partial \log f_i(x_j \,|\, \lambda_i)}{\partial \lambda_i} = \frac{1}{\lambda_i} - x_j \ , \tag{7.10}$$

Substituting Eq. (7.10) into Eq. (7.8) gives the EM equation for λ_i,

$$\sum_{j=1}^{n} z_{1j}^{(k)} \left(\frac{1}{\lambda_1} - x_j \right) \quad = \quad 0 \ . \tag{7.11}$$

Rearrangement yields the iterative estimate for λ_i:

$$\boxed{\lambda_i^{(k+1)} = \frac{\sum_{j=1}^{n} z_{ij}^{(k)}}{\sum_{j=1}^{n} z_{ij}^{(k)} \, x_j}} \ . \tag{7.12}$$

This section presented a derivation of an EM equation for parameter estimation of exponential components. Usually a numerical procedure is used to compute iterative estimate of parameters in mixture models.

7.1.1 Outline of Numerical Procedure

Given a data vector $\boldsymbol{x}$, the following is an outline of the numerical procedure for estimation of exponential parameters for a given component of a mixed distribution. Similarly to mixtures of normal densities, exponential components yield EM equations that can be expressed in closed form, and hence present significant numerical simplicity.

- **Initialisation:** Choose exponential parameters $\pi_i^{(0)}$ and $\boldsymbol{\theta}_i^{(0)} = \lambda_i^{(0)}$, noting that the full parameter vector $\boldsymbol{\psi}^{(0)}$ also requires initialisation (which could be made up of further exponential components, or other parametric distributions defined on $\mathbb{R}^+$)
- **E step$^{(1)}$:** Compute indicator variables $(z_{i1}^{(0)}, z_{i2}^{(0)}, ..., z_{in}^{(0)})$ using Eq. (7.7)
- **M step$^{(1)}$:** Compute iterative estimate of exponential component parameters, i.e. $\pi_i^{(1)}$ and $\boldsymbol{\theta}_i^{(1)} = \lambda_i^{(1)}$ using Eq. (7.6) and Eq. (7.12)
- Compute the log-likelihood, $\log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x}) = \sum_{j=1}^{n} \log \big(\sum_{i=1}^{g} \pi_i^{(1)} f_i(\boldsymbol{x} \,|\, \boldsymbol{\theta}_i^{(1)} \big)$, which is dependent on all of the component distributions
- **E step$^{(2)}$:** Recompute indicator variables $(z_{i1}^{(1)}, z_{i2}^{(1)}, ..., z_{in}^{(1)})$, with update $\boldsymbol{\psi}^{(1)}$
- **M step$^{(2)}$:** Compute updated exponential parameters, i.e $\pi_i^{(2)}$ and $\boldsymbol{\theta}_i^{(2)} = \lambda_i^{(2)}$ using methodology specified above
- Compute updated log-likelihood, checking convergence condition $|\log L(\boldsymbol{\psi}^{(2)} \,|\, \boldsymbol{x}) - \log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x})\,| < \epsilon$, where ϵ defines a preset tolerance, i.e. $\epsilon = 10^{-5}$
- Continue iteration until convergence condition is satisfied

$$\vdots$$

- Maximum likelihood estimate obtained for exponential parameters $\hat{\pi}_i$, and $\hat{\boldsymbol{\theta}}_i = \hat{\lambda}_i$

Chapter 8

Gamma Component Distribution

The gamma distribution is a versatile two parameter continuous probability distribution, defined such that the exponential (Chapter 7), chi-squared, and Erlang distributions are all special cases of it. The gamma distribution is commonplace within statistical modelling, and has been used to model daily rainfall [71], and the size of insurance claims [72], among other things.

The probability density function (pdf) of a random variable X, distributed according to a single gamma distribution and making up a component of a mixture model is

$$f_i(x \,|\, K_i,\, \theta_i) = \frac{1}{\Gamma(K_i)\, \theta_i^{K_i}}\, x^{K_i - 1}\, e^{-x/\theta_i}, \quad x \geq 0\ , \tag{8.1}$$

where $\theta_i \in (0, +\infty)$ is the scale parameter, $K_i \in (0, +\infty)$ is the shape parameter, and the distribution is defined on the domain $\mathbb{R}^+ \cup 0$ (i.e. $x \in [0, +\infty)$). The cumulative distribution function (cdf) of the gamma component is

$$F_i(x \,|\, K_i,\, \theta_i) = \frac{1}{\Gamma(K_i)\, \theta_i^{K_i}}\, x^{K_i - 1}\, e^{-x/\theta_i}, \quad x \geq 0\ . \tag{8.2}$$

Figure 8.1 describes the distribution of probability density as well as the cumulative distribution function for various scale and shape parameters.

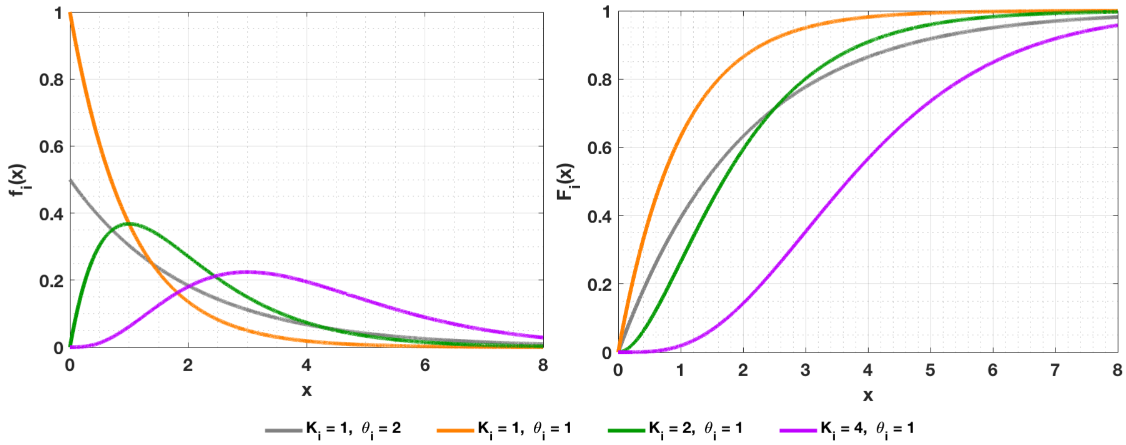


Figure 8.1: pdf and cdf of gamma distribution for various scale and shape parameters.

Knowledge of the parameter vector $\boldsymbol{\theta}_i$ which is estimated from the data is required in order to describe observations drawn from a mixture model. The following section considers parameter estimation for a gamma component distribution.

8.1 EM Equations for Gamma Distribution

Chapter 3 introduced the Expectation Maximisation algorithm and presented the EM framework for maximum likelihood estimation of parameters in mixed distributions. The specific EM equations required for parameter estimation of gamma parametric components will now be derived.

Once again recall the postulated probability density function of the finite mixture used to model the observed data vector $\boldsymbol{x} = (x_1, ..., x_n)$, given as

$$f(x \,|\, \boldsymbol{\psi}) = \sum_{i=1}^{g} \pi_i f_i(x \,|\, \boldsymbol{\theta}_i) \;, \tag{8.3}$$

where at least one mixture component is specified by a gamma distribution. Let the i^{th} parametric component $f_i(x \,|\, \boldsymbol{\theta}_i)$ be assigned the gamma component. The model is parametrised by the vector $\boldsymbol{\psi} = (\boldsymbol{\pi},\, \boldsymbol{\xi})$. As before, $\boldsymbol{\pi} = (\pi_1, ..., \pi_{g-1})$ is a vector containing the $(g$ - $1)$ mixing proportions, and the vector $\boldsymbol{\xi} = (\boldsymbol{\theta_1}, ..., \boldsymbol{\theta_g})$ contains the parameter vectors for each component of the mixture. For the gamma component $f_i(x \,|\, \boldsymbol{\theta}_i)$, the parameter vector is given as $\boldsymbol{\theta_i} = (K_i, \theta_i)$, encapsulating both the shape and scale parameters of the gamma distribution. Recall (from Section 3.2) that given a current estimate $\boldsymbol{\psi}^{(k)}$, the expectation of the log-likelihood function for a finite mixture is given as

$$Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) = \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log \pi_i + \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log f_i(x_j \,|\, \boldsymbol{\theta_i}) + \lambda \left(\sum_{i=1}^{g} \pi_i - 1 \right) \;, \tag{8.4}$$

recalling the use of the Lagrange multiplier λ to deal with the constraint that all mixture proportions sum to one, and $z_{ij} = (\mathbf{z}_i)_j$ is a zero-one indicator variable specifying whether x_j arose or did not arise from the ith component of the mixture ($i = 1, ..., g$ and $j = 1, ..., n$). The EM equation for the mixture proportions is constructed by taking the partial derivative of Eq. (8.4) with respect to π_i, and equating it to zero, i.e.

$$\frac{\partial\, Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})}{\partial\, \pi_i} = 0 \;. \tag{8.5}$$

The value of $\boldsymbol{\pi}^{(k+1)}$ (on the current iteration) is given as that which maximises $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$. A derivation of the following iterative estimate for the mixture proportions was provided in Section 3.2.

$$\boxed{\pi_i^{(k+1)} = \sum_{j=1}^{n} \frac{z_{ij}^{(k)}}{n}} \;, \tag{8.6}$$

where

$$z_{ij}^{(k)} = \frac{\pi_i^{(k)}\, f_i(x_j \,|\, \boldsymbol{\theta_i})}{\sum_{m=1}^{g} \pi_m^{(k)} f_m(x_j \,|\, \boldsymbol{\theta_m})} \;, \qquad i = 1,\, ...,\, g \;. \tag{8.7}$$

The derivation of EM equations for estimation of gamma component parameters $\boldsymbol{\theta_i} = (K_i,\, \theta_i)$ will now be considered. Recall that the EM equations for iterative estimation of the parameter vector $\boldsymbol{\xi}^{(k+1)}$ can be constructed by taking partial derivatives of Eq. (8.4) with respect to $\boldsymbol{\theta_i}$. Selecting the parameter vector $\boldsymbol{\xi}^{(k+1)}$ which maximises $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$ on the current iteration is given as an appropriate solution of the expression,

$$\sum_{j=1}^{n} z_{ij}^{(k)} \frac{\partial \log f_i(x_j \,|\, \boldsymbol{\theta_i})}{\partial\, \boldsymbol{\theta_i}} = 0 \;, \qquad i = 1,\, ...,\, g \;. \tag{8.8}$$

Taking the natural logarithm of the component pdf is

$$\Rightarrow \;\log f_i(x_j \,|\, K_i,\, \theta_i) = -\log \Gamma(K_i) - K_i \log \theta_i + (K_i - 1) \log x_j - \frac{x_j}{\theta_i} \;. \tag{8.9}$$

Differentiation of Eq. (8.9) with respect to the scale parameter θ_i,

$$\Rightarrow \frac{\partial \log f_i(x_j \,|\, K_i,\, \theta_i)}{\partial\, \theta_i} = -\frac{K_i}{\theta_i} + \frac{x_j}{\theta_i^2} \ . \tag{8.10}$$

Substituting Eq. (8.10) into Eq. (8.8) gives the EM equation for θ_i:

$$\Rightarrow \sum_{j=1}^{n} z_{ij}^{(k)} \Big(-\frac{K_i}{\theta_i} + \frac{x_j}{\theta_i^2} \Big) \quad = \quad 0 \tag{8.11}$$

$$\Rightarrow \frac{K_i}{\theta_i} \sum_{j=1}^{n} z_{ij}^{(k)} \quad = \quad \frac{1}{\theta_i^2} \sum_{j=1}^{n} z_{ij}^{(k)}\, x_j \tag{8.12}$$

$$\Rightarrow \boxed{K_i \;=\; \frac{\sum_{j=1}^{n} z_{ij}^{(k)}\, x_j}{\theta_i\, \sum_{j=1}^{n} z_{ij}^{(k)}}}\ , \tag{8.13}$$

noting the coupling with parameter θ_i.

Now consider the partial derivative of Eq. (8.9) with respect to the shape parameter K_i,

$$\Rightarrow \frac{\partial \log f_i(x_j \,|\, K_i,\, \theta_i)}{\partial\, K_i} = -\psi^{(0)}(K_i) - \log\theta_i + \log x_j \ , \tag{8.14}$$

where $\psi^{(0)}$ is the digamma function, defined as the logarithmic derivative of the gamma function,

$$\psi^{(0)}(x) = \frac{d}{dx} \log\Gamma(x) \ . \tag{8.15}$$

Substitution of Eq. (8.14) into Eq. (8.8) gives the EM equation for K_i:

$$\Rightarrow \sum_{j=1}^{n} z_{ij}^{(k)} \Big(-\psi^{(0)}(K_i) - \log\theta_i + \log x_j \Big) \quad = \quad 0\ , \tag{8.16}$$

and using Eq. (8.13) yields

$$\Rightarrow \boxed{\sum_{j=1}^{n} z_{ij}^{(k)} \Bigg(-\psi^{(0)} \bigg(\frac{\sum_{j=1}^{n} z_{ij}^{(k)} x_j}{\theta_i\, \sum_{j=1}^{n} z_{ij}^{(k)}} \bigg) - \log\theta_i + \log x_j \Bigg) = 0}\ . \tag{8.17}$$

Hence finding maximum likelihood estimates $\hat{K}_i$ and $\hat{\theta}_i$ for the parameters of a gamma distribution (as a parametric component of a finite mixture model) using the EM algorithm reduces to numerically finding a solution for $\theta_i^{(k+1)}$ on the $(k+1)^{\text{th}}$ iteration and then substituting for a solution for $K_i^{(k+1)}$ using

$$\Rightarrow \boxed{K_i^{(k+1)} = \frac{\sum_{j=1}^{n} z_{ij}^{(k)} x_j}{\theta_i^{(k+1)}\ \sum_{j=1}^{n} z_{ij}^{(k)}}} \tag{8.18}$$

$$\Rightarrow \boxed{\sum_{j=1}^{n} z_{ij}^{(k)} \Bigg(-\psi^{(0)} \big(\frac{\sum_{j=1}^{n} z_{ij}^{(k)} x_j}{\theta_i^{(k+1)}\ \sum_{j=1}^{n} z_{ij}^{(k)}} \big) - \log\theta_i^{(k+1)} + \log x_j \Bigg) = 0}\ . \tag{8.19}$$

8.1.1 Outline of Numerical Procedure

Given a data vector $\boldsymbol{x}$, the following is an outline of the numerical procedure for estimation of gamma parameters for a given component of a mixed distribution.

- **Initialisation:** Choose gamma parameters $\pi_i^{(0)}$ and $\boldsymbol{\theta}_i^{(0)} = (K_i^{(0)},\, \theta_i^{(0)})$, noting that the full parameter vector $\boldsymbol{\psi}^{(0)}$ also requires initialisation (which could be made up of further gamma components, or other parametric distributions defined on $\mathbb{R}^+$)

- **E step$^{(1)}$:** Compute indicator variables $(z_{i1}^{(0)}, z_{i2}^{(0)}, ..., z_{in}^{(0)})$ using Eq. (8.7)
- **M step$^{(1)}$:**
  - Compute iterative estimate of mixing proportion $\pi_i^{(1)}$ using Eq. (8.6)
  - Compute iterative estimate of scale parameter $\theta_i^{(1)}$ using Eq. (8.19) via a numerical procedure. This work used the *fzero* [73] function available in MATLAB for finding roots of nonlinear functions
  - Compute iterative estimate of shape parameter $K_i^{(1)}$ using Eq. (8.18), and current estimate for parameter $\theta_i^{(1)}$
- Compute the log-likelihood, $\log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x}) = \sum_{j=1}^{n} \log \left(\sum_{i=1}^{g} \pi_i^{(1)} f_i(x_j \,|\, \boldsymbol{\theta}_i^{(1)} \right)$, which is dependent on all of the component distributions
- **E step$^{(2)}$:** Recompute indicator variables $(z_{i1}^{(1)}, z_{i2}^{(1)}, ..., z_{in}^{(1)})$, with update $\boldsymbol{\psi}^{(1)}$
- **M step$^{(2)}$:** Compute updated gamma parameters, i.e $\pi_i^{(2)}$ and $\boldsymbol{\theta}_i^{(2)} = (K_i^{(2)}, \theta_i^{(2)})$ using methodology specified above
- Compute updated log-likelihood, checking convergence condition $|\log L(\boldsymbol{\psi}^{(2)} \,|\, \boldsymbol{x}) - \log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x})\,| < \epsilon$, where ϵ defines a preset tolerance, i.e. $\epsilon = 10^{-5}$
- Continue iteration until convergence condition is satisfied

$$\vdots$$

- Maximum likelihood estimate obtained for gamma parameters $\hat{\pi}_i$ and $\hat{\boldsymbol{\theta}}_i = (\hat{K}_i, \hat{\theta}_i)$

Chapter 9

Loglogistic Component Distribution

The loglogistic distribution is a continuous probability distribution associated with the more widely known logistic distribution. The distribution is defined such that a random variable which is distributed by the loglogistic distribution has a logarithm which is distributed by the logistic distribution. The loglogistic distribution is commonplace within many statistical domains. In economics, where it is known as the Fisk distribution, it is used to model the distribution of wealth within a population [74]. The distribution has also been used to model transmission times of data in networks [75], and is also typical within life-time analysis [76].

The probability density function (pdf) of a random variable X, distributed according to a single loglogistic distribution and making up a component of a mixture model is

$$f_i(x \,|\, \alpha_i,\, \beta_i) = \left(\frac{\beta_i}{\alpha_i}\right)\left(\frac{x}{\alpha_i}\right)^{\beta_i - 1} \frac{1}{\left[1 + \left(\frac{x}{\alpha_i}\right)^{\beta_i}\right]^2} \;, \qquad x \geq 0 \;, \tag{9.1}$$

where $\alpha_i \in (0, +\infty)$ is the scale parameter, $\beta_i \in (0, +\infty)$ is the shape parameter, and the distribution is defined on the domain $\mathbb{R}^+ \cup 0$ (i.e. $x \in [0, +\infty)$). Correspondingly the logistic distribution is supported on the domain $\mathbb{R}$ (i.e. $x \in (-\infty, +\infty)$).

The cumulative distribution function (cdf) of the loglogistic component is

$$F(x \,|\, \alpha_i,\, \beta_i) = \frac{1}{1 + \left(\dfrac{x}{\alpha_i}\right)^{-\beta_i}} \;, \qquad x \geq 0 \;. \tag{9.2}$$

Figure 9.1 describes the distribution of probability density and cumulative density for various scale and shape parameters.

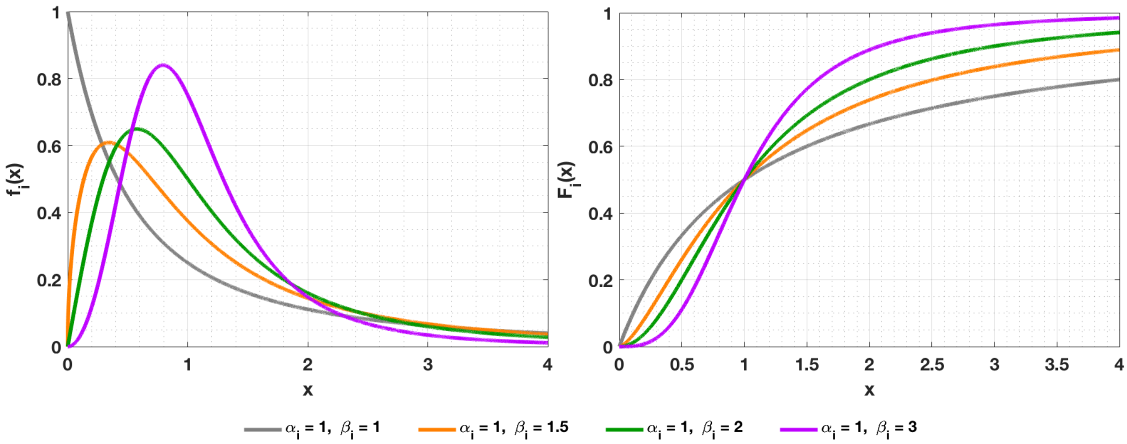


Figure 9.1: pdf and cdf of loglogistic distribution for various scale and shape parameters.

Describing observations drawn from a mixture model requires knowledge of the parameter vector $\boldsymbol{\theta}_i$ which is estimated from the data. The following section considers estimation for a loglogistic component distribution.

9.1 EM Equations for Loglogistic Distribution

Chapter 3 introduced the Expectation Maximisation algorithm and presented the EM framework for maximum likelihood estimation of parameters in mixed distributions. The specific EM equations required for parameter estimation of loglogistic parametric components will now be derived.[1]

Consider the postulated probability density function of the finite mixture used to model the observed data vector $\boldsymbol{x} = (x_1, ..., x_n)$, given as

$$f(x \,|\, \boldsymbol{\psi}) = \sum_{i=1}^{g} \pi_i f_i(x \,|\, \boldsymbol{\theta}_i) \ , \tag{9.3}$$

where at least one mixture component is specified by a loglogistic distribution. Let the i^{th} parametric component $f_i(x \,|\, \boldsymbol{\theta}_i)$ now be defined as loglogistic. The model is parametrised by the vector $\boldsymbol{\psi} = (\boldsymbol{\pi}, \boldsymbol{\xi})$. As before, $\boldsymbol{\pi} = (\pi_1, ..., \pi_{g-1})$ is a vector containing the (g - 1) mixture proportions, and the vector $\boldsymbol{\xi} = (\boldsymbol{\theta_1}, ..., \boldsymbol{\theta_g})$ contains the parameter vectors for each component of the mixture. For the loglogistic component $f_i(x \,|\, \boldsymbol{\theta}_i)$, the parameter vector is given as $\boldsymbol{\theta_i} = (\alpha_i, \beta_i)$, encapsulating both the shape and scale parameters of the loglogistic distribution. Recall (from Section 3.2) that given a current estimate $\boldsymbol{\psi}^{(k)}$, the expectation of the log-likelihood function for a finite mixture is given as

$$Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) = \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log \pi_i + \sum_{j=1}^{n} \sum_{i=1}^{g} z_{ij}^{(k)} \log f_i(x_j \,|\, \boldsymbol{\theta_i}) + \lambda \left(\sum_{i=1}^{g} \pi_i - 1 \right) \ , \tag{9.4}$$

once again recalling the use of the Lagrange multiplier λ to deal with the constraint that all mixture proportions sum to one, and $z_{ij} = (\mathbf{z}_i)_j$ is a zero-one indicator variable specifying whether x_j arose or did not arise from the ith component of the mixture ($i = 1, ..., g$ and $j = 1, ..., n$). The EM equation for the mixture proportions is constructed by taking the partial derivative of Eq. (9.4) with respect to π_i, and equating it to zero, i.e.

$$\frac{\partial \, Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})}{\partial \, \pi_i} = 0 \ . \tag{9.5}$$

The value of $\boldsymbol{\pi}^{(k+1)}$ (on the current iteration) is given as that which maximises $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$. A derivation of the following iterative estimate for the mixture proportions was provided in Section 3.2.

$$\boxed{\pi_i^{(k+1)} = \sum_{j=1}^{n} \frac{z_{ij}^{(k)}}{n}} \ , \tag{9.6}$$

where

$$z_{ij}^{(k)} = \frac{\pi_i^{(k)} \, f_i(x_j \,|\, \boldsymbol{\theta_i})}{\sum_{m=1}^{g} \pi_m^{(k)} f_m(x_j \,|\, \boldsymbol{\theta_m})} \ , \qquad i = 1, \, ..., \, g \ . \tag{9.7}$$

The derivation of EM equations for loglogistic component parameter estimation of $\boldsymbol{\theta_i} = (\alpha_i, \, \beta_i)$ will now be specifically addressed. Recall that EM equations for iterative estimation of the parameter vector $\boldsymbol{\xi}^{(k+1)}$ can be constructed by taking partial derivatives of Eq. (9.4) with respect to $\boldsymbol{\theta_i}$. Selecting the parameter vector $\boldsymbol{\xi}^{(k+1)}$ which maximises $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$ on the current iteration is given as an appropriate solution of the expression,

$$\sum_{j=1}^{n} z_{ij}^{(k)} \frac{\partial \log f_i(x_j \,|\, \boldsymbol{\theta_i})}{\partial \, \boldsymbol{\theta_i}} = 0 \ , \qquad i = 1, \, ..., \, g \ . \tag{9.8}$$

[1]The following formulation is the result of the collaborative effort of the author, Dr Markus Kreer of Goethe University Frankfurt, and Dr Ayse Kizilersu of University of Adelaide.

Taking the natural logarithm of the component pdf gives

$$\Rightarrow \log f_i(x_j \,|\, \alpha_i,\, \beta_i) = \log \beta_i - \log \alpha_i + (\beta_i - 1)\log x_j - (\beta_i - 1)\log \alpha_i - 2\log\left[1 + \left(\frac{x_j}{\alpha_i}\right)^{\beta_i}\right]. \tag{9.9}$$

Take derivatives of Eq. (9.9) with respect to both the scale and shape parameters α_1 and β_1,

$$\Rightarrow \frac{\partial}{\partial \alpha_i} \log f_i(x_j \,|\, \alpha_i,\, \beta_i) = \left(\frac{\beta_i}{\alpha_i}\right)\left[\frac{2\left(\frac{x_j}{\alpha_i}\right)^{\beta_i}}{1 + \left(\frac{x_j}{\alpha_i}\right)^{\beta_i}} - 1\right], \tag{9.10}$$

$$\Rightarrow \frac{\partial}{\partial \beta_i} \log f_i(x_j \,|\, \alpha_i,\, \beta_i) = \frac{1}{\beta_i} + \log x_j - \log \alpha_i - \frac{2\left(\frac{x_j}{\alpha_i}\right)^{\beta_i} \log\left(\frac{x_j}{\alpha_i}\right)}{1 + \left(\frac{x_j}{\alpha_i}\right)^{\beta_i}}\,. \tag{9.11}$$

Substitution of the above equations into Eq. (9.8) yields the EM equations for α_1 and β_1:

$$\sum_{j=1}^{n} z_{ij}^{(k)} \left(\left(\frac{\beta_i}{\alpha_i}\right)\left[\frac{2\left(\frac{x_j}{\alpha_i}\right)^{\beta_i}}{1 + \left(\frac{x_j}{\alpha_i}\right)^{\beta_1}} - 1\right]\right) \quad = \quad 0\,, \tag{9.12}$$

$$\sum_{j=1}^{n} z_{ij}^{(k)} \left(\frac{1}{\beta_i} + \log x_j - \log \alpha_i - \frac{2\left(\frac{x_j}{\alpha_i}\right)^{\beta_i} \log\left(\frac{x_j}{\alpha_i}\right)}{1 + \left(\frac{x_j}{\alpha_i}\right)^{\beta_i}}\right) \quad = \quad 0\,. \tag{9.13}$$

These are intractable highly coupled equations. Consider the following reformulation which defines a re-parametrisation of the parameters which allows finding $\hat{\alpha}$ and $\hat{\beta}$ estimates for the loglogistic distribution with greater numerical ease. Let us define for simplicity $\lambda_i = \alpha_i^{\beta_i}$. Using this alternate parametrisation Eq. (9.9) becomes

$$\Rightarrow \log f_i(x_j \,|\, \lambda_i,\, \beta_i) = \log \beta_i + (\beta_i - 1)\,\log x_j - \log \lambda_i - 2\log\left[1 + \left(\frac{x_j^{\beta_i}}{\lambda_i}\right)\right]. \tag{9.14}$$

Re-write Eq. (9.12) and Eq. (9.13) as

$$\Rightarrow \sum_{j=1}^{n} z_{ij}^{(k)} \;=\; 2\sum_{j=1}^{n} z_{ij}^{(k)} \frac{{x_j}^{\beta_i}}{\alpha_i^{\beta_i} + {x_j}^{\beta_i}} \;=\; 2\sum_{j=1}^{n} z_{ij}^{(k)} \frac{{x_j}^{\beta_i}}{\lambda_i + {x_j}^{\beta_i}}\,, \tag{9.15}$$

$$\Rightarrow \sum_{j=1}^{n} z_{ij}^{(k)} \left(\frac{1}{\beta_i} + \log x_j - 2\,\frac{x_j^{\beta_i} \log x_j}{\lambda_i + x_j^{\beta_i}}\right) \quad = \quad 0\,. \tag{9.16}$$

It can be shown that Eq. (9.15) defines a unique function $\lambda_i\,(\,\cdot \,|\, (x_1, ..., x_n),\, (z_{i1}^{(k)}, ..., z_{in}^{(k)}))$ with argument $\beta_i > 0$, by first defining the right-hand side as a function of λ_i for fixed $\beta_i > 0$, fixed data sample $\boldsymbol{x} = (x_1, ..., x_n)$, and fixed indicator variables $\boldsymbol{z}_i = (z_{i1}^{(k)}, ..., z_{in}^{(k)})$.

$$h(\lambda_i) = 2\sum_{j=1}^{n} z_{ij}^{(k)} \frac{x_j^{\beta_i}}{\lambda_i + x_j^{\beta_i}}\,. \tag{9.17}$$

The function $h(\,\cdot\,)$ is continuous with respect to its argument for $\lambda_i > 0$, and monotonically decreasing,

$$h'(\lambda_i) = -2\sum_{j=1}^{n} z_{ij}^{(k)} \frac{x_j^{\beta_i}}{(\lambda_i + x_j^{\beta})^2} < 0\,, \tag{9.18}$$

with

$$\lim_{\lambda_i \to 0^+} h(\lambda_i) = 2 \sum_{j=1}^{n} z_{ij}^{(k)} , \tag{9.19}$$

$$\lim_{\lambda_i \to +\infty} h(\lambda_i) = 0 . \tag{9.20}$$

Hence by the intermediate value theorem Eq. (9.15) can be written as

$$\sum_{j=1}^{n} z_{ij}^{(k)} = h(\lambda_i) ,$$

and has exactly one solution $\lambda_i = \lambda_i(\beta_i)$. From here, the value of the indicator variables $\boldsymbol{z}_i = (z_{i1}^{(k)}, ..., z_{in}^{(k)})$ is specified on the current iteration, and one can determine a smooth function

$$\lambda_i = \lambda_i(\beta_i \,|\, (x_1, ..., x_n),\, (z_{i1}^{(k)}, ..., z_{in}^{(k)})) \tag{9.21}$$

by solving for all $\beta_i > 0$ (which may be further constrained to a given parameter space) using Eq (9.15). Consequently a function for the scale parameter can also be determined,

$$\alpha_i = \alpha_i(\beta_i) = \lambda_i(\beta_i \,|\, (x_1, ..., x_n),\, (z_{i1}^{(k)}, ..., z_{in}^{(k)}))^{1/\beta_i}. \tag{9.22}$$

From here the parameter combination $\boldsymbol{\theta}_i = (\alpha_i, \beta_i)$ is chosen such that Eq. (9.23), the log-likelihood contribution of the loglogistic component, is maximised. Noting that the terms without parameter dependence have been ignored.

$$\log L'(\pi_i, \beta_i \,|\, \boldsymbol{x}) = \sum_{j=1}^{n} z_{ij}^{(k)} \left\{ \log\left(\frac{\beta_i}{\alpha_i(\beta_i)}\right) + (\beta_i - 1)\log\left(\frac{x_j}{\alpha_i(\beta_i)}\right) - 2\log\left[1 + \left(\frac{x_j}{\alpha_i(\beta_i)}\right)^{\beta_i}\right]\right\} . \tag{9.23}$$

If the parameter space of the loglogistic shape parameter β_i can be restricted, and represented on a discrete grid, the corresponding λ_i estimates can be found using Eq. (9.15), which has been rearranged and revisited for clarity,

$$\Rightarrow \boxed{\sum_{j=1}^{n} z_{ij}^{(k)} \left(\frac{2\,{x_j}^{\beta_i}}{\lambda_i + {x_j}^{\beta_i}} - 1\right) = 0} . \tag{9.24}$$

The corresponding estimate of the scale parameter α_i is given as $\alpha_i = \lambda_i^{1/\beta_i}$. This procedure defines iterative estimation of $\alpha_i^{(k)}$ and $\beta_i^{(k)}$. After the convergence condition is satisfied, estimates $\hat{\alpha}_i$ and $\hat{\beta}_i$ result. The following section outlines a numerical procedure for computing estimates of the loglogistic parameters using the EM framework presented. Subsequently an alternate procedure is also presented which computes parameter estimates via direct search of the loglikelihood function.

9.1.1 Outline of Numerical Procedure: Iterative Discrete Search

Given a data vector $\boldsymbol{x}$, the following is an outline of the numerical procedure for iterative search of estimates for loglogistic parameters, for a given component of a mixed distribution.

- **Initialisation:** Choose loglogistic parameters $\pi_i^{(0)}$ and $\boldsymbol{\theta}_i^{(0)} = (\alpha_i^{(0)}, \beta_i^{(0)})$, noting that the full parameter vector $\boldsymbol{\psi}^{(0)}$ also requires initialisation (which could be made up of further loglogistic components, or other parametric distributions defined on $\mathbb{R}^+$)
- **E step$^{(1)}$:** Compute indicator variables $(z_{i1}^{(0)}, z_{i2}^{(0)}, ..., z_{in}^{(0)})$ using Eq. (9.7)
- **M step$^{(1)}$:**
  - Compute iterative estimate of mixing proportion $\pi_i^{(1)}$ using Eq. (9.6)
  - Consider a vector of possible shape parameters, i.e. $\beta_i = (0.01, 0.02, ..., 99.99)$, computing corresponding $\lambda_i = \lambda_i(\beta_i)$ using Eq. (9.24). This work used Newton-Raphson numerical root finding for this task.

- Compute corresponding $\alpha_i = \alpha_i(\beta_i)$ using Eq. (9.22)

- Calculate the corresponding log-likelihood contributions, $\log L'(\pi_1, \beta_1 \,|\, \boldsymbol{x})$, for each parameter combination $\boldsymbol{\theta}_i = (\alpha_i(\beta_i), \beta_i)$ using Eq. (9.23)

- Choose the parameter combination which maximises the log-likelihood contribution as iterative estimate $\boldsymbol{\theta}_i^{(1)} = (\alpha_i^{(1)},\, \beta_i^{(1)})$

- Compute the full log-likelihood, $\log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x}) = \sum_{j=1}^{n} \log\big(\sum_{i=1}^{g} \pi_i^{(1)} f_i(x_j \,|\, \boldsymbol{\theta}_i^{(1)}\big)$, which is dependent on all of the component distributions

- **E step$^{(2)}$:** Recompute indicator variables $(z_{i1}^{(1)}, z_{i2}^{(1)}, ..., z_{in}^{(1)})$, with update $\boldsymbol{\psi}^{(1)}$

- **M step$^{(2)}$:** Compute updated loglogistic parameters, i.e $\pi_i^{(2)}$ and $\boldsymbol{\theta}_i^{(2)} = (\alpha_i^{(2)},\, \beta_i^{(2)})$ using methodology specified above

- Compute updated log-likelihood, checking convergence condition
  $|\log L(\boldsymbol{\psi}^{(2)} \,|\, \boldsymbol{x}) - \log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x})\,| < \epsilon$, where ϵ defines a preset tolerance, i.e. $\epsilon = 10^{-5}$

- Continue iteration until convergence condition is satisfied

$$\vdots$$

- Maximum likelihood estimate is obtained for the loglogistic parameters $\hat{\pi}_i$ and $\hat{\boldsymbol{\theta}}_i = (\hat{\alpha}_i,\, \hat{\beta}_i)$

A downside of numerical procedures of this nature (procedures which utilise some form of discrete iterative search) is that they are numerically expensive. In order to maintain numerical rigour, it is advisable to utilise a grid of potential β values which defines a large enough region of the parameter space such that any dataset $\boldsymbol{x}$ which may arise from the relevant application can be described. It is also beneficial to ensure the discretisation is fine enough to capture the full nature of the log-likelihood function. Especially for mixture modelling, log-likelihood functions may contain multiple maxima requiring the discrete grid to be fine enough to capture the full information. It is advisable to dynamically adjust both the extent of the region and size of the grid spacing as iteration continues. This can benefit both the speed and accuracy of the numerical algorithm.

9.1.2 Outline of Numerical Procedure: Direct Search

Particularly with large datasets, this work still had difficulty optimising iterative discrete search algorithms such that runtime was appropriate. An alternate approach was to estimate parameters via direct search of the log-likelihood function. This was done using *fminsearch* [77], a derivative free numerical procedure used to find minima of unconstrained multivariable functions, provided by MATLAB. The main difficulty using numerical methods provided by packages like those available in MATLAB is that much of the "back-end" operation of the algorithm is either hidden from the user, or so convoluted it is too difficult to follow. As a result, careful testing of such procedures is required to ensure appropriate performance, with both consistent and reasonable estimates. The use of simulated datasets with known true parameters, or cross validation with the previous discrete iterative search procedure can be useful. The following is an outline of a direct search algorithm which can be utilised for a given data vector $\boldsymbol{x}$, to obtain loglogistic parameter estimates.

- **Initialisation:** Choose loglogistic parameters $\pi_i^{(0)}$ and $\boldsymbol{\theta}_i^{(0)} = (\alpha_i^{(0)},\, \beta_i^{(0)})$, noting that the full parameter vector $\boldsymbol{\psi}^{(0)}$ also requires initialisation (which could be made up of further loglogistic components, or other parametric distributions defined on $\mathbb{R}^+$)

- **E step$^{(1)}$:** Compute indicator variables $(z_{i1}^{(0)}, z_{i2}^{(0)}, ..., z_{in}^{(0)})$ using Eq. (9.7)

- **M step$^{(1)}$:**

  - Compute iterative estimate of mixing proportion $\pi_i^{(1)}$ using Eq. (9.6)

  - Choose loglogistic parameters $\boldsymbol{\theta}_i^{(1)} = (\alpha_i^{(1)},\, \beta_i^{(1)})$, which maximise the log-likelihood contribution, $\log L'(\pi_1, \beta_1 \,|\, \boldsymbol{x})$, defined by Eq. (9.23). This is done using the MATLAB *fminsearch* algorithm to numerically minimise the negative of the log-likelihood contribution

- Compute the full log-likelihood, $\log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x}) = \sum_{j=1}^{n} \log\big(\sum_{i=1}^{g} \pi_i^{(1)} f_i(x_j \,|\, \boldsymbol{\theta}_i^{(1)}\big)$, which is dependent on all of the component distributions
- **E step$^{(2)}$:** Recompute indicator variables $(z_{i1}^{(1)}, z_{i2}^{(1)}, ..., z_{in}^{(1)})$, with update $\boldsymbol{\psi}^{(1)}$
- **M step$^{(2)}$:** Compute updated loglogistic parameters, i.e $\pi_i^{(2)}$ and $\boldsymbol{\theta}_i^{(2)} = (\alpha_i^{(2)}, \beta_i^{(2)})$ using methodology specified above
- Compute updated log-likelihood, checking convergence condition
  $|\log L(\boldsymbol{\psi}^{(2)} \,|\, \boldsymbol{x}) - \log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x})\,| < \epsilon$, where ϵ defines a preset tolerance, i.e. $\epsilon = 10^{-5}$
- Continue iteration until convergence condition is satisfied

$$\vdots$$

- Maximum likelihood estimate is obtained for the loglogistic parameters $\hat{\pi}_i$ and $\hat{\boldsymbol{\theta}}_i = (\hat{\alpha}_i, \hat{\beta}_i)$

Chapter 10

Uniform Component Distribution

The continuous uniform distribution is a family of symmetric probability distributions which display constant probability density. That is, any interval of equal length on the support of the distribution is equally probable. Distributions of this form are central to many sampling algorithms, like the inverse transform method which has been presented in this thesis. The uniform distribution can be used to describe a scenario where all outcomes, ranging from the minimum to maximum value, are equally likely. Trivial examples include; at a random instance of time the number of seconds elapsed since the last minute, or a discrete example, the number shown on a tossed die. The continuous uniform distribution is denoted $U(a, b)$, hence the standard uniform is defined as $U(0, 1)$.

The probability density function (pdf) of a random variable X, distributed according to a generally parametrised continuous uniform distribution and making up a component of a mixture model is

$$f_i(x \,|\, a_i,\, b_i) = \begin{cases} \dfrac{1}{b_i - a_i} \ , & \text{if } x \in [a_i, b_i] \ , \\ \\ 0 \ , & \text{otherwise} \ , \end{cases} \tag{10.1}$$

where a_i and b_i represent the corresponding minimum and maximum values of the region of support $x \in [a, b]$, defined such that $-\infty < a_i < b_i < \infty$. The corresponding cumulative distribution function (cdf) of the uniform component is

$$F_i(x \,|\, a_i, b_I) = \begin{cases} 0 \ , & x < a_i \ , \\ \\ \dfrac{x - a_i}{b_i - a_i} \ , & \text{if } x \in [a_i, b_i) \ , \\ \\ 1 \ , & x \geq b_i \ . \end{cases} \tag{10.2}$$

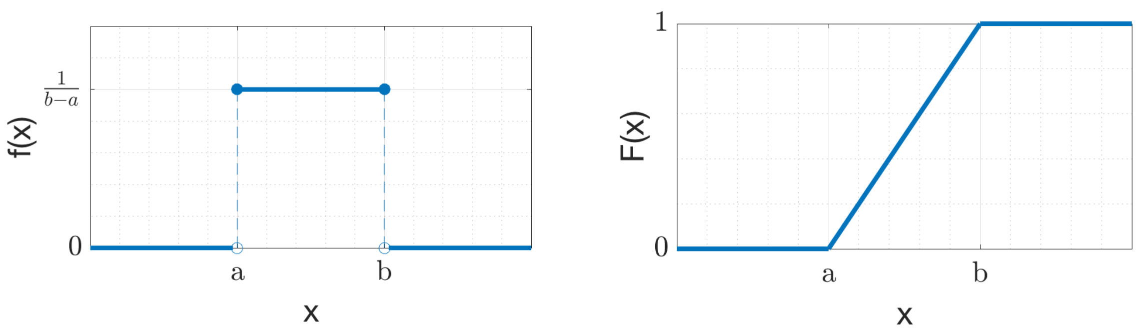


Figure 10.1: pdf and cdf of continuous uniform distribution where a and b are the corresponding minimum and maximum values of the positive probability region.

Eq. (10.1) and Eq. (10.2) can be rewritten in terms of the heaviside step function $H(x)$ as

$$f_i(x \,|\, a_i,\, b_i) = \frac{H(x - a_i) - H(x - b_i)}{b_i - a_i} \;, \tag{10.3}$$

and

$$F_i(x \,|\, a_i,\, b_i) = \frac{(x - a_i)H(x - a_i) - (x - b_i)H(x - b_i)}{b_i - a_i} \;. \tag{10.4}$$

For the purpose of this thesis, consider a one parameter reparametrisation of the continuous uniform distribution, defined such that $a_i = 0$ and $b_i = \xi_i$. Hence,

$$f_i(x \,|\, \xi_i) = \frac{H(x) - H(x - \xi_i)}{\xi_i} = \frac{H(\xi_i - x)}{\xi_i} \;, \tag{10.5}$$

and

$$F_i(x \,|\, \xi_i) = \frac{x\, H(x) - (x - \xi_i)H(x - \xi_i)}{\xi_i} \;. \tag{10.6}$$

In order to consider mixed distributions described by parametric components defined on the domain $\mathbb{R}^+ \cup 0$ (i.e. $x \in [0, +\infty)$) it is required to restrict the uniform distribution to this same region by ignoring the zero density region $x < 0$.

10.1 Adapting the MLE Procedure

The following procedure can be used to find maximum likelihood estimates of the parameter vector $\boldsymbol{\psi} = (\pi_1, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$. The parameter vector describes a mixture of a uniform distribution with an additional unspecified smooth distribution. The formulation which is presented is an adaption of a procedure developed by Kreer [78]. The probability density function (pdf) of a random variable X for such two-component mixtures is given as

$$f(x \,|\, \boldsymbol{\psi}) = \pi_1\, f_1(x \,|\, \boldsymbol{\theta}_1) + \pi_2\, f_2(x \,|\, \boldsymbol{\theta}_2) \;, \tag{10.7}$$

where the first component $f_1(x \,|\, \boldsymbol{\theta}_1)$ describes the uniform distribution, and the second component $f_2(x \,|\, \boldsymbol{\theta}_2)$ describes the additional unspecified smooth distribution. The uniform component is singly parametrised, i.e. $\boldsymbol{\theta_1} = \xi_1$, and $\boldsymbol{\theta_2}$ defines the parametrisation for the unspecified smooth distribution. Estimation of parameter vector $\boldsymbol{\psi} = (\pi_1, \xi_1, \boldsymbol{\theta}_2)$ is required from the data sample $\boldsymbol{x} = (x_1, ..., x_n)$.

Parameters known *a priori*, mixture proportion requiring estimation:

Consider the case where both ξ_1 and $\boldsymbol{\theta}_2$ the parameters of the uniform and unspecified smooth distribution are known *a priori*. Estimation in this case is only required for one unknown mixture proportion π_1. The cumulative distribution function of the mixture takes the familiar form,

$$F(x \,|\, \pi_1,\, \xi_1,\, \boldsymbol{\theta}_2) = \pi_1\, F_1(x \,|\, \xi_1) \;+\; (1 - \pi_1)\; F_2(x \,|\, \boldsymbol{\theta}_2) \;. \tag{10.8}$$

The probability of the random variable taking a value less than or equal to the uniform parameter ξ_1, i.e. $P(X \leq \xi_1)$, is given as

$$\begin{aligned} F(\xi_1 \,|\, \pi_1,\, \xi_1,\, \boldsymbol{\theta}_2) \;&=\; \pi_1\, F_1(\xi_1 \,|\, \xi_1) \;+\; (1 - \pi_1)\; F_2(\xi_1 \,|\, \boldsymbol{\theta}_2) &(10.9)\\ &=\; \pi_1 \;+\; (1 - \pi_1)\; F_2(\xi_1 \,|\, \boldsymbol{\theta}_2) &(10.10)\end{aligned}$$

using the fact that the entirety of the uniform density is located within $(0,\, \xi_1)$.

Hence,

$$\begin{aligned} 1 - F(\xi_1 \,|\, \pi_1,\, \xi_1,\, \boldsymbol{\theta}_2) \;&=\; (1 - \pi_1)\Big(1 - F_2(\xi_1 \,|\, \boldsymbol{\theta}_2)\Big) \\ &=\; (1 - \pi_1) \int_{\xi_1}^{\infty} f_2(s \,|\, \boldsymbol{\theta}_2)\, ds. &(10.11)\end{aligned}$$

Using the Glivenko-Cantelli Theorem [46], Eq. (10.11) can be written as

$$\frac{\#\{x_j > \xi_1\}}{n} \approx (1-\pi_1) \int_{\xi_1}^{\infty} f_2(s \,|\, \boldsymbol{\theta}_2)\, ds \; . \tag{10.12}$$

Rearrangement yields $\hat{\pi}_1$, an estimate of the unknown mixing proportion.

$$\boxed{\hat{\pi}_1 = 1 - \frac{1}{\int_{\xi_1}^{\infty} f_2(s \,|\, \boldsymbol{\theta}_2)\, ds} \frac{\#\{x_j > \xi_1\}}{n}} \; . \tag{10.13}$$

Uniform parameter known *a priori*, additional parameter vector and mixture proportion require estimation:

Now consider the case where ξ_1 the parameter of the uniform distribution is known *a priori*, but both $\boldsymbol{\theta}_2$ the parameter vector of the unspecified smooth distribution, and π_1 require estimation.

The simplest approach to estimation of the parameter vector $\hat{\boldsymbol{\theta}}_2$ for the unspecified smooth distribution is to consider a left-truncated maximum likelihood approach with left-truncation point $\tau_L = \xi_1$. From here, the unknown mixture proportion π_1 can be estimated with Eq. (10.13), making use of the estimate $\hat{\boldsymbol{\theta}}_2$ and ξ_1 which is known *a priori*. An example is provided in Section 10.1.1 where the smooth distribution is specified as a Weibull component.

Estimation of full parameter vector:

Finally consider estimation of the full parameter vector $\boldsymbol{\psi} = (\pi_1, \xi_1, \boldsymbol{\theta}_2)$. The general log-likelihood function of a finite mixture is given as

$$\log L(\boldsymbol{\psi} \,|\, \boldsymbol{x}) = \sum_{j=1}^{n} \log f(x_j \,|\, \boldsymbol{\psi}) \; , \tag{10.14}$$

where for this case,

$$\begin{aligned} f(x_j \,|\, \boldsymbol{\psi}) \quad = \quad & \left(1 - \frac{1}{\int_{\xi_1}^{\infty} f_2(s \,|\, \hat{\boldsymbol{\theta}}_2)\, ds} \frac{\#\{x_j > \xi_1\}}{n}\right) \frac{H(x) - H(x-\xi_1)}{\xi_1} \\ & + \left(\frac{1}{\int_{\xi_1}^{\infty} f_2(s \,|\, \hat{\boldsymbol{\theta}}_2)\, ds} \frac{\#\{x_j > \xi_1\}}{n}\right) f_2(x \,|\, \hat{\boldsymbol{\theta}}_2) \; , \end{aligned} \tag{10.15}$$

using the fact that for $x > 0$

$$\frac{H(x) - H(x-\xi_1)}{\xi_1} = \frac{H(\xi_1 - x)}{\xi_1} . \tag{10.16}$$

A maximum likelihood estimate $\hat{\xi}_1$ can be obtained for the uniform distribution, restricted to the interval $\hat{\xi}_1 \in (0,\, x_{\max})$ where $x_{\max}$ is the largest observation in the sample, by finding a solution of the following equation,

$$\frac{\partial}{\partial \xi_1} \log L(\boldsymbol{\psi} \,|\, \boldsymbol{x}) = \sum_{j=1}^{n} \frac{\partial}{\partial \xi_1} \log f(x_j \,|\, \boldsymbol{\psi}) = 0 \; . \tag{10.17}$$

Noting that the likelihood function will have 'jumps' due to the occurrence of the heaviside function $H(x)$. Once an estimate $\hat{\xi}_1$ has been obtained, one can fix a new left-truncation $\tau_L = \hat{\xi}_1$, and a new left-truncated MLE procedure can be started to obtain a new estimate for the parameter vector $\boldsymbol{\theta}_2$. Thus an iterative procedure can be defined to estimate $\xi_1, \boldsymbol{\theta}_2$, and π_1, to the precision allowed by the sample size n.

10.1.1 Example: Uniform/Weibull Mixture

Consider the following example where the unspecified smooth distribution is described by a Weibull distribution, with corresponding parameter vector given as $\boldsymbol{\theta}_2 = (\alpha_2,\, \beta_2)$. The respective probability density and cumulative distribution functions are given as

$$f_2(x_j \,|\, \alpha_2,\, \beta_2) = \frac{\beta_2}{\alpha_2}\left(\frac{x_j}{\alpha_2}\right)^{\beta_2 - 1} e^{-(x/\alpha_2)^{\beta_2}} \;, \tag{10.18}$$

$$F_2(x_j \,|\, \alpha_2,\, \beta_2) = 1 - e^{-(x/\alpha_2)^{\beta_2}}, \qquad x \geq 0 \;. \tag{10.19}$$

The form of the corresponding probability density for a mixture of uniform and Weibull component distributions is shown in Fig. 10.2. The noticeable 'jump' in the density function occurs at $x = \xi_1$, with the height of the jump being dependent on the mixture proportion, given as π_1/ξ_1.

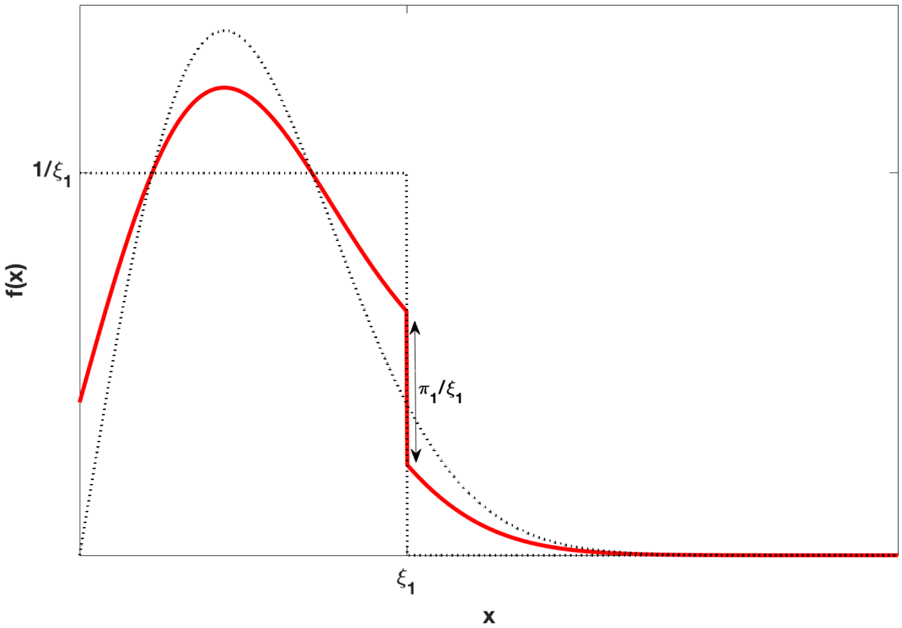


Figure 10.2: PDF of uniform/weibull mixture.

The specific details of maximum likelihood estimation of parameters for a mixture of uniform and Weibull densities will now be provided, consistent with the general methodology presented in the previous section. Estimation will be considered for $\boldsymbol{\psi} = (\pi_1,\, \xi_1,\, (\alpha_2,\, \beta_2)\,)$ from a data sample $\boldsymbol{x} = (x_1, ..., x_n)$.

<u>Parameters known *a priori*, mixture proportion requiring estimation:</u>

Estimation of the mixture proportion $\hat{\pi}_1$ is given by Eq. (10.13), which for this case, now that the Weibull distribution is specified, is given as

$$\hat{\pi}_1 = 1 - \frac{1}{\int_{\xi_1}^{\infty} f_2(s \,|\, \boldsymbol{\theta}_2)\, ds} \frac{\#\{x_j > \xi_1\}}{n} \tag{10.20}$$

where

$$\begin{aligned} \Rightarrow \int_{\xi_1}^{\infty} f_2(s \,|\, \boldsymbol{\theta}_2)\, ds &= \int_0^{\infty} f_2(s \,|\, \alpha_2,\, \beta_2)\, ds - \int_0^{\xi_1} f_2(s \,|\, \alpha_2,\, \beta_2)\, ds \\ &= 1 - F_2(\xi_1 \,|\, \alpha_2,\, \beta_2) \\ &= e^{-(\xi_1/\alpha_2)^{\beta_2}}. \end{aligned} \tag{10.21}$$

Hence,

$$\hat{\pi}_1 = 1 - e^{(\xi_1/\alpha_2)^{\beta_2}} \, \frac{\#\{x_j > \xi_1\}}{n} \;. \tag{10.22}$$

Uniform parameter known *a priori*, additional parameter vector and mixture proportion require estimation:

The maximum likelihood solution for a left-truncated Weibull distribution, where $\tau_L = \xi_1$ (known *a priori*) is given by Eq. (10.23) and Eq. (10.24) [5,6,20]. A simultaneous solution to these equations yields estimates $\hat{\boldsymbol{\theta}}_2 = (\hat{\alpha}_2,\, \hat{\beta}_2)$. A solution for $\hat{\beta}_2$ can first be found by numerically solving Eq. (10.24), before substitution into Eq. (10.23) yields $\hat{\alpha}_2$.

$$\hat{\alpha}_2 = \left(\frac{1}{n} \sum_{j=1}^{n} \left(x_j^{\hat{\beta}_2} - \xi_1^{\hat{\beta}_2} \right) \right)^{1/\hat{\beta}_2} , \tag{10.23}$$

$$\frac{1}{\hat{\beta}_2} - \frac{\sum_{j=1}^{n} \left(\frac{x_j}{\xi_1}\right)^{\hat{\beta}_2} \log\left(\frac{x_j}{\xi_1}\right)}{\sum_{j=1}^{n} \left[\left(\frac{x_j}{\xi_1}\right)^{\hat{\beta}_2} - 1 \right]} + \frac{1}{n} \sum_{j=1}^{n} \log\left(\frac{x_j}{\xi_1}\right) = 0 \,. \tag{10.24}$$

The mixture proportion estimate $\hat{\pi}_1$ in this case is similarly given as

$$\hat{\pi}_1 = 1 - e^{(\xi_1/\hat{\alpha}_2)^{\hat{\beta}_2}} \frac{\#\{x_j > \xi_1\}}{n} . \tag{10.25}$$

noting the dependence on the estimates $\hat{\alpha}_2$ and $\hat{\beta}_2$.

Estimation of full parameter vector:

This case requires also finding an estimate $\hat{\xi}_1$. This is done by finding a solution of Eq (10.17) where for this example $f(x_j \,|\, \boldsymbol{\psi})$ is given by

$$\begin{aligned} f(x_j \,|\, \boldsymbol{\psi}) \;=\; & \left(1 - e^{(\xi_1/\hat{\alpha}_2)^{\hat{\beta}_2}} \frac{\#\{x_j > \xi_1\}}{n} \right) \frac{H(\xi_1 - x)}{\xi_1} \\ & + \left(e^{(\xi_1/\hat{\alpha}_2)^{\hat{\beta}_2}} \frac{\#\{x_j > \xi_1\}}{n} \right) \frac{\hat{\beta}_2}{\hat{\alpha}_2} \left(\frac{x}{\hat{\alpha}_2} \right)^{\hat{\beta}_2 - 1} e^{-(x/\hat{\alpha}_2)^{\hat{\beta}_2}} . \end{aligned} \tag{10.26}$$

The partial derivative of the natural logarithm of the pdf with respect to ξ_1 is required,

$$\sum_{j=1}^{n} \frac{\partial}{\partial \xi_1} \log f(x_j \,|\, \boldsymbol{\psi}) \;=\; \sum_{j=1}^{n} \frac{\frac{\partial}{\partial \xi_1} f(x_j \,|\, \boldsymbol{\psi})}{f(x_j \,|\, \boldsymbol{\psi})} = 0 \,, \tag{10.27}$$

where the derivative term is given as

$$\begin{aligned} \frac{\partial}{\partial \xi_1} f(x_j \,|\, \boldsymbol{\psi}) \;=\; & \frac{\delta(\xi_1 - x)}{\xi_1} - \frac{H(\xi_1 - x)}{\xi_1^2} - e^{(\xi_1/\hat{\alpha}_2)^{\hat{\beta}_2}} \frac{\#\{x_j > \xi_1\}}{n} \frac{\delta(\xi_1 - x)}{\xi_1} \\ & + e^{(\xi_1/\hat{\alpha}_2)^{\hat{\beta}_2}} \frac{\#\{x_j > \xi_1\}}{n} \frac{H(\xi_1 - x)}{\xi_1^2} \\ & - e^{(\xi_1/\hat{\alpha}_2)^{\hat{\beta}_2}} \frac{\#\{x_j > \xi_1\}}{n} \frac{H(\xi_1 - x)}{\xi_1} \frac{\hat{\beta}_2}{\hat{\alpha}_2} \left(\frac{\xi_1}{\hat{\alpha}_2} \right)^{\hat{\beta}_2 - 1} \\ & + e^{(\xi_1/\hat{\alpha}_2)^{\hat{\beta}_2}} \frac{\#\{x_j > \xi_1\}}{n} \left(\frac{\hat{\beta}_2}{\hat{\alpha}_2} \right)^2 \left(\frac{x}{\hat{\alpha}_2} \right)^{\hat{\beta}_2 - 1} e^{-(x/\hat{\alpha}_2)^{\hat{\beta}_2}} \left(\frac{\xi_1}{\hat{\alpha}_2} \right)^{\hat{\beta}_2 - 1} , \end{aligned} \tag{10.28}$$

which simplifies to

$$\Rightarrow \quad \frac{\partial}{\partial \xi_1} f(x_j \,|\, \boldsymbol{\psi}) \;=\; \hat{\pi}_1 \left[\frac{\delta(\xi_1 - x)}{\xi_1} - \frac{H(\xi_1 - x)}{\xi_1^2} \right] + \hat{\pi}_2 \left[- \frac{H(\xi_1 - x)}{\xi_1} \frac{\hat{\beta}_2}{\hat{\alpha}_2} \left(\frac{\xi_1}{\hat{\alpha}_2} \right)^{\hat{\beta}_2 - 1} + \left(\frac{\hat{\beta}_2}{\hat{\alpha}_2} \right)^2 \left(\frac{x}{\hat{\alpha}_2} \right)^{\hat{\beta}_2 - 1} e^{-(x/\hat{\alpha}_2)^{\hat{\beta}_2}} \left(\frac{\xi_1}{\hat{\alpha}_2} \right)^{\hat{\beta}_2 - 1} \right] . \tag{10.29}$$

An estimate $\hat{\xi}_1$ can be obtained numerically, from there one can fix a new left-truncation $\tau_L = \hat{\xi}_1$ and a new left-truncated MLE procedure can be started to obtain a new estimate for the parameter vector $\boldsymbol{\theta}_2$. Thus an iterative procedure is defined to estimate $\xi_1, \boldsymbol{\theta}_2$, and π_1.

Chapter 11

Censored Expectation Maximisation Algorithm

An observation is classified as *censored* if only partial information about its value is known. The concept of censoring is best explained by example, consider the following:

- An experiment exists in which a scientist is measuring temperature with an apparatus which is calibrated to only make recordings above a certain lower bound T_l. If the temperature value is below the specified detection limit, the observation is considered censored. The exact value of the temperature is unknown, but it is known to be lower than T_l. In this case the observation is considered *left-censored*, with T_l referred to as the left censor limit.

- Censored data is standard within survival analysis. Consider a study where subjects are followed over a specific period of time, and the time until a particular event occurs observed. Typically within survival analysis this event is death. *Right-censoring* occurs when a subject does not experience the particular event in the duration of the study, or leaves the study early. In this case the exact time at which the event occurs is unknown, but the value is known to exceed a particular censor limit (either the time at the end of the study, or the time the subject leaves the study).

Interval-censoring also exists in which the value of an observation is unknown, but known to exist within a certain interval. The concept of censoring is closely related, but shouldn't be confused with that of *truncation*. The exact values of all observations are known in a truncated dataset, with observations beyond particular limits either not measured or excluded entirely from the analysis. Acknowledgment of the presence of data beyond these limits, even though exact values are not known, is what differentiates censoring from truncation.

Very few studies exist dealing with censored data within an EM framework. Chauveau [17] presented EM equations to deal with a two-component mixture where the unknown parameter is the mixture proportion, and Verbelen et al. [18] derived censored EM equations for a mixture of Erlang distributions. This chapter will present a complete EM framework for dealing with censored data and derive censored EM equations for the exponential and Weibull component distributions. This is the first study to formulate censored EM equations for the Weibull distribution.

The reader is encouraged to consider the following formulation with reference to the EM framework presented in Section 3.2 (for the uncensored case). The focus of this chapter is to present the details pertinent to censoring, with much of the general formulation not repeated.

11.1 Censored EM Framework

Consider a random variable X taking values in $\mathbb{R}^+$. The realisation vector of the random variable is data sample $\boldsymbol{x} = (x_1, ..., x_n)$, which has a corresponding sample size n. However, the data actually observed is censored, taking exact values in the region $\mathcal{I}_0 \subset \mathbb{R}^+$, whereas censored observations take values in an interval $\mathcal{I}_\ell \subset \mathbb{R}^+$, for $\ell = 1, 2, ..., L$. These censored observations can be represented by discrete values $c_\ell \in \mathcal{I}_\ell$. Let the censored representation of random variable X be given as Y. This random variable represents a corresponding data sample $\boldsymbol{y} = (y_1, ..., y_n)$, where observations can take values in $\mathcal{I}_0 \cup \{c_1, ..., c_L\}$. The complete data is represented by (X, Z) where random variable Z describes the missing data related to the mixture component each datum originates. The formal representation of random variable Y as in [17] is

$$Y = X\mathbb{I}_{\mathcal{I}_0}(X) + \sum_{\ell=1}^{L} c_\ell \mathbb{I}_{\mathcal{I}_\ell}(X) \ . \tag{11.1}$$

The E-step requires finding the expectation of the log-likelihood function, conditioned on the current parameter estimates and the observed data. For data in the uncensored region $\mathcal{I}_0$ this requires finding the values of the familiar indicator variables

$$z_{ij}^{(k)} = \mathbb{E}_{\boldsymbol{\psi}^{(k)}}(Z_{ij} \,|\, \boldsymbol{x} \in \mathcal{I}_0) \tag{11.2}$$

$$= \frac{\pi_i^{(k)} f_i(x_j \,|\, \boldsymbol{\theta}_i^{(k)})}{\sum_{m=1}^{g} \pi_m^{(k)} f_m(x_j \,|\, \boldsymbol{\theta}_m^{(k)})} \qquad i = 1, ..., g. \tag{11.3}$$

For censored observations the form of the indicator variables is slightly modified. Firstly, let N_ℓ denote the number of observations within the censored interval $\mathcal{I}_\ell$, i.e. for which observations take the value c_ℓ. The corresponding indicator variables are

$$\tilde{z}_{ij\ell}^{(k)} = \mathbb{E}_{\boldsymbol{\psi}^{(k)}}(Z_{ij} \,|\, \boldsymbol{x} \in \mathcal{I}_\ell) \tag{11.4}$$

$$= \frac{\pi_i^{(k)} \int_{\mathcal{I}_\ell} dy \;\; f_i(\, y \,|\, \boldsymbol{\theta}_i^{(k)}\,)}{\sum_{m=1}^{g} \pi_m^{(k)} \int_{\mathcal{I}_\ell} dy \;\; f_m(\, y \,|\, \boldsymbol{\theta}_m^{(k)}\,)} \ , \tag{11.5}$$

noting the use of the tilde representation for censoring. The total sample size n is given as the sum of censored and uncensored (N) sample sizes, defined as

$$n = N + \sum_{\ell=1}^{L} N_\ell \ . \tag{11.6}$$

The M-step for iterative estimation of mixture proportions is given by the following expression

$$\begin{aligned} \pi_i^{(k+1)} = \; & \frac{1}{n} \sum_{j=1}^{N} \frac{\pi_i^{(k)} f_i(\, x_j \,|\, \boldsymbol{\theta}_i^{(k)}\,)}{\sum_{m=1}^{g} \pi_m^{(k)} f_m(\, x_j \,|\, \boldsymbol{\theta}_m^{(k)}\,)} \\ & + \frac{1}{n} \sum_{\ell=1}^{L} N_\ell \frac{\pi_i^{(k)} \int_{\mathcal{I}_\ell} dy \;\; f_i(\, y \,|\, \boldsymbol{\theta}_i^{(k)}\,)}{\sum_{m=1}^{g} \pi_m^{(k)} \int_{\mathcal{I}_\ell} dy \;\; f_m(\, y \,|\, \boldsymbol{\theta}_m^{(k)}\,)} \ , \end{aligned} \tag{11.7}$$

which can be written in terms of the indicator variables as

$$\pi_i^{(k+1)} = \frac{1}{n} \left(\sum_{j=1}^{N} z_{ij}^{(k)} + \sum_{\ell=1}^{L} N_\ell \tilde{z}_{ij\ell}^{(k)} \right) \ , \tag{11.8}$$

where the first term recovers the familiar form for the uncensored case. The iterative estimate of the parameter vector $\boldsymbol{\xi}^{(k+1)}$ is chosen such that

$$\boldsymbol{\xi}^{(k+1)} = \underset{\boldsymbol{\xi}}{\operatorname{argmax}} \; Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) \ , \tag{11.9}$$

where $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$ is analogous to the expression presented in Eq. (3.31) for the uncensored case. For simplicity, only terms of the $Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)})$ function with component parameter dependence are included in the expression given below,

$$\begin{aligned} Q(\boldsymbol{\psi}\,|\,\boldsymbol{\psi}^{(k)}) &= \sum_{i=1}^{g}\sum_{j=1}^{N} z_{ij}^{(k)} \log f_i(\,x_j\,|\,\boldsymbol{\theta}_i^{(k)}\,) \\ &+ \sum_{i=1}^{g}\sum_{\ell=1}^{L} N_\ell\, \tilde{z}_{i\ell}^{(k)} \int_{I_\ell} dy\;\log f_i(\,y\,|\,\boldsymbol{\theta}_i^{(k)}\,)\; h_i(\,y\,|\,c_\ell,\,\boldsymbol{\theta}_i^{(k-1)})\;, \end{aligned} \tag{11.10}$$

where ℓ is the index variable for the censor regions, i the component distributions, and j the exact observations. The function $h_i(\,y\,|\,c_\ell,\,\boldsymbol{\theta}_i^{(k-1)})$ is defined as

$$h_i(\,y\,|\,c_\ell,\,\boldsymbol{\theta}_i^{(k-1)}) = \frac{f_i(\,y\,|\,\boldsymbol{\theta}_i^{(k-1)})}{\int_{I_\ell} dy\; f_i(\,y\,|\,\boldsymbol{\theta}_i^{(k-1)})}\;, \tag{11.11}$$

such that

$$\mathbb{E}\,(\,\log f_i(\,y\,|\,\boldsymbol{\theta}_i^{(k)})\,|\,\boldsymbol{x}\in\mathcal{I}_l,\,\boldsymbol{\theta}_i^{(k)}) = \int_{I_\ell} dy\;\log f_i(\,y\,|\,\boldsymbol{\theta}_i^{(k)}\,)\; h_i(\,y\,|\,c_\ell,\,\boldsymbol{\theta}_i^{(k-1)})\;. \tag{11.12}$$

The first term in Eq. (11.10) corresponds to standard maximisation of the uncensored case, with the censored contribution represented by the second term. Censored EM equations will now be derived for exponential and Weibull component distributions.

11.2 Exponential Component Censored EM Equations

The general formulation of an EM framework was presented for maximum likelihood estimation of mixed distribution parameters when data is considered censored. This section applies the framework to the exponential component distribution. Consider the exponential probability density function,

$$f_i(\,x_j\,|\,\lambda_i^{(k)}) = \lambda_i^{(k)}\, e^{-\,\lambda_i^{(k)}\,x_j}\;, \tag{11.13}$$

and censor regions denoted $\mathcal{I}_l = (\xi_{l-1},\,\xi_l)$ for $l = 1, ..., L$. The conditional density of the exponential distribution on the censor region $\mathcal{I}_l$ is given as

$$\begin{aligned} h_i(\,y\,|\,\boldsymbol{\theta}_i^{(k)}) &= \frac{f_i(\,y\,|\,\boldsymbol{\theta}_i^{(k)})}{\int_{I_\ell} dy\; f_i(\,y\,|\,\boldsymbol{\theta}_i^{(k)})} \\ &= \frac{\lambda_i^{(k)}\, e^{-\,\lambda_i^{(k)} y}}{\int_{\xi_{l-1}}^{\xi^l} dy\;\lambda_i^{(k)}\, e^{-\,\lambda_i^{(k)} y}} \\ &= \frac{\lambda_i^{(k)}\, e^{-\,\lambda_i^{(k)} y}}{e^{-\,\lambda_i^{(k)}\,\xi_{l-1}} - e^{-\,\lambda_i^{(k)}\,\xi_l}}\;. \end{aligned} \tag{11.14}$$

The following integral is required in calculations to follow,

$$\begin{aligned} \int_{\mathcal{I}_l} dy\; h_i(\,y\,|\,\boldsymbol{\theta}_i^{(k)}) &= \int_{\xi_{l-1}}^{\xi^l} dy\; h_i(\,y\,|\,\lambda_i^{(k)}) \\ &= \int_{\xi_{l-1}}^{\xi^l} dy\;\frac{\lambda_i^{(k)}\, e^{-\,\lambda_i^{(k)} y}}{e^{-\,\lambda_i^{(k)}\,\xi_{l-1}} - e^{-\,\lambda_i^{(k)}\,\xi_l}} \\ &= 1\;. \end{aligned} \tag{11.15}$$

We are also required to compute the following expectation given by Eq. (11.12),

$$\mathbb{E}\,(\log f_i(\,y\,|\,\lambda_i^{(k)})\,|\,\boldsymbol{x} \in \mathcal{I}_l,\,\lambda_i^{(k)}) \;=\; \int_{I_\ell} dy\;\log\,f_i(\,y\,|\,\lambda_i^{(k)}\,)\;h_i(\,y\,|\,c_\ell,\,\lambda_i^{(k)})$$

$$= \int_{\xi_{l-1}}^{\xi^l} dy\;\log\,(\lambda_i^{(k)}\,e^{-\,\lambda_i^{(k)}\,y}\,)\;h_i(\,y\,|\,c_\ell,\,\lambda_i^{(k)})$$

$$= \log\lambda_i^{(k)}\int_{\xi_{l-1}}^{\xi^l} dy\;h_i(\,y\,|\,c_\ell,\,\lambda_i^{(k)}) - \lambda_i^{(k)}\int_{\xi_{l-1}}^{\xi^l} dy\;y\;h_i(\,y\,|\,c_\ell,\,\lambda_i^{(k)}) \tag{11.16}$$

and using Eq. (11.15) we obtain

$$\mathbb{E}\,(\log f_i(\,y\,|\,\lambda_i^{(k)})\,|\,\boldsymbol{x} \in \mathcal{I}_l,\,\lambda_i^{(k)}) \;=\; \log\lambda_i^{(k)} - \lambda_i^{(k)}c_{il}^{(k-1)}, \tag{11.17}$$

where

$$c_{il}^{(k-1)} = \int_{\xi_{l-1}}^{\xi^l} dy\;y\;h_i(\,y\,|\,c_\ell,\,\lambda_i^{(k)}). \tag{11.18}$$

Substituting Eq. (11.14) into Eq. (11.18) yields

$$c_{il}^{(k-1)} \;=\; \int_{\xi_{l-1}}^{\xi^l} dy\;y\;\frac{\lambda_i^{(k-1)}\,e^{-\,\lambda_i^{(k-1)}y}}{e^{-\,\lambda_i^{(k-1)}\,\xi_{l-1}} - e^{-\,\lambda_i^{(k-1)}\,\xi_l}} \tag{11.19}$$

$$\;=\; \frac{1}{\lambda_i^{(k-1)}} + \frac{\xi_{l-1}\,e^{-\,\lambda_i^{(k-1)}\xi_{l-1}} - \xi_l\,e^{-\,\lambda_i^{(k-1)}\xi_l}}{e^{-\,\lambda_i^{(k-1)}\xi_{l-1}} - e^{-\,\lambda_i^{(k-1)}\xi_l}}\;. \tag{11.20}$$

Hence, iterative selection of $\lambda_i^{(k)}$ is such that the following expression is maximised

$$Q(\boldsymbol{\psi}\,|\,\boldsymbol{\psi}^{(k)}) \;=\; \sum_{i=1}^{g}\sum_{j=1}^{N} z_{ij}^{(k)}\;\log\,f_i(\,x_j\,|\,\lambda_i^{(k)}\,)$$

$$+\sum_{i=1}^{g}\sum_{\ell=1}^{L} N_\ell\;\tilde{z}_{ij\ell}^{(k)}\int_{I_\ell} dy\;\log\,f_i(\,y\,|\,\lambda_i^{(k)}\,)\;h_i(\,y\,|\,c_\ell,\,\lambda_i^{(k-1)}) \tag{11.21}$$

$$= \sum_{i=1}^{g}\sum_{j=1}^{N} z_{ij}^{(k)}\left(\log\,\lambda_i^{(k)} - \lambda_i^{(k)}\,x_j\right)$$

$$+\sum_{i=1}^{g}\sum_{\ell=1}^{L} N_\ell\;\tilde{z}_{ij\ell}^{(k)}\left(\log\lambda_i^{(k)} - \lambda_i^{(k)}c_{il}^{(k-1)}\right)\;. \tag{11.22}$$

Differentiating Eq. (11.22) with respect to $\lambda_i^{(k)}$ and setting the resultant expression to zero gives

$$\sum_{i=1}^{g}\sum_{j=1}^{N} z_{ij}^{(k)}\left(\frac{1}{\lambda_i^{(k)}} - x_j\right) + \sum_{i=1}^{g}\sum_{\ell=1}^{L} N_\ell\;\tilde{z}_{ij\ell}^{(k)}\left(\frac{1}{\lambda_i^{(k)}} - c_{il}^{(k-1)}\right) = 0\;. \tag{11.23}$$

Rearrangement yields the iterative estimation procedure for maximum likelihood estimation of the λ_i parameter:

$$\boxed{\lambda_i^{(k)} = \frac{\sum_{j=1}^{N} z_{ij}^{(k)} \;+\; \sum_{\ell=1}^{L} N_\ell\,\tilde{z}_{ij\ell}^{(k)}}{\sum_{j=1}^{N} z_{ij}^{(k)}\;x_j \;+\; \sum_{\ell=1}^{L} N_\ell\,\tilde{z}_{ij\ell}^{(k)}\,c_{i\ell}^{(k-1)}}}\;, \tag{11.24}$$

where

$$z_{ij}^{(k)} \;=\; \frac{\pi_i^{(k)}\,f_i(x_j\,|\,\boldsymbol{\theta}_i^{(k)})}{\sum_{m=1}^{g}\pi_m^{(k)}\,f_m(x_j\,|\,\boldsymbol{\theta}_m^{(k)})} \tag{11.25}$$

$$\;=\; \frac{\pi_i^{(k)}\,\lambda_i^{(k)}\,e^{-\,\lambda_i^{(k)}\,x_j}}{\sum_{m=1}^{g}\pi_m^{(k)}\,f_m(x_j\,|\,\boldsymbol{\theta}_m^{(k)})}\;, \tag{11.26}$$

and

$$\tilde{z}_{ij\ell}^{(k)} = \frac{\pi_i^{(k)} \int_{\mathcal{I}_\ell} dy \;\; f_i(\, y \,|\, \boldsymbol{\theta}_i^{(k)} \,)}{\sum_{m=1}^{g} \pi_m^{(k)} \int_{\mathcal{I}_\ell} dy \;\; f_m(\, y \,|\, \boldsymbol{\theta}_m^{(k)})} \tag{11.27}$$

$$= \frac{\pi_i^{(k)} \left(e^{-\,\lambda_i^{(k)}\,\xi_{l-1}} - e^{-\,\lambda_i^{(k)}\,\xi_l} \right)}{\lambda_i^{(k)} \sum_{m=1}^{g} \pi_m^{(k)} \int_{\mathcal{I}_\ell} dy \;\; f_m(\, y \,|\, \boldsymbol{\theta}_m^{(k)})} \,. \tag{11.28}$$

In practice a numerical procedure is utilised to solve these equations.

11.2.1 Outline of Numerical Procedure

Consider a data vector $\boldsymbol{x}$, taking exact values in the region $\mathcal{I}_0 \subset \mathbb{R}^+$, whereas censored values in intervals $\mathcal{I}_\ell \subset \mathbb{R}^+$ for $\ell = 1, 2, ..., L$. These intervals are defined by boundaries $\xi_{\ell-1}$ and ξ_ℓ for $\ell = 1, ..., L$. The following is an outline of the numerical procedure which can be utilised for estimation of exponential parameters for a given component of a mixed distribution, for the case when data is censored. These equations can be expressed in closed form which presents significant numerical simplicity.

- Identify uncensored observations x_j for $j = 1, ..., N$. Determine N_ℓ, the number of censored observations in each censor interval $\mathcal{I}_\ell$ for $\ell = 1, ..., L$

- **Initialisation:** Choose exponential parameters $\pi_i^{(0)}$ and $\boldsymbol{\theta}_i^{(0)} = \lambda_i^{(0)}$ noting that the full parameter vector $\boldsymbol{\psi}^{(0)}$ also requires initialisation (which could be made up of further exponential components, or other parametric distributions defined on $\mathbb{R}^+$)

- **E step$^{(1)}$:** Compute indicator variables $(z_{i1}^{(0)}, z_{i2}^{(0)}, ..., z_{in}^{(0)})$ and $\Big((\tilde{z}_{i11}^{(0)}, \tilde{z}_{i21}^{(0)}, ..., \tilde{z}_{in1}^{(0)}), (\tilde{z}_{i12}^{(0)}, \tilde{z}_{i22}^{(0)}, ..., \tilde{z}_{in2}^{(0)}), ..., (\tilde{z}_{i1L}^{(0)}, \tilde{z}_{i2L}^{(0)}, ..., \tilde{z}_{inL}^{(0)}) \Big)$ using Eq. (11.26) and Eq. (11.28)

- **M step$^{(1)}$:**

  - Compute $(c_{i1}^{(0)}, ..., c_{iL}^{(0)})$ using Eq. (11.20)

  - Compute iterative estimate of exponential component parameters, i.e. $\pi_i^{(1)}$ and $\boldsymbol{\theta}_i^{(1)} = \lambda_i^{(1)}$ using Eq. (11.8) and Eq. (11.24)

- Compute the log-likelihood $\log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x}) = \sum_{j=1}^{N} \log \big(\sum_{i=1}^{g} \pi_i^{(1)} f_i(x_j \,|\, \boldsymbol{\theta}_i^{(1)}) \;\; + \sum_{\ell=1}^{L} N_\ell \log \big(\sum_{i=1}^{g} \pi_i^{(1)} \int_{\mathcal{I}_\ell} dy \;\, f_i(\, y \,|\, \boldsymbol{\theta}_i^{(1)})\big)$ which is dependent on all of the component distributions

- **E step$^{(2)}$:** Recompute indicator variables $(z_{i1}^{(1)}, z_{i2}^{(1)}, ..., z_{in}^{(1)})$, $\Big((\tilde{z}_{i11}^{(1)}, \tilde{z}_{i21}^{(1)}, ..., \tilde{z}_{in1}^{(1)}), (\tilde{z}_{i12}^{(1)}, \tilde{z}_{i22}^{(1)}, ..., \tilde{z}_{in2}^{(1)}), ..., (\tilde{z}_{i1L}^{(1)}, \tilde{z}_{i2L}^{(1)}, ..., \tilde{z}_{inL}^{(1)}) \Big)$ with updated $\boldsymbol{\psi}^{(1)}$

- **M step$^{(2)}$:**

  - Recompute $(c_{i1}^{(1)}, ..., c_{iL}^{(1)})$

  - Compute updated exponential parameter vector, i.e $\pi_i^{(2)}$, and, $\boldsymbol{\theta}_i^{(2)} = \lambda_i^{(2)}$

- Compute updated log-likelihood, checking convergence condition $|\log L(\boldsymbol{\psi}^{(2)} \,|\, \boldsymbol{x}) - \log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x})\,| < \epsilon$, where ϵ defines a preset tolerance, i.e. $\epsilon = 10^{-5}$

- Continue iteration until convergence condition is satisfied

$$\vdots$$

- Maximum likelihood estimate obtained for exponential parameters, $\hat{\pi}_i$, and $\hat{\boldsymbol{\theta}}_i = \hat{\lambda}_i$

11.3 Weibull Component Censored EM Equations

This section derives censored EM equations for Weibull component distributions[1]. Consider the Weibull probability density function

$$f_i(\,x_j\,|\,\alpha_i^{(k)},\,\beta_i^{(k)}) = \frac{\beta_i^{(k)}}{\alpha_i^{(k)}}\left(\frac{x_j}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}-1} e^{-\left(\frac{x_j}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}}}, \tag{11.29}$$

and censor regions denoted $\mathcal{I}_l = (\xi_{l-1}, \xi_l)$ for $l = 1, ..., L$. The conditional density of the Weibull distribution on the censor region $\mathcal{I}_l$ is given as

$$\begin{aligned}
h_i(\,y\,|\,\boldsymbol{\theta}_i^{(k)}) &= \frac{f_i(\,y\,|\,\boldsymbol{\theta}_i^{(k)})}{\int_{I_\ell} dy\; f_i(\,y\,|\,\boldsymbol{\theta}_i^{(k)})} \\
&= \frac{\frac{\beta_i^{(k)}}{\alpha_i^{(k)}}\left(\frac{y}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}-1} e^{-\left(\frac{y}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}}}}{\int_{\xi_{l-1}}^{\xi^l} dy\; \frac{\beta_i^{(k)}}{\alpha_i^{(k)}}\left(\frac{y}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}-1} e^{-\left(\frac{y}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}}}} \\
&= \frac{\frac{\beta_i^{(k)}}{\alpha_i^{(k)}}\left(\frac{y}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}-1} e^{-\left(\frac{y}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}}}}{e^{-\left(\frac{\xi_{\ell-1}}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}}} - e^{-\left(\frac{\xi_\ell}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}}}}\,.
\end{aligned} \tag{11.30}$$

Introduce for simplicity $\zeta_\ell = (\xi_\ell/\alpha_i^{(k-1)})^{\beta_i^{(k-1)}}$. We are required to compute the following expectation given by Eq. (11.12),

$$\begin{aligned}
&\mathbb{E}\,(\,\log f_i(\,y\,|\,\alpha_i^{(k)},\,\beta_i^{(k)})\,|\,\boldsymbol{x}\in\mathcal{I}_l,\,\alpha_i^{(k)},\,\beta_i^{(k)}) \\
&= \int_{I_\ell} dy\;\log\, f_i(\,y\,|\,\alpha_i^{(k)},\,\beta_i^{(k)}\,)\; h_i(\,y\,|\,c_\ell,\,\alpha_i^{(k-1)},\,\beta_i^{(k-1)}) \\
&= \int_{\xi_{l-1}}^{\xi^l} dy\;\log\left(\frac{\beta_i^{(k)}}{\alpha_i^{(k)}}\left(\frac{y}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}-1} e^{-\left(\frac{y}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}}}\right) \\
&\quad\times\left(\frac{\frac{\beta_i^{(k-1)}}{\alpha_i^{(k-1)}}\left(\frac{y}{\alpha_i^{(k-1)}}\right)^{\beta_i^{(k-1)}} e^{-\left(\frac{y}{\alpha_i^{(k-1)}}\right)^{\beta_i^{(k-1)}}}}{e^{-\zeta_{\ell-1}} - e^{-\zeta_\ell}}\right) \\
&= \log\left(\frac{\beta_i^{(k)}}{\alpha_i^{(k)}}\right) + (\beta_i^{(k)} - 1)\log\left(\frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}}\right) + \\
&\quad \frac{1}{e^{-\zeta_{\ell-1}} - e^{-\zeta_\ell}}\left\{\left(\frac{\beta_i^{(k)}-1}{\beta_i^{(k-1)}}\right)\left[e^{-\zeta_{\ell-1}}\log\zeta_{\ell-1} - e^{-\zeta_\ell}\log\zeta_\ell\right] + \right. \\
&\quad \left(\frac{\beta_i^{(k)}-1}{\beta_i^{(k-1)}}\right)\left[\Gamma(0,\zeta_{\ell-1}) - \Gamma(0,\zeta_\ell)\right] - \\
&\quad \left.\left(\frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}}\left[\Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}}+1,\zeta_{\ell-1}\right) - \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}}+1,\zeta_\ell\right)\right]\right\}.
\end{aligned} \tag{11.31}$$

[1] Much of this derivation was outlined by Dr Ayse Kizilersu (supervisor of this thesis) of University of Adelaide. Synthesis of resultant numerical algorithms was undertaken by the author.

Specifically when the lower censor limit $\xi_{l-1} = 0$ for the first corresponding censor region ($l = 1$), we find that

$$
\begin{aligned}
&\mathbb{E}\left(\log f_i(y \,|\, \alpha_i^{(k)},\, \beta_i^{(k)}) \,|\, \boldsymbol{x} \in \mathcal{I}_l,\, \alpha_i^{(k)},\, \beta_i^{(k)} \right) \\
= \;& \log\left(\frac{\beta_i^{(k)}}{\alpha_i^{(k)}}\right) + (\beta_i^{(k)} - 1)\log\left(\frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}}\right) + \frac{1}{1 - e^{-\zeta_\ell}} \left\{ \left(\frac{\beta_i^{(k)} - 1}{\beta^{(k-1)}}\right) \left(-\gamma - e^{-\zeta_\ell}\log\zeta_\ell - \Gamma(0,\zeta_\ell)\right) \right. \\
& \left. - \left(\frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}} \left[\Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1\right) - \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1, \zeta_\ell\right) \right] \right\} , \qquad (11.32)
\end{aligned}
$$

where $\Gamma\,(a, b)$ denotes the incomplete gamma function defined as

$$
\Gamma\,(a, b) = \int_b^\infty t^{a-1} e^{-t} dt \; . \qquad (11.33)
$$

Hence, iterative selection of $\alpha_i^{(k)}$ and $\beta_i^{(k)}$ is such that the following expression is maximised

$$
\begin{aligned}
Q(\boldsymbol{\psi} \,|\, \boldsymbol{\psi}^{(k)}) \;=\; & \sum_{i=1}^{g}\sum_{j=1}^{N} z_{ij}^{(k)} \, \log\, f_i(\, x_j \,|\, \alpha_i^{(k)},\, \beta_i^{(k)} \,) \\
& + \sum_{i=1}^{g}\sum_{\ell=1}^{L} N_\ell \; \tilde{z}_{ij\ell}^{(k)} \int_{I_\ell} dy \;\, \log\, f_i(\, y \,|\, \alpha_i^{(k)},\, \beta_i^{(k)} \,) \; h_i(\, y \,|\, c_\ell,\, \alpha_i^{(k-1)},\, \beta_i^{(k-1)}) \\
=\; & \sum_{j=1}^{n} z_{ij}^{(k)} \left[\log\left(\frac{\beta_i^{(k)}}{\alpha_i^{(k)}}\right) + (\beta_i^{(k)} - 1)\log\left(\frac{x_j}{\alpha_i^{(k)}}\right) - \left(\frac{x_j}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}} \right] \\
& + \sum_{\ell=1}^{L} N_\ell \; \tilde{z}_{ij\ell}^{(k)} \left\{ \log\left(\frac{\beta_i^{(k)}}{\alpha_i^{(k)}}\right) + (\beta_i^{(k)} - 1)\log\left(\frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}}\right) \right. \\
& + \left(\frac{\beta_i^{(k)} - 1}{\beta_i^{(k-1)}}\right) \frac{e^{-\zeta_{\ell-1}}\log\zeta_{\ell-1} - e^{-\zeta_\ell}\log\zeta_\ell + \Gamma(0,\zeta_{\ell-1}) - \Gamma(0,\zeta_\ell)}{e^{-\zeta_{\ell-1}} - e^{-\zeta_\ell}} \\
& \left. - \left(\frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}} \frac{\Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1, \zeta_{\ell-1}\right) - \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1, \zeta_\ell\right)}{e^{-\zeta_{\ell-1}} - e^{-\zeta_\ell}} \right\}, \qquad (11.34)
\end{aligned}
$$

noting that when the lower censor limit $\xi_{l-1} = 0$ for $\ell = 1$ ($\xi_0 = 0$), the following simplification can be made

$$
\frac{e^{-\zeta_{\ell-1}}\log\zeta_{\ell-1} - e^{-\zeta_\ell}\log\zeta_\ell + \Gamma(0,\zeta_{\ell-1}) - \Gamma(0,\zeta_\ell)}{e^{-\zeta_{\ell-1}} - e^{-\zeta_\ell}} = -\frac{\gamma + e^{-\zeta_1}\log\zeta_1 + \Gamma(0,\zeta_1)}{1 - e^{-\zeta_1}} \; .
$$

Maximisation can either be undertaken directly from Eq. (11.34) or by considering differentiation with respect to both parameters $\alpha_i^{(k)}$ and $\beta_i^{(k)}$ and setting the resultant expressions to zero. First consider differentiation with respect to $\alpha_i^{(k)}$

$$
\begin{aligned}
0 \;=\; & \sum_{j=1}^{n} z_{ij}^{(k)} \left[-1 + \left(\frac{x_j}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}} \right] \\
& + \sum_{\ell=1}^{L} N_\ell \; \tilde{z}_{ij\ell}^{(k)} \left\{ -1 - \left(\frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}} \frac{\Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1, \zeta_{\ell-1}\right) - \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1, \zeta_\ell\right)}{e^{-\zeta_{\ell-1}} - e^{-\zeta_\ell}} \right\} \qquad (11.35)
\end{aligned}
$$

and now differentiation with respect to $\beta_i^{(k)}$

$$\begin{aligned}
0 &= \sum_{j=1}^{n} z_{ij}^{(k)} \left[\frac{1}{\beta_i^{(k)}} + \log \frac{x_j}{\alpha_i^{(k)}} - \left(\frac{x_j}{\alpha_i^{(k)}} \right)^{\beta_i^{(k)}} \log \frac{x_j}{\alpha_i^{(k)}} \right] \\
&+ \sum_{\ell=1}^{L} N_\ell \, \tilde{z}_{ij\ell}^{(k)} \left\{ \frac{1}{\beta_i^{(k)}} + \log \frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}} + \frac{D_{i\ell}^{(k)}}{e^{-\zeta_{\ell-1}} - e^{\zeta_\ell}} \right\}, \qquad (11.36)
\end{aligned}$$

where $D_{i\ell}^{(k)}$ has been introduced and is given as

$$\begin{aligned}
D_{i\ell}^{(k)} &= \frac{e^{-\zeta_{\ell-1}} \log \zeta_{\ell-1} - e^{-\zeta_\ell} \log \zeta_\ell + \Gamma(0, \zeta_{\ell-1}) - \Gamma(0, \zeta_\ell)}{\beta_i^{(k-1)}} \\
&- \frac{1}{\beta_i^{(k-1)}} \left(\frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}} \right)^{\beta_i^{(k)}} \left[\sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \frac{\zeta_{\ell-1}^{\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}}+1+p} - \zeta_\ell^{\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}}+1+p}}{\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1 + p \right)^2} \right. \\
&- \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1 \right) (\log \zeta_{\ell-1} - \log \zeta_\ell) \\
&+ \left. \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1, \zeta_{\ell-1} \right) \log \zeta_{\ell-1} - \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1, \zeta_\ell \right) \log \zeta_\ell \right] \\
&- \log \frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}} \left(\frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}} \right)^{\beta_i^{(k)}} \left\{ \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1, \zeta_{\ell-1} \right) - \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1, \zeta_\ell \right) \right\}, \\
&\qquad (11.37)
\end{aligned}$$

and for $\xi_0 = 0$

$$\begin{aligned}
D_{i1}^{(k)} &= \frac{-\gamma - e^{-\zeta_1} \log \zeta_1 - \Gamma(0, \zeta_1)}{\beta_i^{(k-1)}} \\
&- \frac{1}{\beta_i^{(k-1)}} \left(\frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}} \right)^{\beta_i^{(k)}} \left[-\sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \frac{\zeta_1^{\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}}+1+p}}{\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1 + p \right)^2} \right. \\
&+ \left. \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1 \right) \log \zeta_1 - \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1, \zeta_1 \right) \log \zeta_1 \right] \\
&- \log \frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}} \left(\frac{\alpha_i^{(k-1)}}{\alpha_i^{(k)}} \right)^{\beta_i^{(k)}} \left\{ \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1 \right) - \Gamma\left(\frac{\beta_i^{(k)}}{\beta_i^{(k-1)}} + 1, \zeta_1 \right) \right\} \\
&\qquad (11.38)
\end{aligned}$$

The indicator variables for the Weibull component are given as

$$\begin{aligned}
z_{ij}^{(k)} &= \frac{\pi_i^{(k)} f_i(x_j \,|\, \boldsymbol{\theta}_i^{(k)})}{\sum_{m=1}^{g} \pi_m^{(k)} f_m(x_j \,|\, \boldsymbol{\theta}_m^{(k)})} \\
&= \frac{\pi_i^{(k)} \frac{\beta_i^{(k)}}{\alpha_i^{(k)}} \left(\frac{x_j}{\alpha_i^{(k)}} \right)^{\beta_i^{(k)}-1} e^{-\left(\frac{x_j}{\alpha_i^{(k)}} \right)^{\beta_i^{(k)}}}}{\sum_{m=1}^{g} \pi_m^{(k)} f_m(x_j \,|\, \boldsymbol{\theta}_m^{(k)})} \qquad (11.39)
\end{aligned}$$

and

$$\tilde{z}_{ij\ell}^{(k)} = \frac{\pi_i^{(k)} \int_{\mathcal{I}_\ell} dy \;\; f_i(\, y \,|\, \boldsymbol{\theta}_i^{(k)} \,)}{\sum_{m=1}^{g} \pi_m^{(k)} \int_{\mathcal{I}_\ell} dy \;\; f_m(\, y \,|\, \boldsymbol{\theta}_m^{(k)})} \tag{11.40}$$

$$= \frac{\pi_i^{(k)} \left(e^{-\left(\frac{\xi_{\ell-1}}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}}} - e^{-\left(\frac{\xi_{\ell}}{\alpha_i^{(k)}}\right)^{\beta_i^{(k)}}} \right)}{\sum_{m=1}^{g} \pi_m^{(k)} \int_{\mathcal{I}_\ell} dy \;\; f_m(\, y \,|\, \boldsymbol{\theta}_m^{(k)})}. \tag{11.41}$$

The resultant censored EM equations for the Weibull distribution are non-trivial. For solutions of these equations to be found, a numerical procedure is required to be used.

11.3.1 Outline of Numerical Procedure

Consider a data vector $\boldsymbol{x}$ taking exact values in the region $\mathcal{I}_0 \subset \mathbb{R}^+$, whereas censored values in intervals $\mathcal{I}_\ell \subset \mathbb{R}^+$, for $\ell = 1, 2, ..., L$. These intervals are defined by boundaries $\xi_{\ell-1}$ and ξ_ℓ for $\ell = 1, ..., L$. The following is an outline of the numerical procedure which can be utilised for estimation of Weibull parameters for a given component of a mixed distribution, for the case when data is censored. This work opted for direct search of Eq. (11.34) in order to determine the parameter combination which maximises the expression and hence corresponds to an iterative estimate. This was undertaken with *fminsearch* [77] a derivative free numerical procedure which can be used to find minima of unconstrained multivariable functions provided by MATLAB.

- Identify uncensored observations x_j, for $j = 1, ..., N$. Determine N_ℓ, the number of censored observations in each censor interval $\mathcal{I}_\ell$, for $\ell = 1, ..., L$

- **Initialisation:** Choose Weibull component parameters, i.e. $\pi_i^{(0)}$ and $\boldsymbol{\theta}_i^{(0)} = (\alpha_i^{(0)}, \beta_i^{(0)})$, noting that the full parameter vector $\boldsymbol{\psi}^{(0)}$ also requires initialisation (which could be made up of further Weibull components, or other parametric distributions defined on $\mathbb{R}^+$)

- **E step$^{(1)}$:** Compute indicator variables $(z_{i1}^{(0)}, z_{i2}^{(0)}, ..., z_{in}^{(0)})$ and $\Big((\tilde{z}_{i11}^{(0)}, \tilde{z}_{i21}^{(0)}, ..., \tilde{z}_{in1}^{(0)}), (\tilde{z}_{i12}^{(0)}, \tilde{z}_{i22}^{(0)}, ..., \tilde{z}_{in2}^{(0)}), ..., (\tilde{z}_{i1L}^{(0)}, \tilde{z}_{i2L}^{(0)}, ..., \tilde{z}_{inL}^{(0)}) \Big)$ using Eq. (11.39) and Eq. (11.40)

- **M step$^{(1)}$:**

  - Compute iterative estimate of mixing proportion, $\pi_i^{(1)}$, using Eq. (11.8)

  - Choose Weibull parameters $\boldsymbol{\theta}_i^{(1)} = (\alpha_i^{(1)}, \beta_i^{(1)})$, which maximise Eq. (11.34) using the MATLAB *fminsearch* algorithm (to numerically minimise the negative of the function)[2]

- Compute the log-likelihood $\log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x}) = \sum_{j=1}^{N} \log \left(\sum_{i=1}^{g} \pi_i^{(1)} f_i(x_j \,|\, \boldsymbol{\theta}_i^{(1)}) \right. + \sum_{\ell=1}^{L} N_\ell \log \left(\sum_{i=1}^{g} \pi_i^{(1)} \int_{\mathcal{I}_\ell} dy \;\; f_i(\, y \,|\, \boldsymbol{\theta}_i^{(1)}) \right)$ which is dependent on all of the component distributions

- **E step$^{(2)}$:** Recompute indicator variables indicator variables, $(z_{i1}^{(1)}, z_{i2}^{(1)}, ..., z_{in}^{(1)})$, $\Big((\tilde{z}_{i11}^{(1)}, \tilde{z}_{i21}^{(1)}, ..., \tilde{z}_{in1}^{(1)}), (\tilde{z}_{i12}^{(1)}, \tilde{z}_{i22}^{(1)}, ..., \tilde{z}_{in2}^{(1)}), ..., (\tilde{z}_{i1L}^{(1)}, \tilde{z}_{i2L}^{(1)}, ..., \tilde{z}_{inL}^{(1)}) \Big)$ with updated $\boldsymbol{\psi}^{(1)}$

- **M step$^{(2)}$:** Compute updated Weibull parameter vector, i.e $\pi_i^{(2)}$ and $\boldsymbol{\theta}_i^{(2)} = (\alpha_i^{(2)}, \beta_i^{(2)})$ using methodology specified above

[2]MATLAB provides inbuilt functions, *igamma*, *gammainc*, and *gamma*, to deal with the complete and incomplete gamma functions, defined as $\Gamma(a, b) = igamma\,(a, b) = gamma\,(a)\,(1 - gammainc\,(b, a)\,)$. The runtime for function *gammainc* is substantially quicker than *igamma*, but problematic when $a = 0$. The following series expression for the incomplete gamma function for these cases can be useful. $\Gamma(0, b) = -\gamma - \log b + \sum_{n=0}^{\infty} \frac{b^{n+1}(-1)^n}{n!(n+1)^2}$. The rate of convergence of the series is high, with numerical tractability, so a finite approximation of the series is appropriate.

- Compute updated log-likelihood, checking convergence condition $|\log L(\boldsymbol{\psi}^{(2)} \,|\, \boldsymbol{x}) - \log L(\boldsymbol{\psi}^{(1)} \,|\, \boldsymbol{x})\,| < \epsilon$, where ϵ defines a preset tolerance, i.e. $\epsilon = 10^{-5}$
- Continue iteration until convergence condition is satisfied

$$\vdots$$

- Maximum likelihood estimate obtained for Weibull parameters, $\hat{\pi}_i$, and $\hat{\boldsymbol{\theta}}_i = (\hat{\alpha}_i,\, \hat{\beta}_i)$

Chapter 12

Application: London Stock Exchange Order Book Data

Mixture models are often employed to model data in situations when single parametric distributions cannot provide an adequate result. Typically this data is distributed in a complex manner, lending itself to mixture models because of the flexibility they offer. The Expectation Maximisation (EM) algorithm is a typical statistical tool utilised to obtain parameter estimates in such models.
Chapter 2 and Chapter 3 presented a thorough analysis of mixture modelling and parameter estimation via the EM algorithm. Model selection via information criteria was introduced in Chapter 4 as a tool for goodness-of-fit testing of mixtures. This chapter analyses electronic order book data obtained for a range of stocks on the London Stock Exchange (LSE), leveraging the statistical methodology presented thus far in the thesis.

12.1 Background

Stock markets allow buyers and sellers to transact various financial instruments, typically the fractional ownership of listed companies, referred to as stocks. A stock exchange facilitates the exchange of these financial instruments with the market functioning as a "continuous auction", meaning buyers and sellers execute trades simultaneously from a central location. In the past this location was the floor of the stock exchange, but because of the benefit of increased speed and reduced overhead costs, the majority of stock exchanges now function electronically. Stock exchanges of this type manage the flow of transactions and orders via the electronic order book (EOB). The EOB allows traders who wish to buy, i.e. exchange a certain amount of money for shares of a given stock (at a certain time and at a certain price), and those who wish to sell, to transact in a regulated environment. These transactions occur by first submitting an order of which there are two typical types, limit orders (LO), and market orders (MO) [19].

- **Limit orders:** represent the intent to buy or sell a quantity of stock at a specific price. Sellers wish to sell stock at the highest possible price, with the price they are willing to accept referred to as the "ask-price". Likewise, buyers wish to buy at the lowest possible price, the price which they intend to pay is the "bid-price". If the market does not reach the stated limit price the transaction will not be executed and typically cancelled at the end of the trading day.

- **Market orders:** are executed almost immediately at the current "best-price" (or market price). Market orders are matched with limit orders, meaning the "best-bid" (highest "bid-price") or "best-ask" (lowest "ask-price") will be cancelled by the trade once executed. Orders of this type are used when certainty of execution is a priority over the price, and can only be accepted inside market hours. This is in comparison to limit orders which can be placed whenever, and will be queued if placed outside of market hours.

The market price of a particular stock is determined by typical supply and demand parameters. The difference between the "best-bid" and the "best-ask" is referred to as the current spread, and the **mid-price** (which is typically just referred to as the price) is the average of "best-bid" and "best-ask", rounded to the nearest valid tradable price. The mid-price **waiting time** is defined as the time difference between consecutive price changes.

The stock exchange sends all EOB data as a stream to vendors who distribute the listed orders to various traders. Information provided to the end users for each listed order includes the stock ticker (used to uniquely identify a particular publicly listed stock), type of order (bid or ask), limit price, the proposed volume of shares, a *time-stamp* of the arrival time of the order, and a unique order identification number [19]. Limit orders can also be cancelled or amended (on the assumption that the order has not already been executed). Because of this, the limit order status is also included in the EOB data ("addition" for newly arrived orders, "cancellation" for cancelled orders, and "modification" for amended orders). The last key data field provided in the EOB is a trade indicator stating whether the proposed transaction has taken place or not.

Stock can be exchanged in the manner described when listed on a given stock exchange which are located at various geographic locations around the world. Some include, the Australian Securities Exchange (ASX), the New York Stock Exchange (NYSE), the NASDAQ Stock Market (NASDAQ), and the exchange in which the EOB data dealt with in this thesis originates, the London Stock Exchange (LSE).

Understanding the stochastic nature of the stock market is of great importance to traders, not just because of the prospect of direct financial gain, but also in the process of assessing the stability of the market. The frequency in which orders are submitted to an exchange is related to the rate of change of the stock price. Understanding the temporal nature of orders placed on an exchange is therefore of great value to financial institutions. Research undertaken towards this thesis largely builds upon the existing work of Kizilersü et al. [19] and Guscott [20]. Existing work attempted to describe the distribution of inter-arrival times (of both limit and market orders) using left-truncated distributions. Tick-by-tick data is recorded by the EOB with microsecond resolution, but the data made available for use in this thesis (and the existing work of [19, 20]) is rounded with only millisecond resolution. Careful treatment is required when using continuous distributions to describe data which is discretely binned. Truncation of data at small time scales can be used to avoid many of the resultant difficulties. If we are observing time differences which are on the order of 1 millisecond, the binning of the arrival times has a significant affect on the distribution of inter-arrival times. As the truncation level is increased, this significance decreases because the ratio of bin width to inter-arrival times decreases [20]. Truncation also deals with the issue of "zero inflated" data[1]. If orders arrive with a time separation of less than a half millisecond (0.5 ms), the data released records these orders as simultaneous, i.e. with the same time-stamp, so the corresponding recorded time difference is zero. Any non-zero τ_l value (truncation value) is sufficient to remove the "zero inflated" data. Tables 12.1 and 12.2 provide data statistics describing the proportion of data which is "zero inflated" for the stocks of interest to this work.[2] Kizilersü et al. [19] suggested that the short time scale inter-arrival times are a result of ultra-high frequency trading algorithms operating in the range of microseconds at locations in close proximity to the exchange. Many dummy limit orders are placed and cancelled by such algorithms in an attempt to manipulate certain behaviours of the stock market. The main findings of Kizilersü et al. [19] and Guscott [20] are summarised as follows:

- If the short scale differences in inter-arrival times of both limit and market orders are excluded (those orders generated by ultra-high frequency trading algorithms) via left-truncation, a left-truncated Weibull distribution best describes the time separation of such orders.

- The scale parameter α of the Weibull distribution corresponds to the inverse of the activity of the stock, hence is variable on both stock and time.

- The shape parameter β of the Weibull distribution does not change for different stocks and has been found to be universal, corresponding to the maximum continuous information entropy given by the Euler-Mascheroni constant ($\gamma \approx 0.57722$). The work empirically justified that the Euler-Mascheroni constant fell within the uncertainty margins for all maximum likelihood estimates of the β parameter.

[1] A "zero inflated" model allows for frequent zero-valued observations.

[2] The specific data made available to this work will be formally introduced in Section 12.2.

- **The modelled left-truncated Weibull distribution if extrapolated, in an attempt to describe the data in the pre-existing truncated region, doesn't account for the excess of inter-arrival times with differences in the small time scales.**
- As truncation level τ_L increases, the left-truncated Weibull distribution better describes the difference in inter-arrival times of limit and market orders. The left-truncated Weibull distribution also better describes data for stocks of greater liquidity.

Kizilersü et al. [19] theorised that behaviour in the short time scale region (that dominated by high-frequency trading, approx less than 10ms) may be described by a mixed distribution. The focus of research undertaken towards this thesis has been the application of mixture models in an attempt to describe the full distribution of time differences of tick-by-tick data. The boldfaced finding above explains that differences in inter-arrival times in the intermediate and tail region of the distribution can be well described by a Weibull distribution. Extrapolation of the single distribution into the small time scale region shows the single Weibull distribution inadequately describes the excess data apparent at small time scales. The mixed distributions considered in this work exhibit a Weibull component to describe the intermediate and tail regions of the distribution. A mixture of additional components will attempt to describe data in the short time scale region of the distribution, whilst attempting to minimise the compromise to the Weibull explained region. Fig. 12.1 shows a sketch of the nature of the distribution of probability density. Data considered for the analysis is limit order inter-arrival times and mid-price waiting times.

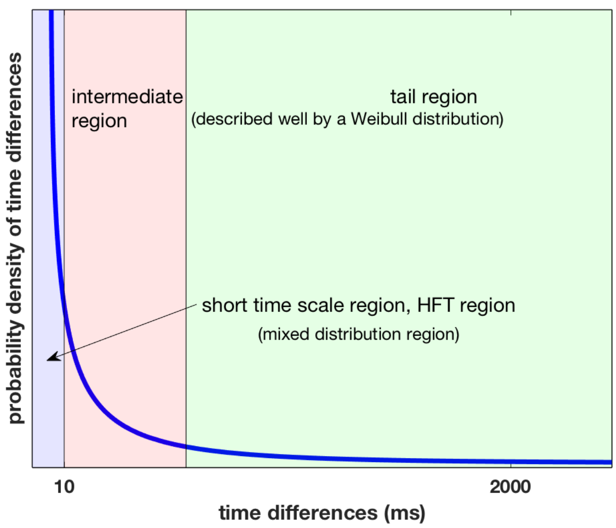


Figure 12.1: Sketch of the distribution of probability density for time differences of tick-by-tick data. Short time scale, intermediate, and tail regions are shown.

12.2 Limit-Order and Mid-Price Data

This study had access to EOB data in millisecond resolution for all limit-orders placed, cancelled, and/or amended on the London Stock Exchange (LSE) between 1st June 2010 and 30th September 2010 for the stocks listed below. Additionally, mid-price data (all price changes and associated timestamps) were also available from the same exchange for the period of 1st June 2010 until 30th September 2011, again for the same stocks.

- **ABF:** Associated British Foods
- **BARC:** Barclays
- **RIO:** Rio Tinto
- **RR:** Rolls-Royce Holding
- **SSE:** Scottish and Southern Energy Company
- **VOD:** Vodafone Group

Limit-Order Data:

The time differences between two consecutive limit-order arrivals are given as

$$x_i^{\mathrm{LO}} = t_i^{\mathrm{LO}} - t_{i-1}^{\mathrm{LO}}, \qquad i = 1, 2, ..., N_{\mathrm{EOB}}^{\mathrm{LO}} \ , \tag{12.1}$$

where t_i^{LO} denotes the i^{th} arrival time (measured in milliseconds) of a limit-order placed on the exchange for a particular stock, and $N_{\mathrm{EOB}}^{\mathrm{LO}}$ denotes the total number of limit-order arrivals recorded in the EOB for the time period considered. Consider a data sample $\boldsymbol{x_i^{\mathrm{LO}}}$ where the data is made up of n consecutive x_i^{LO} measurements,

$$\boldsymbol{x_i^{\mathrm{LO}}} = (x_i^{\mathrm{LO}}, x_{i+1}^{\mathrm{LO}}, ..., x_{n+i-1}^{\mathrm{LO}}) \ . \tag{12.2}$$

For a given data sample $\boldsymbol{x_i^{\mathrm{LO}}}$ it is required that all x_i^{LO} originate from the same trading day. This is to avoid irregularities associated with large time differences where two consecutive time-stamps span over two trading days.

The full dataset can be partitioned into consecutive intervals of a given sample size n. During a trading day, activity is typically highest at market opening and closure, with a noticeable decrease during the middle of the day. The Weibull scale parameter α is inversely proportional to the activity of the stock, therefore is not constant throughout a trading day. Figure 12.2 displays the Kizilersü et al. [19] $\hat{\alpha}$ estimates for limit order arrival BARC data on the trading day of June 1st 2010 for a sample size of $n = 100$. The blue curve represents the entirety of the interval estimates and the red curve gives a moving mean across 50 consecutive intervals.

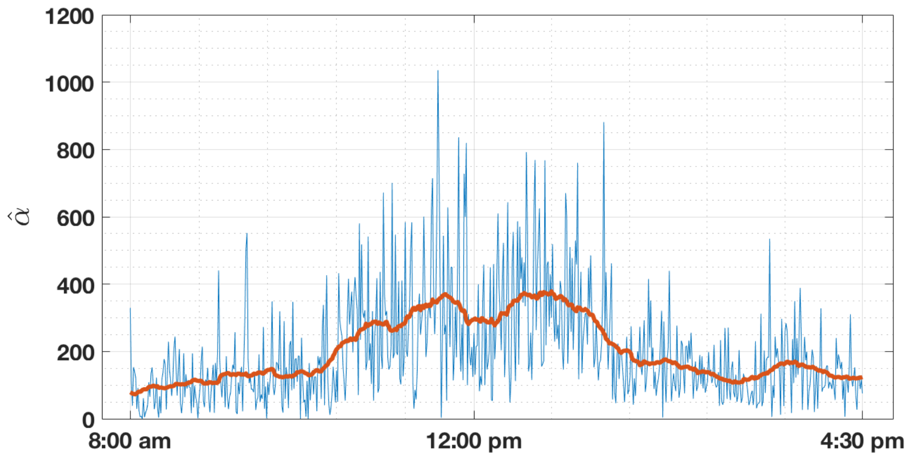


Figure 12.2: $\hat{\alpha}$ estimates throughout trading day June 1st 2010 for BARC stock. Moving mean with window size of 50 intervals is shown in red.

Table 12.1 displays the relevant data statistics for the limit order arrivals. $N_{\mathrm{EOB}}^{\mathrm{LO}}$/day is the daily average number of limit orders recorded by the EOB, a measure of the activity of a given stock, the mean inter-arrival time for each stock is given as $\overline{x_i^{\mathrm{LO}}}$, and the table also indicates the percentage of inter-arrival times in the short time scale region. From brief examination of the dataset, the following observations can be made:

- Stocks in order of activity, categorised by the highest volume of orders are; RIOTINTO (large stock), BARC (large stock), VOD (medium stock), RRLN (medium stock), SSELN (small stock), and ABFLN (small stock).

- More active stocks have a smaller mean inter-arrival time $\overline{x_i^{\mathrm{LO}}}$.

- **Significant density exists in the short time scale region of the distribution, including over half the data being "zero inflated", corresponding to a time difference of consecutive orders of less than 0.5 ms.**

Stock	$N_{\text{EOB}}^{\text{LO}}$/day	$\overline{x_i^{\text{LO}}}$ ms	$N[x_i^{\text{LO}} = 0\,\text{ms}]\%$	$N[x_i^{\text{LO}} \leq 5\,\text{ms}]\%$	$N[x_i^{\text{LO}} \leq 10\,\text{ms}]\%$
ABFLN	25,537	1,196	**53.27%**	60.09%	61.96%
BARC	237,245	129	**51.69%**	66.91%	70.85%
RIOTINTO	280,367	108	**56.85%**	69.23%	72.94%
RRLN	35,665	846	**52.63%**	60.49%	62.76%
SSELN	28,713	1,605	**52.63%**	58.91%	60.73%
VOD	103,063	297	**56.73%**	66.43%	69.05%

Table 12.1: Statistics for limit order arrival data.

Mid-Price Data:

Limit-orders and market-orders make up the input for the EOB, the output being the variation of mid-price. The mid-price is given as the arithmetic mean of "best-bid" and "best-ask" (which is determined by the nature of the limit-orders) but a stock dependent tick size dictates the minimum possible mid-price change. Any increment in the mid price is recorded in the EOB alongside an associated time-stamp. Denote the stock price of a particular stock at time t as S_t. Beginning with $t_0^{\text{MP}} = 0$ and $S_{t_0^{\text{MP}}} > 0$ the stochastic process which describes the stock price is given as

$$S_t = S_{t_0^{\text{MP}}} + X_{t_1^{\text{MP}}} + X_{t_2^{\text{MP}}} + ... + X_{t_N^{\text{MP}}} \ , \tag{12.3}$$

where $X_{t_i^{\text{MP}}}$ is the i^{th} incremental change in mid-price at time t_i^{MP}. Figure 12.3 displays the mid-price of BARC stock from 1st June 2010 to 30th September 2011.

The mid-price waiting time can be defined as the difference between the time stamps of two consecutive mid-price changes,

$$x_i^{\text{MP}} = t_i^{\text{MP}} - t_{i-1}^{\text{MP}} \ , \qquad i = 1, 2, ..., N_{\text{EOB}}^{\text{MP}} \ , \tag{12.4}$$

where $N_{\text{EOB}}^{\text{MP}}$ represents the total number of mid-price changes recorded in the EOB. Consider a data sample $\boldsymbol{x_i^{\text{MP}}}$ made up of n consecutive x_i^m measurements,

$$\boldsymbol{x_i^{\text{MP}}} = (x_i^{\text{MP}}, x_{i+1}^{\text{MP}}, ..., x_{n+i-1}^{\text{MP}}) \ . \tag{12.5}$$

The construction of data samples for mid-price waiting times $\boldsymbol{x_i^{\text{MP}}}$ is identical to that of limit-order arrival data samples $\boldsymbol{x_i^{\text{LO}}}$. This analysis treats the mid-price waiting time as a stochastic process, although Scalas [79] suggested that, in general, the mid-price and the mid-price waiting time are not in fact independent of each other.

Figure 12.3: Mid-price S_t for BARC stock from 1st June 2010 to 30th September 2011.

Table 12.2 displays the relevant statistics for mid-price waiting time data. $N^{\text{MP}}_{\text{EOB}}$/day represents the daily average number of mid-price changes, and the mean waiting time is given as $\overline{x^{\text{MP}}_i}$. The following observations can be made directly from the data:

- The distribution (general shape) of mid-price waiting times is similar to that of limit order inter-arrival times.
- More active stocks which are characterised by a higher volume of orders change price more frequently.
- **"Zero inflated" data exists within mid-price waiting times (categorised by a price change within less than 0.5 ms) although far less regular than limit order inter-arrival times.**
- **Significant density exists in the short time scale region of the distribution of waiting times, although far less than limit order inter-arrival times.**

Stock	$N^{\text{MP}}_{\text{EOB}}$/day	$\overline{x^{\text{MP}}_i}$ ms	$N[x^{\text{MP}}_i = 0\,\text{ms}]\%$	$N[x^{\text{MP}}_i \leq 5\,\text{ms}]\%$	$N[x^{\text{MP}}_i \leq 10\,\text{ms}]\%$
ABFLN	1,067	28,017	**8.27%**	24.80%	29.70%
BARC	32,157	931	**13.07%**	44.07%	51.84%
RIOTINTO	41,563	718	**15.30%**	45.33%	52.51%
RRLN	2,738	10956	**9.67%**	39.48%	45.72%
SSELN	1,362	22,020	**8.49%**	24.89%	29.85%
VOD	8,179	3,628	**10.86%**	30.87%	37.38%

Table 12.2: Statistics for mid-price waiting time data.

As previously discussed, the excess data in the short time scale region of the distribution of limit order inter-arrival times is due to many dummy orders placed by high-frequency trading algorithms. Significantly less density in these regions is therefore expected for the distribution of mid-price waiting times, which is consistent with the observations made directly from the data.

12.3 Towards Modelling EOB Data

EOB data comprises of three fundamental components; time, price, and volume. In order to model the dynamics of the EOB one needs to understand how all of these components behave. This section outlines a modelling procedure to describe the time information of the EOB. The mixed distributions considered to model the full distribution of limit order inter-arrival times and mid-price waiting times will consist of **Weibull**, **exponential**, **gamma**, **loglogistic**, and/or **uniform** parametric components. The use of a Weibull distribution as the primary component of these mixtures was motivated by Kizilersü et al. [19]. The additional distributions were chosen because they share the property that for certain parameter combinations they can exhibit large densities in the short time scale region and decay quickly. This property ensures minimal compromise to the intermediate and tail region of the distribution which is well described by a Weibull component. Chapters [6 - 10] were dedicated to each of these distributions, with a full derivation of the relevant EM equations provided. The following procedure outlines several contrasting techniques to deal with the issue of "zero inflation".

12.3.1 Removal of Zeros: First Attempt at Dealing with "zero inflation"

Removal of "zero inflated" data: refers to the artificial removal of all "zero inflated" data. Clearly this technique is rudimentary and by far the least statistically rigorous of those considered. Direct removal of "zero inflated" data is equivalent to placing a left-truncation point at 0.5 ms, but still considering mixtures with un-truncated component distributions. Removal of all density corresponding to "zero inflated" data provided an adequate test case for initial application of the mixed distributions to the real dataset. The following two-component mixtures were considered for all stocks:

- Exponential/Weibull
- Gamma/Weibull
- Loglogistic/Weibull
- Weibull/Weibull

} ABFLN, BARC, RIOTINTO, RRLN, SSELN, VOD.

12.3.2 Replacement of Zeros: Second Attempt at Dealing with "zero inflation"

- **Distributing "zero inflated" data uniformly:** is motivated by the fact that all "zero inflated" data corresponds to time differences less than 0.5 ms. Recall that the data available is restricted to millisecond resolution, thus all time differences of less than 0.5 ms are recorded as zero. This technique assumes that the true distribution of time differences in this region is approximated by uniformly distributed "zero inflated" data. For this technique, three component mixtures are considered. The distinction being the addition of a uniform component with fixed parameter $\theta = 0.5$[3]. All other parameters including the mixture proportion of the uniform density are estimated from the data. The following three-component mixtures were considered for all stocks:

  - Exponential/Uniform/Weibull
  - Gamma/Uniform/Weibull
  - Loglogistic/Uniform/Weibull
  - Weibull/Uniform/Weibull

  } ABFLN, BARC, RIOTINTO, RRLN, SSELN, VOD.

- **Distributing "zero inflated" data exponentially:** is motivated by the same premise, but assumes that the true distribution of time differences is approximated by exponentially distributed "zero inflated" data instead. An exponential distribution with rate parameter $\lambda = 8$ distributes "zero inflated" data with almost the entirety of density within the [0, 0.5] ms region. i.e. $\int_0^{0.5} 8\,e^{-8\,x}\,dx \approx 0.98$, therefore was considered suitable for choice distributing the "zero inflated" data. For this case three component mixtures are considered with the additional distribution now being exponential. For this case all parameters were estimated from the data. The following three-component mixtures were considered for all stocks:

  - Exponential/Exponential/Weibull
  - Gamma/Exponential/Weibull
  - Loglogistic/Exponential/Weibull
  - Weibull/Exponential/Weibull

  } ABFLN, BARC, RIOTINTO, RRLN, SSELN, VOD.

12.3.3 g-component Exponential Mixture: Third Attempt at Dealing with "zero inflation"

The next attempt at dealing with "zero inflated" data considers g-component exponential mixtures. For this case no Weibull components were considered, unlike the previous approaches. The rationale

[3] Other θ values were also considered in this analysis but the results presented will be for $\theta = 0.5$

for this was provided by Scalas [79], who proposed a finite exponential mixture to model the distribution of waiting times. These mixtures were primarily theorised to deal with the variability of stock activity throughout a trading day, with T exponential components corresponding to T intervals. This work considers an alternate abstraction, g-component exponential mixtures on each interval, but follows the same notion that multiple exponential distributions each with different parameter values are required to model the distribution of time differences. The following g-component ($g = 2, 4, 6, 10$) exponential mixed distributions were considered for all stocks:

- 2-component Exponential
- 4-component Exponential
- 6-component Exponential
- 10-component Exponential

} ABFLN, BARC, RIOTINTO, RRLN, SSELN, VOD.

12.3.4 Censoring: Final Attempt at Dealing with "zero inflation"

Censoring "zero inflated" data is the culmination of the analysis. Censoring acknowledges the presence of data within a certain region, without requiring full specification. Orders are recorded by the EOB at microsecond resolution but only collected at millisecond resolution, effectively binned or rounded. Because the ratio of bin width to time increases for smaller time differences, this binning has a significant effect on the reliability of the estimator for small time differences. Censoring data in these regions offers a statistically rigorous technique for mitigating these difficulties in addition to dealing with the "zero inflation". A single censored region $[0, 0.5]$ms, and a mutli-censored region $[0, 0.5, 1.5, 2.5, 10]$ms were considered for exponential and Weibull mixture components[4] for all stocks:

- 3-comp exp: censored on $[0, 0.5]$ms
- 4-comp exp: censored on $[0, 0.5]$ms
- 3-comp exp: censored on $[0, 0.5, 1.5, 2.5, 10]$ms
- 4-comp exp: censored on $[0, 0.5, 1.5, 2.5, 10]$ms
- exp/exp/Weibull: censored on $[0, 0.5]$ms

} ABFLN, BARC, RIOTINTO, RRLN, SSELN, VOD.

12.4 Numerical Procedure for Modelling EOB Data

All of the mixed distributions proposed within the procedure outlined above were used to model the full distribution of limit order inter-arrival times and mid-price waiting times. Parameter estimation was undertaken from the data using the Expectation Maximisation algorithm (Chapter 3). The algorithm was initialised with fixed values which were dependent on the stock and distribution, but independent of the sample size n. These values were determined by repeated analysis of a four trading day subset of the full dataset, in addition to the previous parameter estimates of Kizilersü et al. [19]. Although complete runs were undertaken with several initial values, implementing a more robust statistically rigorous initialisation procedure would be an area where this analysis could improve. Goodness-of-fit testing was undertaken via information criteria tests, those considered were the Akaike information criterion and the Bayesian information criterion.

Although it is understood that estimation procedures perform better for larger data samples, the activity of a stock varies throughout a trading day. For this reason the full dataset was split into intervals. The data sample of each interval $\boldsymbol{x_i}$ contains n consecutive time differences. Reliable estimation of parameters requires the dataset be identically distributed, i.e. scale parameter α is required to be close to constant on each interval. Scalas [79], who proposed a compound Poisson process to model financial waiting time data, also dealt with the variability in stock activity by

[4] Derivation of censored EM equations is not trivial, in fact it is one of the accomplishments of this thesis to present censored EM equations for both the exponential and Weibull distribution. Unfortunately time constraints related to the preparation of this thesis meant deriving censored EM equations for the loglogistic and gamma distribution was not possible. These derivations will be presented in a subsequent paper. Consequently it was not possible to include gamma/exponential/Weibull, and loglogistic/exponential/Weibull mixed distributions within the censored framework.

dividing data from a particular trading day into intervals. A compromise between having enough data and minimising the variability of the parameter α is required for reliable estimation. The sample sizes studied in this work were:

- mixtures with removal of zeros: Section 12.3.1
- mixtures with replacement of zeros: Section 12.3.2
- g-component exponential mixtures: Section 12.3.3
- exponential censored mixtures: Section 12.3.4

$\Big\}$ $n = \{50, 100, 200, 500, 1000\}$.

- exp/exp/Weibull censored mixtures: Section 12.3.4 $\Big\}$ $n = \{200, 500, 1000, 2000, 5000\}$[5].

Chapters 13 and 14 present results and discussions of the outlined procedure for limit order inter-arrival time and mid-price waiting time data respectively. Mean, median, and percentile parameter estimates are presented for each mixture, sample size, and stock. Goodness-of-fit testing results are also provided and discussed.

[5] When data is censored, information is lost, as such larger data samples are required for reliable estimation. Note that censored mixtures which include a Weibull component considered different sample sizes to all other mixtures.

Chapter 13

Limit-Order Arrival Time Results

Both the specific data and an analysis procedure was introduced in Chapter 12 for the financial application central to this thesis. This study had access to limit order data between 1st June 2010 and 30th September 2010 for the following stocks:

- **ABF:** Associated British Foods
- **BARC:** Barclays
- **RIO:** Rio Tinto
- **RR:** Rolls-Royce Holding
- **SSE:** Scottish and Southern Energy Company
- **VOD:** Vodafone Group

Section 12.3 outlined a procedure to describe the full distribution of limit order inter-arrival times by use of mixed distributions. Because much of this data ($> 50\%$) is in fact "zero inflated", time differences of consecutive limit orders are difficult to model. This chapter presents the relevant results and discussions of the analysis. A representative sample of estimates for **BARC** stock and sample sizes $\boldsymbol{n = \{50, 100, 200, 500, 1000\}}$ is provided within this chapter, however the full set of estimates are provided in Appendix C.

13.1 Removal of "zero inflated" Data

Tabulated Estimates:
Tables [13.1 - 13.4] (Tables [C.1 - C.24] in appendix) present parameter estimates for the following mixtures and sample sizes,

- Exponential/Weibull
- Gamma/Weibull
- Loglogistic/Weibull
- Weibull/Weibull

$$\left.\right\} n = \{50, 100, 200, 500, 1000\} \,, \tag{13.1}$$

for the case when "zero inflated" data is removed entirely from the dataset. The mean and standard deviation parameter estimates (with respect to the interval estimates) are tabulated. Because the nature of data in many intervals (especially for smaller sample sizes) is largely variable, many outliers exist within the interval estimates. These outliers can largely distort the quoted mean and contribute towards large standard deviations. For this reason, the 16th percentile, the median, and the 84th percentile estimate are also tabulated, allowing for a better understanding of the full distribution of interval parameter estimates. In analyses of this type, the median estimate is often more useful for comparative purposes than the mean estimate anyway. Additional tabulated data includes the mean and percentile log-likelihood values, and the percentage of non-converging intervals.

n	$\hat\lambda_{ave}$	$\pm\hat\lambda_{ave}$	$\hat\lambda_{median}$	$\hat\lambda_{lower}$	$\hat\lambda_{upper}$	-	-	-	-	-	$\hat\alpha_{ave}$	$\pm\hat\alpha_{ave}$	$\hat\alpha_{median}$	$\hat\alpha_{lower}$	$\hat\alpha_{upper}$	$\hat\beta_{ave}$	$\pm\hat\beta_{ave}$	$\hat\beta_{median}$	$\hat\beta_{lower}$	$\hat\beta_{upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.259	0.164	0.223	0.108	0.413	-	-	-	-	-	336.670	293.082	254.628	121.761	538.605	0.832	0.858	0.728	0.603	0.966	0.410	0.184	0.394	0.590	0.184	0.606	-270.857	-317.996	-223.867	0.001
100	0.250	0.131	0.227	0.129	0.369	-	-	-	-	-	300.646	213.779	243.715	132.092	462.866	0.703	0.245	0.673	0.587	0.799	0.385	0.154	0.373	0.615	0.154	0.627	-546.249	-625.776	-467.494	0.000
200	0.244	0.104	0.228	0.148	0.338	-	-	-	-	-	280.715	167.575	238.790	140.189	418.536	0.655	0.098	0.645	0.579	0.727	0.370	0.132	0.359	0.630	0.132	0.641	-1098.073	-1235.960	-960.475	0.000
500	0.239	0.080	0.229	0.165	0.312	-	-	-	-	-	268.016	135.916	235.944	149.190	386.206	0.629	0.057	0.627	0.575	0.683	0.360	0.113	0.351	0.640	0.113	0.649	-2755.391	-3050.904	-2460.992	0.000
1000	0.236	0.068	0.229	0.173	0.299	-	-	-	-	-	263.249	122.237	235.324	153.957	371.598	0.619	0.047	0.617	0.574	0.664	0.356	0.102	0.347	0.644	0.102	0.653	-5519.866	-6060.022	-4983.277	0.000

Table 13.1: Exponential-Weibull mixture on BARC limit order arrival data: removal of zeros.

n	$\hat k_{ave}$	$\pm\hat k_{ave}$	$\hat k_{median}$	$\hat k_{lower}$	$\hat k_{upper}$	$\hat\theta_{ave}$	$\pm\hat\theta_{ave}$	$\hat\theta_{median}$	$\hat\theta_{lower}$	$\hat\theta_{upper}$	$\hat\alpha_{ave}$	$\pm\hat\alpha_{ave}$	$\hat\alpha_{median}$	$\hat\alpha_{lower}$	$\hat\alpha_{upper}$	$\hat\beta_{ave}$	$\pm\hat\beta_{ave}$	$\hat\beta_{median}$	$\hat\beta_{lower}$	$\hat\beta_{upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.037	1.894	2.404	1.984	3.669	0.931	0.148	0.952	0.784	1.080	235.748	208.967	175.358	81.343	395.595	0.659	0.257	0.634	0.543	0.765	0.281	0.160	0.251	0.719	0.160	0.749	-269.425	-316.921	-219.263	0.014
100	2.749	1.421	2.357	1.979	3.062	0.949	0.127	0.968	0.825	1.071	222.063	163.933	176.643	92.657	346.442	0.613	0.096	0.605	0.536	0.687	0.270	0.139	0.247	0.730	0.139	0.753	-544.480	-624.958	-464.293	0.001
200	2.558	1.016	2.337	1.991	2.843	0.961	0.107	0.976	0.856	1.063	214.267	135.153	179.638	102.190	323.001	0.593	0.062	0.590	0.534	0.650	0.265	0.123	0.247	0.735	0.123	0.753	-1094.691	-1234.630	-955.714	0.000
500	2.411	0.607	2.327	2.011	2.695	0.969	0.087	0.981	0.882	1.052	208.922	113.245	182.196	110.964	304.165	0.579	0.046	0.578	0.534	0.623	0.262	0.108	0.249	0.738	0.108	0.751	-2746.742	-3045.769	-2449.117	0.000
1000	2.365	0.438	2.331	2.016	2.646	0.971	0.078	0.983	0.891	1.045	206.493	102.478	183.359	116.343	294.513	0.573	0.040	0.573	0.534	0.612	0.261	0.101	0.249	0.739	0.101	0.751	-5502.352	-6050.760	-4962.160	0.000

Table 13.2: Gamma-Weibull mixture on BARC limit order arrival data: removal of zeros.

n	$\hat\alpha_{log\ ave}$	$\pm\hat\alpha_{log\ ave}$	$\hat\alpha_{log\ median}$	$\hat\alpha_{log\ lower}$	$\hat\alpha_{log\ upper}$	$\hat\beta_{log\ ave}$	$\pm\hat\beta_{log\ ave}$	$\hat\beta_{log\ median}$	$\hat\beta_{log\ lower}$	$\hat\beta_{log\ upper}$	$\hat\alpha_{ave}$	$\pm\hat\alpha_{ave}$	$\hat\alpha_{median}$	$\hat\alpha_{lower}$	$\hat\alpha_{upper}$	$\hat\beta_{ave}$	$\pm\hat\beta_{ave}$	$\hat\beta_{median}$	$\hat\beta_{lower}$	$\hat\beta_{upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.354	5.468	2.686	1.001	6.846	17.129	33.285	2.106	1.312	82.163	383.288	404.188	273.724	130.131	622.169	0.908	1.294	0.715	0.586	1.020	0.402	0.214	0.378	0.598	0.214	0.622	-268.743	-316.903	-219.395	0.008
100	3.622	4.173	2.483	1.000	5.365	21.119	36.434	2.070	1.342	86.576	324.877	284.561	250.490	127.770	509.938	0.728	0.591	0.661	0.567	0.814	0.374	0.188	0.356	0.626	0.188	0.644	-536.213	-620.973	-451.850	0.001
200	3.048	2.968	2.345	1.000	4.444	24.248	38.112	2.071	1.406	87.298	284.134	196.962	233.899	124.205	440.088	0.652	0.211	0.631	0.558	0.732	0.349	0.163	0.337	0.651	0.163	0.663	-1068.603	-1222.809	-916.096	0.000
500	2.658	2.080	2.274	1.000	3.778	25.442	38.123	2.055	1.496	85.706	258.532	151.111	224.311	122.874	392.417	0.617	0.070	0.609	0.554	0.681	0.329	0.141	0.328	0.671	0.141	0.672	-2669.069	-3012.470	-2329.633	0.000
1000	2.508	1.745	2.263	1.000	3.505	24.950	37.356	2.034	1.562	84.060	250.635	136.631	224.276	123.172	373.418	0.606	0.056	0.600	0.554	0.660	0.322	0.130	0.325	0.678	0.130	0.675	-5347.303	-5985.339	-4710.460	0.000

Table 13.3: Loglogistic-Weibull mixture on BARC limit order arrival data: removal of zeros.

n	$\hat\alpha_{1ave}$	$\pm\hat\alpha_{1ave}$	$\hat\alpha_{1median}$	$\hat\alpha_{1lower}$	$\hat\alpha_{1upper}$	$\hat\beta_{1ave}$	$\pm\hat\beta_{1ave}$	$\hat\beta_{1median}$	$\hat\beta_{1lower}$	$\hat\beta_{1upper}$	$\hat\alpha_{2ave}$	$\pm\hat\alpha_{2ave}$	$\hat\alpha_{2median}$	$\hat\alpha_{2lower}$	$\hat\alpha_{2upper}$	$\hat\beta_{2ave}$	$\pm\hat\beta_{2ave}$	$\hat\beta_{2median}$	$\hat\beta_{2lower}$	$\hat\beta_{2upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	5.792	7.128	3.963	2.142	11.457	1.740	1.388	1.434	1.031	2.993	322.536	295.388	238.297	114.025	683.363	0.807	0.755	0.699	0.588	1.081	0.382	0.184	0.359	0.618	0.184	0.641	-269.206	-314.595	-201.647	0.078
100	5.258	5.989	3.836	2.128	9.535	1.572	0.808	1.363	1.020	2.671	292.267	230.365	231.293	122.796	565.474	0.692	0.308	0.650	0.569	0.846	0.361	0.158	0.343	0.639	0.158	0.657	-544.308	-621.208	-433.882	0.072
200	4.723	4.310	3.740	2.132	8.244	1.517	0.600	1.335	1.029	2.486	266.748	174.454	223.641	129.141	491.592	0.637	0.118	0.620	0.556	0.749	0.342	0.134	0.329	0.658	0.134	0.671	-1095.092	-1229.027	-901.183	0.074
500	4.252	3.055	3.631	2.154	7.300	1.499	0.568	1.332	1.046	2.446	246.227	131.576	215.455	135.651	435.880	0.604	0.065	0.597	0.546	0.692	0.325	0.109	0.318	0.675	0.109	0.682	-2748.939	-3035.417	-2314.729	0.081
1000	4.041	2.386	3.616	2.231	6.982	1.476	0.535	1.330	1.062	2.332	238.476	116.440	213.175	138.820	414.087	0.592	0.053	0.586	0.543	0.668	0.317	0.096	0.312	0.683	0.096	0.688	-5508.060	-6029.852	-4678.674	0.086

Table 13.4: Weibull-Weibull mixture on BARC limit order arrival data: removal of zeros.

Convergence:
Some intervals didn't converge to an estimate, both of the typical form within a numerical root finding procedure, but also deemed if a threshold number of 1000 EM iterations was exceeded.[1] This was to avoid intervals with exceedingly slow convergence which could significantly compromise the overall run-time of the analysis.

The percentage of intervals which didn't converge to an estimate (for the case of removal of zeros) was negligible for all of the mixed distributions considered, i.e. $\sim$ 99% convergence for exponential/Weibull, gamma/Weibull, loglogistic/Weibull mixtures, and $\sim$ 90% for Weibull/Weibull mixtures.

Mixture Proportions:
Mixture proportion estimates displayed stability with respect to varying sample size n and varying stock. Recall Fig. 12.1 provided statistics indicating the percentage of data for each stock which was both zero inflated and fell within the $[0, 10]$ms region. For example, BARC possesses approximately 50% "zero inflated" data, an additional 20% of the data within the $[0, 10]$ms region, and the remaining 30% distributed in the intermediate and tail region. Once "zero inflated" data is removed, the mixture proportion estimates $\boldsymbol{\hat{\pi}_1 \approx 0.35}$ and $\boldsymbol{\hat{\pi}_2 \approx 0.65}$ display remarkable consistency with the ratio of data within the short time scale and intermediate/tail regions. This is an important observation, and is true for all stocks and mixed distributions. Inspection of the component parameter and mixture proportion estimates validates the claim that the first (non-Weibull) component attempts to describe excess data in the short time scale region, whilst the Weibull distribution describes intermediate and tail distribution data. High activity stocks (BARC, RIOTINTO) exhibited fractionally higher densities of data in the short time scale regions, as a consequence $\hat{\pi}_1$ estimates are fractionally greater for larger stocks.

Component Parameters:
Kizilersü et al. [19] found for left-truncated analysis that maximum entropy of the system for the Weibull distribution corresponded to the Euler-Mascheroni constant, which fell within the error bars for estimates of the shape parameter $\hat{\beta}$. For mixed distribution analysis (when data in the short time scale region is not truncated) this observation is recovered when parameter estimates of the first (non-Weibull) component are such that little probability density leaks into the intermediate and tail regions of the distribution. This allows the Weibull component to exclusively model this region with little compromise.

Figure 13.1 provides an example (BARC: first $n = 1000$ interval of 1st June 2010) of the probability density function for the converged mixtures, noting the use of a logarithmic scale for the vertical axis. The excess data apparent in the short time scale region is well represented by the inset figure which gives the axis on a linear scale. For this particular interval, at a macro level all component parameter estimation was such that all mixtures converged to a similar density function.
Figure 13.2 provides the corresponding cumulative distribution function, although the horizontal axis is now represented with a logarithmic scale.

For the case when "zero inflated" data is removed, gamma/Weibull mixtures led to estimates which restricted the most density to the short time scale region, recovering Kizilersü et al. [19] estimates most closely. Figures [13.3 - 13.6] display the Weibull shape parameter estimates for all stocks, sample sizes, and mixed distributions. Median values are represented on the figure with 68% percent of interval estimates falling within the error bars (defined by the lower and upper percentile estimates). The red dotted line represents the Euler-Mascheroni constant, and the black dotted line a weighted estimate, calculated as follows. If the mean estimate over all intervals (for a given stock, given sample size, and a given mixed distribution) is $\bar{\beta}_{s,n}$, where index s represents the stock, and index n the sample size (e.g. $\bar{\beta}_{\mathrm{BARC},100} = 0.613$ for the gamma/Weibull mixed distribution displayed in Table 13.2), the corresponding weighted mean $\bar{\beta}$ is given as

$$\bar{\beta} = \sum_{s,n} w_{s,n}\, \bar{\beta}_{s,n} \Big/ \sum_{s,n} w_{s,n} \ , \tag{13.2}$$

[1] Noting that this is a conservative upper bound. For example, for an exponential/Weibull mixture with BARC stock and a sample size $n = 50$, the mean number of EM iterations required for parameter convergence was 23.

with variance,

$$(\bar{\sigma})^2 = 1 \Big/ \sum_{s,n} w_{s,n} \ , \tag{13.3}$$

where the weights are given as

$$w_{s,n} = 1 \Big/ (\bar{\sigma}_{s,n})^2 \ . \tag{13.4}$$

Some further noteworthy observations of the component parameter estimates can be made. These include: **1.** Weibull scale parameter α is dependent on the inverse activity of the stock, as expected. **2.** Parameter estimates are relatively stable for varying sample size n. **3.** Loglogistic/Weibull estimates display the most significant variability for all mixed distributions. **4.** Gamma distribution estimates yield a scale parameter $\theta \approx 1$ which appears to be universal for all stocks.

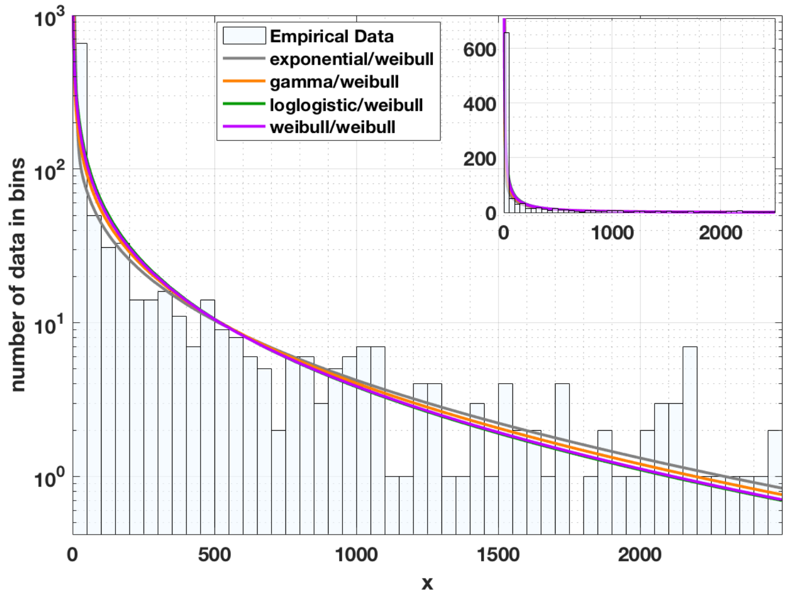


Figure 13.1: Probability density function for mixtures with BARC interval (first $n = 1000$ inter-arrival times from 1st June 2010).

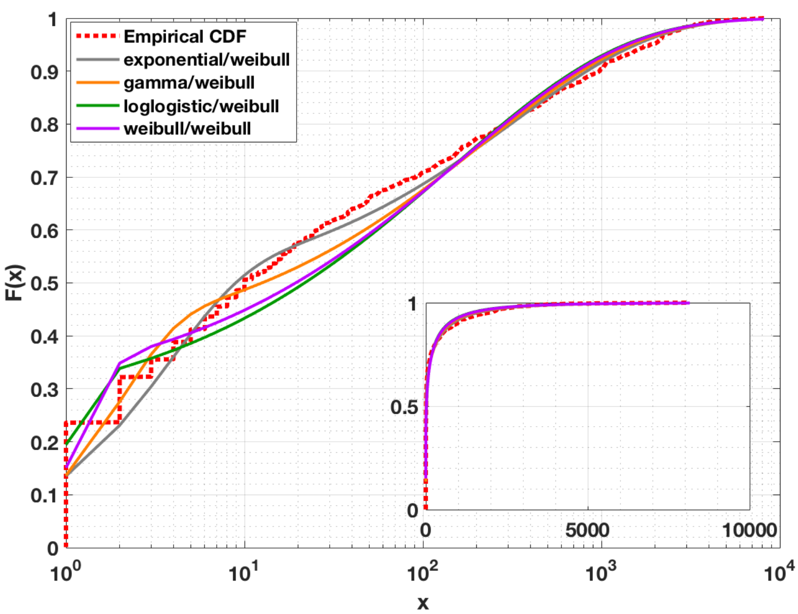


Figure 13.2: Cumulative distribution function for mixtures with BARC interval (first $n = 1000$ inter-arrival times from 1st June 2010).

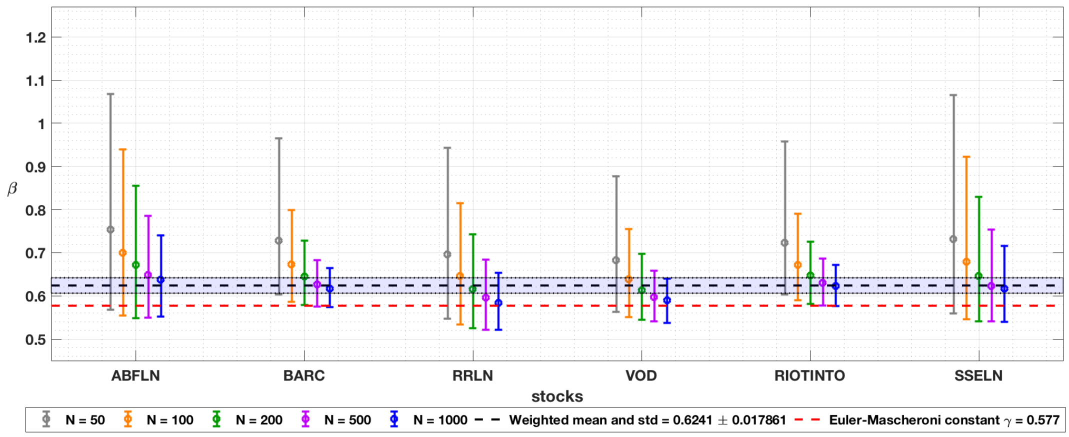


Figure 13.3: Exponential/Weibull mixture: removal of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

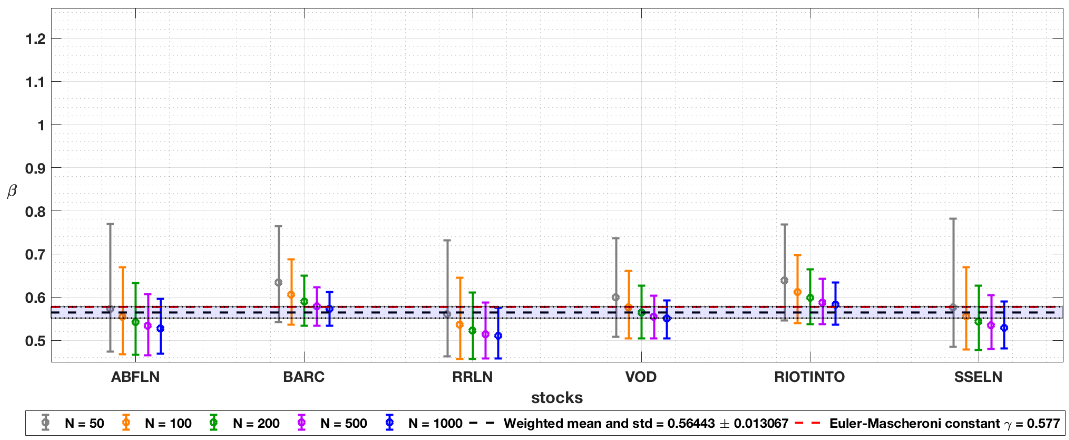


Figure 13.4: Gamma/Weibull mixture: removal of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

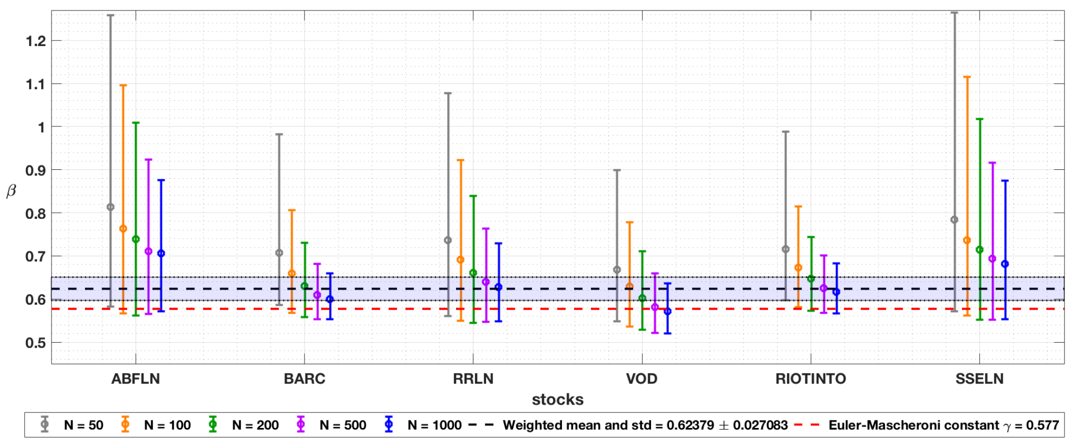


Figure 13.5: Loglogistic/Weibull mixture: removal of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

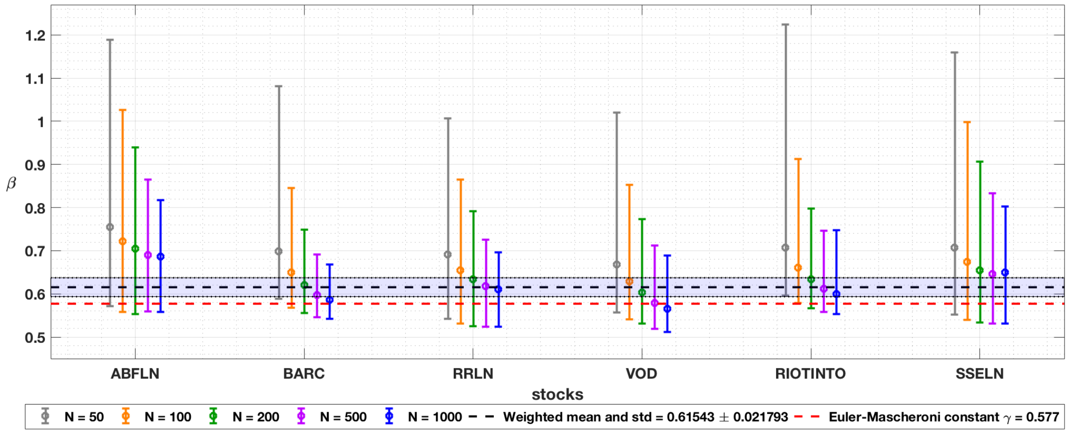


Figure 13.6: Weibull/Weibull mixture: removal of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

Goodness-of-fit Testing:
Model selection was undertaken via information criteria testing. Tables [13.5 - 13.9] (Tables [C.25 - C.54] in appendix) present Akaike Information Criterion and Bayesian Information Criterion results for each mixed distribution. For a given criterion test, quoted numbers indicate the percentage of total intervals for which each mixed distribution performed best. Recall that model selection is carried out by comparing log-likelihood values, but including a penalty function consistent with the complexity of the model. The exponential distribution is a singly parametrised distribution, hence for exponential/Weibull mixtures, k, the number of parameters in the distribution is equal to 4, eg. (λ, α, β, π_1). For all other considered mixed distributions which include doubly parametrised component distributions (gamma, loglogistic, and Weibull), the number of parameters $k = 5$. Once the first preference model has been determined, the second preference can be determined by removing the first from the testing procedure and repeating the process. The third and consequently last preference model can similarly be determined.

For this case, loglogistic/Weibull mixtures yielded estimates which most consistently described the limit order inter-arrival data most appropriately, although interval estimates for this mixture displayed significant variability. The remaining preferences were dependent on sample size n and stock.

	AIC			BIC		
Exponential/Weibull	0.2250	0.4520	-	0.5995	-	-
Gamma/Weibull	0.0169	0.2839	0.3899	0.0055	0.0206	0.3899
Loglogistic/Weibull	0.6807	-	-	0.3753	0.8324	-
Weibull/Weibull	0.0773	0.2639	0.6053	0.0196	0.1469	0.6053

Table 13.5: BARC, N = 50: limit order arrival times, removal of zeros.

	AIC			BIC		
Exponential/Weibull	0.0932	0.3440	0.5022	0.4899	0.7617	-
Gamma/Weibull	0.0086	0.4078	-	0.0041	0.1253	0.4688
Loglogistic/Weibull	0.8647	-	-	0.4979	-	-
Weibull/Weibull	0.0335	0.2482	0.4978	0.0081	0.1129	0.5309

Table 13.6: BARC, N = 100: limit order arrival times, removal of zeros.

	AIC			BIC		
Exponential/Weibull	0.0183	0.2467	0.4235	0.3080	0.6258	-
Gamma/Weibull	0.0027	0.5168	-	0.0016	0.2615	0.5475
Loglogistic/Weibull	0.9696	-	-	0.6880	-	-
Weibull/Weibull	0.0093	0.2365	0.5765	0.0024	0.1126	0.4524

Table 13.7: BARC, N = 200: limit order arrival times, removal of zeros.

	AIC			BIC		
Exponential/Weibull	0.0005	0.1431	0.3212	0.0768	0.4111	0.6326
Gamma/Weibull	0.0003	0.6324	-	0.0003	0.4902	-
Loglogistic/Weibull	0.9982	-	-	0.9225	-	-
Weibull/Weibull	0.0010	0.2244	0.6787	0.0004	0.0985	0.3673

Table 13.8: BARC, N = 500: limit order arrival times, removal of zeros.

	AIC			BIC		
Exponential/Weibull	0.0000	0.0879	0.2597	0.0084	0.2750	0.5231
Gamma/Weibull	0.0002	0.7102	-	0.0001	0.6344	-
Loglogistic/Weibull	0.9995	-	-	0.9913	-	-
Weibull/Weibull	0.0003	0.2018	0.7402	0.0002	0.0905	0.4768

Table 13.9: BARC, N = 1000: limit order arrival times, removal of zeros.

13.2 Distributing "zero inflated" Data Uniformly

Tabulated Estimates:
Tables [13.10 - 13.13] (Tables [C.55 - C.78] in appendix) present parameter estimates for the following mixtures and sample sizes,

- Exponential/Uniform/Weibull
- Gamma/Uniform/Weibull
- Loglogistic/Uniform/Weibull
- Weibull/Uniform/Weibull

$$n = \{50, 100, 200, 500, 1000\}, \qquad (13.5)$$

for the case when "zero inflated" data is distributed by a uniform density with parameter $\theta = 0.5$. Estimation of all parameters was undertaken from the data with the exception of θ, the parameter of the uniform density.[2] This was fixed at $\theta = 0.5$, consistent with the density which distributed the "zero inflated" data. Tabulated data includes the mean, standard deviation, median, and percentile parameter estimates, in addition to the mean and percentile log-likelihood values, and the percentage of non-converging intervals.

[2] Noting that the mixture proportion for the uniform density, π_2, was estimated from the data.

n	θ	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	3.893	584.917	0.474	0.159	1.531	-	-	-	-	-	313.616	395.771	192.760	60.734	554.951	1.040	2.348	0.723	0.514	1.197	0.221	0.157	0.191	0.449	0.196	0.485	0.329	0.152	0.320	-141.324	-191.692	-88.004	0.012
100	0.5	1.253	40.016	0.454	0.181	1.246	-	-	-	-	-	265.181	286.569	179.655	68.262	448.994	0.717	1.209	0.618	0.485	0.853	0.201	0.136	0.172	0.457	0.182	0.494	0.342	0.135	0.334	-286.368	-373.391	-197.106	0.003
200	0.5	0.699	6.774	0.452	0.211	1.125	-	-	-	-	-	228.114	207.399	168.172	74.268	374.763	0.598	0.395	0.565	0.474	0.704	0.185	0.121	0.157	0.459	0.172	0.499	0.355	0.120	0.346	-578.572	-730.191	-421.476	0.002
500	0.5	0.594	0.404	0.457	0.254	1.056	-	-	-	-	-	198.283	148.591	156.681	78.977	314.284	0.541	0.081	0.530	0.469	0.611	0.172	0.108	0.145	0.461	0.163	0.503	0.367	0.107	0.356	-1454.446	-1786.281	-1109.940	0.000
1000	0.5	0.593	0.362	0.457	0.282	1.043	-	-	-	-	-	185.520	122.926	152.130	82.229	287.853	0.523	0.060	0.517	0.468	0.576	0.167	0.102	0.141	0.461	0.159	0.504	0.373	0.101	0.360	-2917.584	-3501.274	-2273.318	0.000

Table 13.10: Exponential-Uniform-Weibull mixture on limit order arrival zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	2.595	2.496	1.871	0.732	4.598	0.952	0.121	0.963	0.868	1.054	234.132	311.374	138.486	42.281	423.049	0.757	1.614	0.623	0.466	0.911	0.166	0.136	0.130	0.458	0.194	0.492	0.376	0.150	0.363	-141.291	-191.533	-86.794	0.018
100	0.5	2.381	2.089	1.835	0.818	3.542	0.981	0.077	0.980	0.921	1.047	199.578	221.088	132.560	50.804	335.728	0.599	1.140	0.552	0.449	0.705	0.149	0.117	0.119	0.463	0.179	0.499	0.388	0.131	0.375	-286.812	-373.704	-197.423	0.004
200	0.5	2.147	1.665	1.784	0.889	2.938	0.994	0.054	0.989	0.948	1.045	175.652	162.359	128.642	57.973	285.017	0.534	0.351	0.518	0.443	0.615	0.139	0.103	0.113	0.465	0.168	0.503	0.396	0.118	0.382	-579.174	-730.832	-421.978	0.003
500	0.5	1.902	1.147	1.762	0.941	2.546	1.000	0.040	0.994	0.964	1.041	158.360	120.213	125.434	65.960	245.973	0.501	0.061	0.497	0.442	0.559	0.132	0.091	0.108	0.467	0.159	0.508	0.401	0.106	0.386	-1455.377	-1787.429	-1111.431	0.000
1000	0.5	1.786	0.848	1.763	0.963	2.410	1.001	0.035	0.997	0.968	1.036	150.863	98.095	124.764	70.865	227.878	0.490	0.048	0.488	0.443	0.537	0.129	0.085	0.107	0.468	0.154	0.509	0.403	0.100	0.387	-2919.108	-3505.272	-2275.038	0.000

Table 13.11: Gamma-Uniform-Weibull mixture on limit order arrival zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{\alpha}_{\text{log ave}}$	$\pm\hat{\alpha}_{\text{log ave}}$	$\hat{\alpha}_{\text{log median}}$	$\hat{\alpha}_{\text{log lower}}$	$\hat{\alpha}_{\text{log upper}}$	$\hat{\beta}_{\text{log ave}}$	$\pm\hat{\beta}_{\text{log ave}}$	$\hat{\beta}_{\text{log median}}$	$\hat{\beta}_{\text{log lower}}$	$\hat{\beta}_{\text{log upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	5.980	95.596	2.240	1.000	9.718	23.023	42.536	2.042	1.039	99.672	416.247	605.919	238.478	75.217	849.747	1.432	3.590	0.795	0.546	1.947	0.225	0.157	0.187	0.473	0.188	0.508	0.303	0.160	0.291	-139.516	-187.962	-79.834	0.048
100	0.5	4.712	55.295	2.143	1.000	6.295	23.024	41.074	1.728	0.955	92.248	368.140	495.404	227.139	78.935	639.195	0.968	1.909	0.675	0.509	1.146	0.214	0.145	0.183	0.479	0.173	0.514	0.308	0.151	0.301	-280.122	-366.509	-187.433	0.012
200	0.5	3.610	25.354	2.059	1.000	4.733	23.696	40.017	1.612	0.935	89.966	311.946	346.376	213.310	84.302	524.440	0.731	0.757	0.611	0.494	0.877	0.201	0.131	0.175	0.481	0.162	0.518	0.318	0.141	0.312	-564.433	-717.047	-404.273	0.004
500	0.5	2.537	3.947	1.960	1.000	3.518	26.352	39.993	1.626	0.967	88.572	256.734	217.725	194.844	88.703	422.942	0.607	0.214	0.566	0.485	0.710	0.182	0.114	0.162	0.484	0.153	0.522	0.334	0.130	0.326	-1416.848	-1751.074	-1067.391	0.000
1000	0.5	2.178	2.515	1.913	1.000	3.038	27.994	39.825	1.651	1.023	86.947	230.606	167.870	184.020	90.989	374.524	0.569	0.159	0.548	0.483	0.645	0.170	0.103	0.156	0.484	0.148	0.524	0.345	0.122	0.334	-2841.210	-3440.933	-2181.504	0.000

Table 13.12: Loglogistic-Uniform-Weibull mixture on limit order arrival zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	8.192	20.527	3.700	1.608	26.447	1.712	3.500	1.057	0.675	3.558	414.612	527.279	259.160	95.127	1202.681	1.545	3.510	0.851	0.588	3.636	0.257	0.160	0.225	0.459	0.191	0.494	0.284	0.154	0.270	-141.834	-187.297	-64.012	0.109
100	0.5	7.622	20.141	3.530	1.713	18.888	1.330	2.132	0.925	0.638	2.584	386.133	462.627	255.605	104.352	913.464	1.064	2.173	0.711	0.537	1.721	0.243	0.145	0.217	0.465	0.178	0.501	0.292	0.146	0.284	-286.764	-366.085	-165.065	0.088
200	0.5	6.484	18.466	3.303	1.807	16.474	1.126	1.088	0.865	0.632	2.347	340.882	367.491	240.716	111.139	795.515	0.802	1.133	0.632	0.512	1.229	0.230	0.129	0.208	0.466	0.167	0.504	0.304	0.137	0.297	-579.505	-715.503	-359.252	0.095
500	0.5	4.758	7.415	3.044	1.922	14.142	1.018	0.670	0.841	0.651	2.366	278.126	237.020	217.917	116.551	676.913	0.635	0.398	0.576	0.495	0.953	0.214	0.110	0.195	0.465	0.158	0.505	0.321	0.125	0.311	-1459.068	-1736.177	-914.848	0.118
1000	0.5	3.913	4.865	2.915	2.008	14.366	0.974	0.440	0.839	0.677	2.524	245.348	165.192	206.239	119.708	647.904	0.578	0.149	0.553	0.490	0.865	0.205	0.099	0.187	0.464	0.155	0.503	0.332	0.116	0.318	-2927.173	-3388.401	-1758.340	0.138

Table 13.13: Weibull-Uniform-Weibull mixture on limit order arrival zero inflated (uniformly distributed) BARC data.

Convergence:
Convergence for this case was similarly strong, $\sim 99\%$ convergence for exponential/uniform/Weibull, gamma/uniform/Weibull, and loglogistic/uniform/Weibull mixtures, and $\sim 95\%$ for Weibull/uniform/Weibull mixtures.

Mixture Proportions:
It is comforting that even when "zero inflated" data is included, mixture proportion estimates display relative stability with respect to both varying sample size n and varying stock. The contribution to the mixture of the second component (the uniform density) is approximately 50% (eg. $\pi_2 \sim 0.5$), consistent with the proportion of data which is "zero inflated".

Component Parameters:
From inspection of the component parameter estimates (and mixture proportion estimates) it can be reasoned that the first component of the mixtures was tasked with describing the transition region between the short time scale data, and the data in the intermediate and tail region. The uniform component was responsible for the short time scale region, whilst the data in the intermediate and tail region was described by the third component, the Weibull component. This is consistent for all sample sizes n and stocks. Figures [13.7 - 13.10] displays the Weibull shape parameter estimates for all stocks, sample sizes, and mixed distributions. Median values are represented on the figure with 68% percent of interval estimates falling within the error bars. The black dotted line represents the weighted estimate, and the red line the Euler-Mascheroni constant. Noting that in an attempt to keep the axis bounds consistent for each plot, some of the upper percentile estimates for loglogistic/uniform/Weibull and Weibull/uniform/Weibull are not displayed. These mixtures exhibited the largest variability in their estimates.

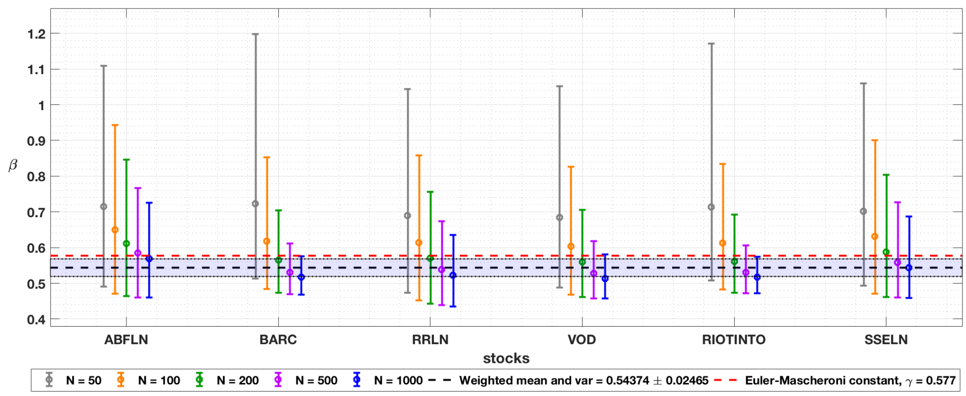


Figure 13.7: Exponential/Uniform/Weibull mixture: uniform distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

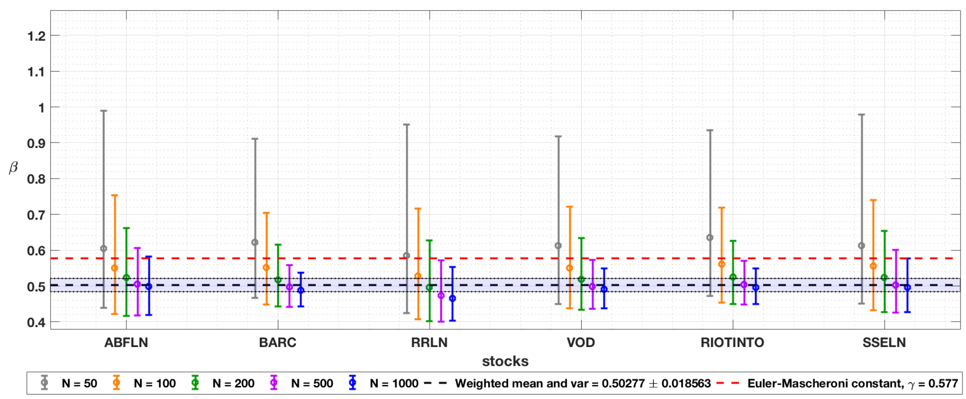


Figure 13.8: Gamma/Uniform/Weibull mixture: uniform distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

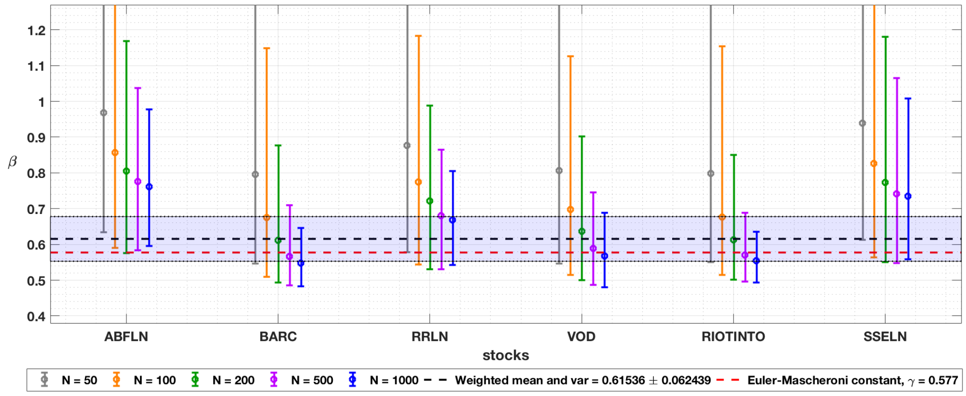


Figure 13.9: Loglogistic/Uniform/Weibull mixture: uniform distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

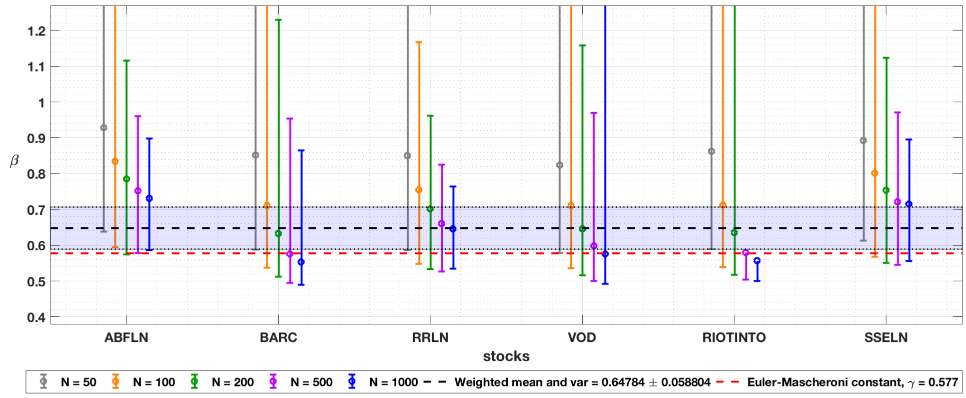


Figure 13.10: Weibull/Uniform/Weibull mixture: uniform distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

Goodness-of-fit Testing:

Tables [13.14 - 13.18] (Tables [C.79 - C.108] in appendix) present Akaike Information Criterion and Bayesian Information Criterion results for each mixed distribution. AIC indicated that both exponential/uniform/Weibull and loglogistic/uniform/Weibull mixtures displayed superior performance describing the full distribution of limit order inter-arrival times. The mixture which included the exponential density performed better for smaller sample sizes, whilst the loglogistic for larger. Model selection according to BIC, which imposes a slightly greater penalty on the complexity of a model, indicated exponential/uniform/Weibull mixtures performed best.

	AIC			BIC		
Exponential/Uniform/Weibull	0.4533	-	-	0.7670	-	-
Gamma/Uniform/Weibull	0.0240	0.0669	0.2495	0.0054	0.0669	0.2495
Loglogistic/Uniform/Weibull	0.3576	0.5262	-	0.2014	0.5262	-
Weibull/Uniform/Weibull	0.1618	0.4031	0.7419	0.0229	0.4031	0.7419

Table 13.14: BARC, N = 50: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
Exponential/Uniform/Weibull	0.4066	0.6795	-	0.7566	-	-
Gamma/Uniform/Weibull	0.0234	0.0615	0.2591	0.0013	0.0525	0.2591
Loglogistic/Uniform/Weibull	0.4236	-	-	0.2287	0.5996	-
Weibull/Uniform/Weibull	0.1449	0.2566	0.7380	0.0119	0.3463	0.7380

Table 13.15: BARC, N = 100: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
Exponential/Uniform/Weibull	0.3192	0.6395	-	0.7279	-	-
Gamma/Uniform/Weibull	0.0251	0.0846	0.2887	0.0008	0.0446	0.2887
Loglogistic/Uniform/Weibull	0.5279	-	-	0.2624	0.6819	-
Weibull/Uniform/Weibull	0.1258	0.2736	0.7089	0.0070	0.2715	0.7089

Table 13.16: BARC, N = 200: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
Exponential/Uniform/Weibull	0.1592	0.5574	-	0.6481	-	-
Gamma/Uniform/Weibull	0.0212	0.1389	0.3329	0.0004	0.0285	0.3329
Loglogistic/Uniform/Weibull	0.7301	-	-	0.3463	0.8098	-
Weibull/Uniform/Weibull	0.0894	0.3035	0.6668	0.0050	0.1616	0.6668

Table 13.17: BARC, N = 500: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
Exponential/Uniform/Weibull	0.0686	0.4854	-	0.5424	-	-
Gamma/Uniform/Weibull	0.0132	0.1823	0.3581	0.0006	0.0160	0.3581
Loglogistic/Uniform/Weibull	0.8592	-	-	0.4518	0.8926	-
Weibull/Uniform/Weibull	0.0589	0.3321	0.6417	0.0051	0.0914	0.6417

Table 13.18: BARC, N = 1000: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

13.3 Distributing "zero inflated" Data Exponentially

Tabulated Estimates:
Tables [13.19 - 13.22] (Tables [C.109 - C.132] in appendix) present parameter estimates for the following mixtures and sample sizes,

- Exponential/Exponential/Weibull
- Gamma/Exponential/Weibull
- Loglogistic/Exponential/Weibull
- Weibull/Exponential/Weibull

$$n = \{50, 100, 200, 500, 1000\}, \tag{13.6}$$

for the case when "zero inflated" data is now distributed by an exponential density. A rate parameter $\lambda = 8$ was chosen such that the distribution of density for the replaced "zero inflated" data was almost entirely restricted to the $[0, 0.5]$ms region. Estimation of all parameters was undertaken from the data, including the rate parameter of the second mixture component (the exponential component). Tabulated data once again includes the mean, standard deviation, median, and percentile parameter estimates, in addition to the mean and percentile log-likelihood values, and the percentage of non-converging intervals.

Convergence:
For this case, convergence is once again strong, although not as strong as previous cases, $\sim 99\%$ convergence for exponential/exponential/Weibull and gamma/exponential/Weibull (gamma/exponential/Weibull not as strong for smaller sample sizes), and $\sim 90\%$ for loglogistic/exponential/Weibull and Weibull/exponential/Weibull mixtures.

Mixture Proportions:
The mixture proportions once again display strong stability with respect to varying sample size n and varying stock. The contribution to the mixture of the second component (now the exponential density) is approximately 50% (eg. $\pi_2 \sim 0.5$), similar to the behaviour of the uniform density in Section 13.2. The mixture proportion estimates for the exponential component are consistent with the proportion of data which is "zero inflated".

Component Parameters:
The convergence of parameter estimates was such that the second component (exponential) provided the density required to describe the excess data in the short time scale region. The first component was responsible for the transition region, and the Weibull component the intermediate and tail regions. This behaviour is consistent for all sample sizes n and stocks. Convergence of parameters for exponential/exponential/Weibull and gamma/exponential/Weibull mixtures were such that the density contribution of the first two components was most restricted to the short time scale region. For these mixtures, the left-truncated estimates of Kizilersü et al. [19] for the Weibull distribution were most closely recovered. Figures [13.11 - 13.14] display the Weibull shape parameter estimates for all stocks, sample sizes, and mixed distributions. Median values are represented on the figure with 68% percent of interval estimates falling within the error bars. The black dotted line once again represents the weighted estimate, and the red dotted line the Euler-Mascheroni constant. Similarly note that in an attempt to keep the axis bounds consistent for each plot some of the upper percentile estimates for the loglogistic/uniform/Weibull and Weibull/uniform/Weibull mixtures are once again not displayed. These mixtures exhibited significant variability in their estimates.

Goodness-of-fit Testing:
Tables [13.23 - 13.27] (Tables [C.133 - C.162] in appendix) present Akaike Information Criterion and Bayesian Information Criterion results for each mixed distribution. AIC indicated that loglogistic/exponential/Weibull mixtures were superior. BIC indicated that for smaller samples sizes ($n = 50, 100$) exponential/exponential/Weibull mixtures performed best under model selection, otherwise loglogistic/exponential/Weibull mixtures prevailed.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	-	-	-	-	-	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	2.075	2.679	0.935	0.276	4.819	-	-	-	-	-	17.060	1457.169	9.939	7.718	14.439	263.137	341.494	158.640	46.841	464.038	0.876	1.962	0.657	0.473	1.021	0.242	0.143	0.216	0.399	0.166	0.423	0.359	0.149	0.353	-130.273	-183.633	-74.464	0.009
100	1.674	2.262	0.800	0.287	2.441	-	-	-	-	-	12.892	67.012	9.740	8.115	12.518	226.059	245.185	150.986	54.017	386.599	0.647	0.900	0.576	0.450	0.781	0.213	0.126	0.187	0.417	0.158	0.446	0.370	0.132	0.365	-264.853	-356.547	-170.728	0.003
200	1.311	1.810	0.694	0.301	1.867	-	-	-	-	-	11.509	31.990	9.584	8.369	11.530	201.095	187.718	145.514	60.169	336.437	0.562	0.308	0.536	0.444	0.667	0.192	0.111	0.166	0.431	0.151	0.463	0.378	0.118	0.371	-535.756	-695.079	-369.267	0.002
500	0.970	1.158	0.611	0.324	1.548	-	-	-	-	-	10.359	9.322	9.448	8.606	10.767	179.886	140.399	139.461	66.893	291.640	0.518	0.081	0.508	0.444	0.591	0.175	0.097	0.151	0.441	0.145	0.476	0.384	0.105	0.376	-1347.705	-1692.080	-983.085	0.000
1000	0.851	0.792	0.577	0.344	1.478	-	-	-	-	-	10.146	4.712	9.401	8.748	10.415	169.396	117.279	136.709	71.002	268.311	0.502	0.060	0.496	0.445	0.559	0.169	0.091	0.145	0.443	0.142	0.480	0.388	0.098	0.378	-2703.942	-3307.073	-2020.932	0.000

Table 13.19: Exponential-Exponential-Weibull mixture on limit order arrival zero inflated (exponentially distributed) BARC data.

n	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	2.646	2.689	1.827	0.576	5.191	0.958	0.104	0.967	0.888	1.046	10.980	85.708	9.127	6.981	12.633	234.566	313.984	136.907	39.487	431.212	0.757	2.130	0.622	0.454	0.927	0.169	0.130	0.138	0.453	0.183	0.477	0.378	0.149	0.366	-130.307	-183.121	-72.585	0.019
100	2.333	2.216	1.737	0.624	3.731	0.984	0.068	0.982	0.931	1.041	10.157	23.369	9.143	7.561	11.458	195.784	222.644	127.641	46.067	333.316	0.592	1.859	0.543	0.432	0.704	0.156	0.111	0.132	0.450	0.168	0.478	0.394	0.131	0.381	-265.178	-356.755	-170.687	0.004
200	2.027	1.761	1.654	0.655	2.854	0.995	0.048	0.990	0.953	1.040	9.752	6.490	9.156	7.977	10.864	168.532	161.946	121.177	52.072	276.223	0.521	0.405	0.505	0.425	0.609	0.147	0.095	0.128	0.449	0.155	0.479	0.404	0.117	0.390	-536.084	-695.483	-369.981	0.002
500	1.720	1.195	1.595	0.673	2.395	1.000	0.037	0.995	0.966	1.037	9.537	2.290	9.141	8.327	10.365	149.199	116.460	116.737	59.265	234.174	0.487	0.065	0.483	0.423	0.550	0.141	0.081	0.124	0.449	0.144	0.481	0.410	0.106	0.396	-1348.327	-1693.472	-984.607	0.000
1000	1.589	0.878	1.601	0.674	2.245	1.001	0.033	0.998	0.969	1.034	9.497	1.896	9.133	8.493	10.125	141.003	94.854	115.586	64.247	215.190	0.476	0.052	0.473	0.424	0.527	0.139	0.073	0.123	0.449	0.138	0.483	0.413	0.099	0.397	-2705.351	-3306.636	-2022.287	0.000

Table 13.20: Gamma-Exponential-Weibull mixture on limit order arrival zero inflated (exponentially distributed) BARC data.

n	$\hat{\alpha}_{\log\ \text{ave}}$	$\pm\hat{\alpha}_{\log\ \text{ave}}$	$\hat{\alpha}_{\log\ \text{median}}$	$\hat{\alpha}_{\log\ \text{lower}}$	$\hat{\alpha}_{\log\ \text{upper}}$	$\hat{\beta}_{\log\ \text{ave}}$	$\pm\hat{\beta}_{\log\ \text{ave}}$	$\hat{\beta}_{\log\ \text{median}}$	$\hat{\beta}_{\log\ \text{lower}}$	$\hat{\beta}_{\log\ \text{upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.497	48.468	1.985	1.000	8.088	27.799	440.428	2.507	1.099	102.843	10.890	40.081	8.941	7.047	12.786	393.901	569.184	223.907	68.254	802.163	1.307	3.006	0.763	0.527	1.753	0.215	0.155	0.174	0.469	0.181	0.499	0.316	0.160	0.307	-128.001	-179.006	-65.609	0.046
100	3.785	27.545	1.833	1.000	5.282	27.485	43.573	1.961	0.970	95.855	16.324	2744.989	8.948	7.554	11.093	348.540	470.105	211.105	68.989	614.447	0.912	2.123	0.652	0.488	1.093	0.210	0.145	0.177	0.468	0.170	0.502	0.322	0.154	0.317	-257.454	-348.280	-159.695	0.012
200	2.900	20.212	1.714	1.000	3.892	28.015	42.308	1.704	0.926	92.541	9.632	7.464	8.969	7.943	10.459	298.606	340.187	198.725	72.785	514.765	0.703	0.674	0.593	0.473	0.856	0.201	0.131	0.175	0.467	0.158	0.502	0.333	0.147	0.327	-519.397	-679.105	-350.735	0.004
500	2.036	2.867	1.611	1.000	2.874	29.222	41.298	1.575	0.928	89.291	9.296	1.877	9.000	8.279	10.040	249.380	221.632	182.557	77.864	423.205	0.592	0.199	0.551	0.466	0.705	0.187	0.112	0.172	0.467	0.148	0.503	0.346	0.138	0.337	-1305.972	-1655.791	-937.810	0.001
1000	1.761	1.562	1.575	1.000	2.492	29.504	40.480	1.522	0.953	87.351	9.235	1.131	9.017	8.412	9.846	228.005	176.633	175.684	79.959	387.027	0.558	0.113	0.536	0.466	0.647	0.179	0.101	0.170	0.467	0.142	0.503	0.354	0.133	0.342	-2622.561	-3242.641	-1930.555	0.001

Table 13.21: Loglogistic-Exponential-Weibull mixture on limit order arrival zero inflated (exponentially distributed) BARC data.

n	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\alpha}_{2\text{ave}}$	$\pm\hat{\alpha}_{2\text{ave}}$	$\hat{\alpha}_{2\text{median}}$	$\hat{\alpha}_{2\text{lower}}$	$\hat{\alpha}_{2\text{upper}}$	$\hat{\beta}_{2\text{ave}}$	$\pm\hat{\beta}_{2\text{ave}}$	$\hat{\beta}_{2\text{median}}$	$\hat{\beta}_{2\text{lower}}$	$\hat{\beta}_{2\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	7.356	16.451	3.308	1.464	32.857	1.830	3.858	1.113	0.662	5.845	11.512	56.256	9.132	7.194	21.287	408.236	523.717	254.595	93.337	1493.408	1.500	3.375	0.841	0.581	5.308	0.257	0.158	0.226	0.453	0.182	0.481	0.289	0.154	0.277	-131.063	-178.071	-36.334	0.130
100	6.703	18.488	3.090	1.533	22.903	1.357	2.490	0.899	0.597	2.831	10.718	32.202	9.147	7.731	14.797	383.687	470.807	252.245	101.909	1097.886	1.052	2.116	0.705	0.527	2.206	0.252	0.146	0.229	0.451	0.170	0.481	0.297	0.148	0.290	-265.991	-347.376	-118.257	0.114
200	5.536	17.411	2.832	1.583	22.027	1.132	14.472	0.796	0.582	2.552	10.200	15.717	9.183	8.139	14.680	339.529	376.434	239.486	108.377	993.351	0.786	1.004	0.628	0.502	1.575	0.245	0.130	0.227	0.447	0.158	0.479	0.308	0.140	0.302	-538.982	-677.359	-259.609	0.125
500	3.947	6.222	2.557	1.663	58.022	0.937	0.551	0.749	0.593	3.302	9.855	10.999	9.210	8.509	31.628	277.933	234.367	218.181	115.193	1727.155	0.623	0.258	0.571	0.486	2.625	0.232	0.108	0.219	0.442	0.148	0.476	0.326	0.129	0.315	-1363.836	-1639.289	-292.540	0.158
1000	3.161	3.817	2.428	1.748	-	0.893	0.478	0.736	0.612	-	9.703	2.080	9.234	8.700	-	247.327	164.690	208.530	120.194	-	0.573	0.170	0.547	0.482	-	0.225	0.093	0.215	0.438	0.143	0.473	0.337	0.122	0.321	-2745.949	-3194.102	-	0.189

Table 13.22: Weibull-Exponential-Weibull mixture on limit order arrival zero inflated (exponentially distributed) BARC data.

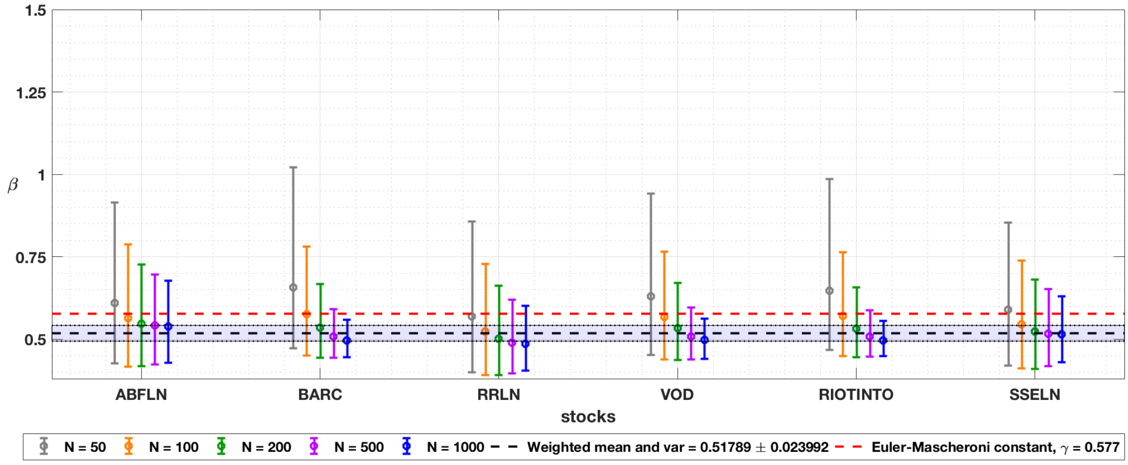


Figure 13.11: Exponential/Exponential/Weibull mixture: exponential distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

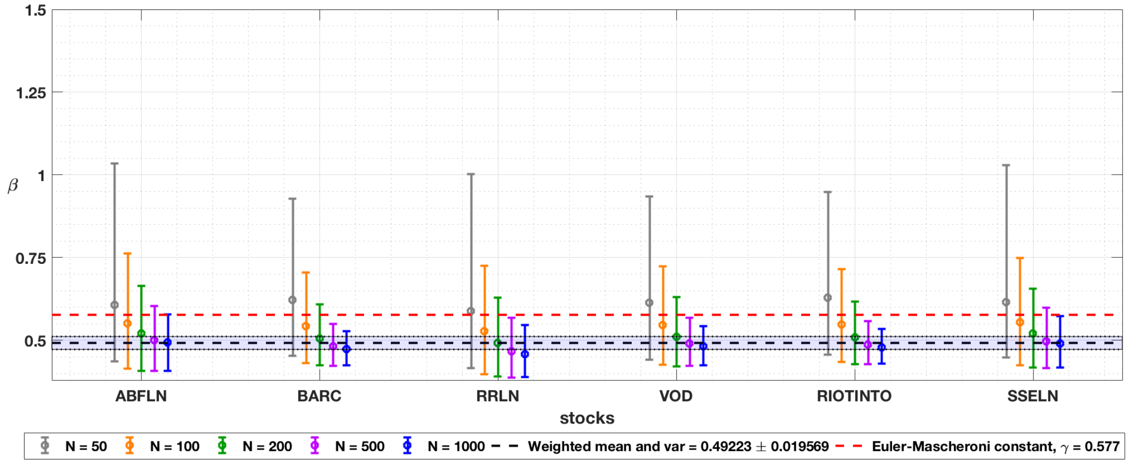


Figure 13.12: Gamma/Exponential/Weibull mixture: exponential distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

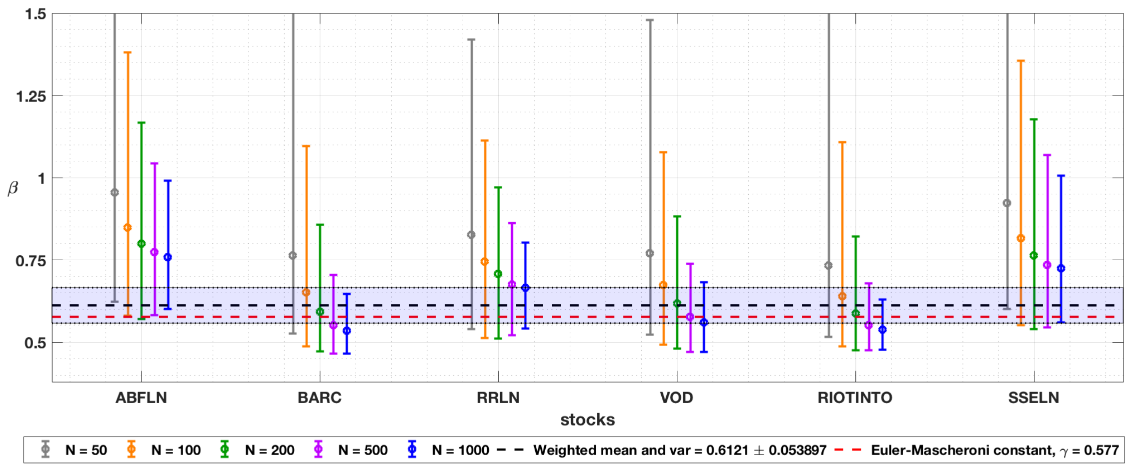


Figure 13.13: Loglogistic/Exponential/Weibull mixture: exponential distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

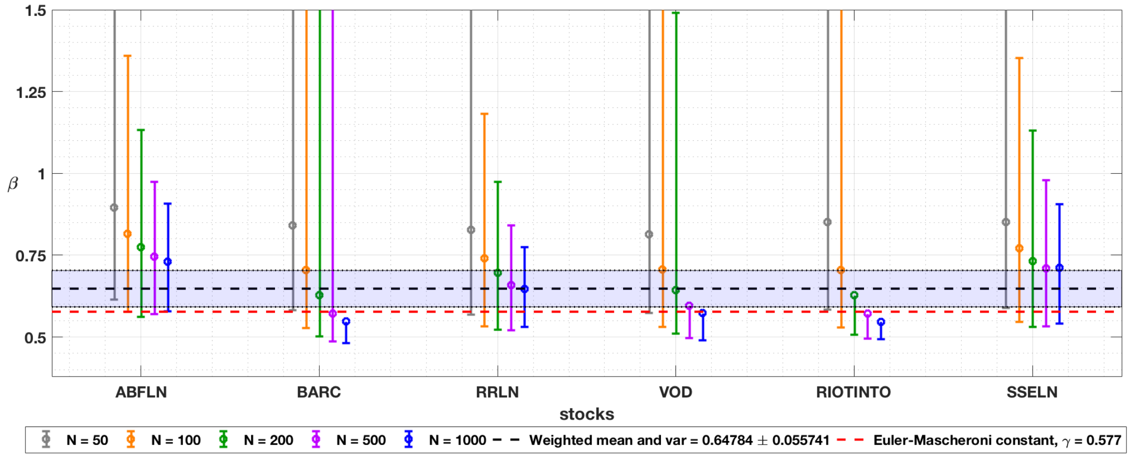


Figure 13.14: Weibull/Exponential/Weibull mixture: exponential distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

	AIC			BIC		
Exponential/Exponential/Weibull	0.2247	0.3834	-	0.3816	-	-
Gamma/Exponential/Weibull	0.1666	0.2845	0.4863	0.1243	0.2356	0.4863
Loglogistic/Exponential/Weibull	0.3963	-	-	0.3335	0.4782	-
Weibull/Exponential/Weibull	0.2094	0.3283	0.5050	0.1575	0.2824	0.5050

Table 13.23: BARC, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
Exponential/Exponential/Weibull	0.2017	0.3721	-	0.3325	0.5455	-
Gamma/Exponential/Weibull	0.1605	0.3006	0.5018	0.1248	0.2134	0.5018
Loglogistic/Exponential/Weibull	0.4390	-	-	0.3858	-	
Weibull/Exponential/Weibull	0.1972	0.3252	0.4953	0.1552	0.2390	0.4953

Table 13.24: BARC, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
Exponential/Exponential/Weibull	0.1853	0.3669	-	0.2845	0.5135	-
Gamma/Exponential/Weibull	0.1516	0.3124	0.5155	0.1249	0.2368	0.5155
Loglogistic/Exponential/Weibull	0.4756	-	-	0.4352	-	-
Weibull/Exponential/Weibull	0.1857	0.3187	0.4822	0.1536	0.2477	0.4822

Table 13.25: BARC, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
Exponential/Exponential/Weibull	0.1668	0.3670	-	0.2341	0.4787	-
Gamma/Exponential/Weibull	0.1400	0.3242	0.5389	0.1214	0.2647	0.5389
Loglogistic/Exponential/Weibull	0.5214	-	-	0.4956	-	-
Weibull/Exponential/Weibull	0.1716	0.3086	0.4608	0.1488	0.2563	0.4608

Table 13.26: BARC, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
Exponential/Exponential/Weibull	0.1603	0.3651	-	0.2128	0.4562	
Gamma/Exponential/Weibull	0.1300	0.3306	0.5519	0.1148	0.2796	0.5519
Loglogistic/Exponential/Weibull	0.5397	-	-	0.5199	-	-
Weibull/Exponential/Weibull	0.1699	0.3041	0.4479	0.1523	0.2641	0.4479

Table 13.27: BARC, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

13.4 g-component Exponential Mixture

Tabulated Estimates:
Tables [13.28 -13.31] (Tables [C.163 - C.186] in appendix) present parameter estimates for the following g-component exponential mixtures and sample sizes,

- 2-component Exponential
- 4-component Exponential
- 6-component Exponential
- 10-component Exponential

$$n = \{50, 100, 200, 500, 1000\} \ . \tag{13.7}$$

Recall the g-component exponential mixture was motivated by the work of Scalas [79] who proposed a finite exponential mixture to model the distribution of waiting times. "Zero inflated" data is once again distributed according to an exponential distribution with rate parameter $\lambda = 8$, as adopted by the previous case presented in Section 13.3. Although the number of components in a mixed distribution was initially motivated by the number of sub-populations within an overall population, this framework views the number of components from a purely modelling perspective. The task at hand is to determine the minimum number of components which adequately describes the data. Estimation of all parameters was undertaken from the data. For mixtures with number of components $g = (2,\ 4)$ the mean, standard deviation, percentile, and median parameter estimates are tabulated. Due to space restrictions on the page, only the mean estimates are quoted for $g = (6,\ 10)$. Mean and percentile log-likelihood values, in addition to the percentage of non-converging intervals is provided for all g-component mixtures.

Convergence:
A numerically tractable closed form expression exists for maximum likelihood estimation of exponential parameters using the EM algorithm. Almost the entirety of intervals therefore contributed converged estimates, i.e. $\sim 100\%$ convergence for $g = (2,\ 4,\ 6,\ 10)$.

Mixture Proportions:
Mixture proportion estimates once again displayed strong stability with respect to both varying sample size n and varying stock.

Component Parameters:
This thesis adopted the convention for parametrising the exponential distribution with rate parameter λ. As a consequence, larger λ parameters correspond to greater density in the short time scale region, whilst smaller λ parameters are responsible for density in the tail region. The labelling convention is such that the λ parameters are in increasing order. Hence, using 4-component mixtures as an example, the first component is responsible for density in the tail region, the next two in the intermediate and transition region, and the final component in the short time scale region. This is in contrast to the convention adopted by the other cases where the final component describes density in the tail region. Consistency could have been maintained by parametrising the exponential distribution by scale parameter $\beta = 1/\lambda$.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.03413	0.37296	0.00456	5.14064	3.58199	4.98311	0.37949	0.13212	0.37284	0.62051	0.13212	0.62716	-139.49738	-195.84216	-83.71608	0.00001
100	0.01877	0.26967	0.00411	4.49593	2.70410	4.67423	0.37277	0.11057	0.36887	0.62723	0.11057	0.63113	-287.34362	-384.77984	-191.74625	0.00000
200	0.00756	0.11396	0.00381	4.15122	2.16371	4.53845	0.37057	0.09485	0.36721	0.62943	0.09485	0.63279	-585.60133	-758.24213	-415.70564	0.00000
500	0.00435	0.00317	0.00358	3.93425	3.48873	4.43607	0.37110	0.08013	0.36779	0.62890	0.08013	0.63221	-1486.32475	-1868.67972	-1109.09630	0.00002
1000	0.00406	0.00254	0.00347	3.87449	1.76310	4.40151	0.37200	0.07169	0.36726	0.62800	0.07169	0.63274	-2993.64110	-3682.77178	-2277.71842	0.00000

Table 13.28: 2-component Exponential mixture on limit order arrival BARC data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.02998	0.38940	0.00283	0.49703	1.18234	0.13931	2.83978	3.02980	1.46666	16.07222	234.86039	10.06810	0.23900	0.12918	0.22608	0.18046	0.09700	0.16503	0.17883	0.11514	0.15157	0.40172	0.16313	0.43415	-129.14818	-183.62245	-74.44368	0.00004
100	0.01544	0.26810	0.00232	0.22557	0.63270	0.07656	2.08935	2.49102	1.13813	12.80196	84.55765	9.85461	0.21279	0.10831	0.20122	0.17974	0.08086	0.16929	0.18445	0.11190	0.15935	0.42302	0.15604	0.45710	-263.60518	-356.58086	-170.12925	0.00008
200	0.00497	0.11446	0.00200	0.10216	0.32763	0.04592	1.35430	1.72541	0.85523	11.80192	298.52004	9.66260	0.19003	0.08971	0.18025	0.17771	0.06730	0.16957	0.19021	0.10531	0.16763	0.44205	0.15051	0.47598	-533.68889	-695.22141	-368.78767	0.00008
500	0.00211	0.00152	0.00177	0.04657	0.10416	0.03121	0.84629	0.86469	0.63146	10.26019	25.26063	9.47419	0.17175	0.07217	0.16368	0.17568	0.05508	0.16900	0.19782	0.09612	0.17478	0.45475	0.14574	0.49062	-1347.30019	-1693.76091	-982.25405	0.00017
1000	0.00193	0.00127	0.00167	0.03482	0.03671	0.02707	0.71186	0.52768	0.55049	10.03039	4.21464	9.39380	0.16453	0.06428	0.15611	0.17529	0.04948	0.16871	0.20134	0.09110	0.17866	0.45884	0.14261	0.49685	-2705.85656	-3312.19432	-2020.66859	0.00030

Table 13.29: 4-component Exponential mixture on limit order arrival BARC data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.02930	0.12509	0.39150	2.27136	4.35217	22.98642	0.19108	0.13210	0.08839	0.09884	0.12262	0.36697	-128.45945	-182.79798	-73.97341	0.00002
100	0.01515	0.07331	0.28778	1.79972	4.21927	16.45713	0.17370	0.13970	0.09603	0.09529	0.11197	0.38331	-262.56093	-355.10328	-169.32387	0.00003
200	0.00479	0.04471	0.22412	1.42958	4.11062	12.58396	0.15423	0.14490	0.10526	0.09551	0.10150	0.39860	-532.11056	-693.20259	-367.36941	0.00007
500	0.00170	0.01964	0.15281	1.11347	3.93793	10.51970	0.12758	0.14842	0.11637	0.10182	0.09320	0.41260	-1343.42722	-1689.22459	-979.90349	0.00010
1000	0.00146	0.01199	0.11224	0.97613	3.75846	10.32437	0.11035	0.14989	0.12083	0.10756	0.09255	0.41882	-2698.03162	-3304.22431	-2014.24654	0.00015

Table 13.30: 6-component Exponential mixture on limit order arrival BARC data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\lambda}_{7\text{ ave}}$	$\hat{\lambda}_{8\text{ ave}}$	$\hat{\lambda}_{9\text{ ave}}$	$\hat{\lambda}_{10\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	$\hat{\pi}_{7\text{ ave}}$	$\hat{\pi}_{8\text{ ave}}$	$\hat{\pi}_{9\text{ ave}}$	$\hat{\pi}_{10\text{ ave}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.02940	0.11588	0.28110	1.06126	1.73925	2.91689	4.97921	6.76868	10.26250	42.16893	0.18851	0.12611	0.06611	0.04867	0.04810	0.05121	0.06170	0.07767	0.15466	0.17725	-128.32281	-182.52133	-73.65309	0.00052
100	0.01534	0.06909	0.20929	0.84188	1.41631	2.47410	5.01668	7.20746	10.10208	34.37364	0.17060	0.13459	0.07379	0.04964	0.04839	0.04986	0.05849	0.07627	0.15790	0.18047	-262.28164	-354.60498	-168.99956	0.00083
200	0.00460	0.04092	0.16737	0.72933	1.19482	2.01596	5.15060	7.66971	10.11495	19.91667	0.15041	0.14084	0.08446	0.05123	0.04942	0.04950	0.05541	0.07534	0.15830	0.18508	-531.70872	-692.11054	-366.49899	0.00048
500	0.00166	0.01798	0.11963	0.65160	1.05827	1.52852	5.32124	8.09218	10.19384	13.16869	0.12279	0.14562	0.09968	0.05489	0.05183	0.05057	0.05205	0.07481	0.15740	0.19036	-1343.06282	-1688.04561	-978.84875	0.00012
1000	0.00142	0.01095	0.09144	0.60937	1.00914	1.28981	5.43168	8.24653	10.21141	10.57059	0.10542	0.14702	0.10651	0.05929	0.05424	0.05240	0.05051	0.07506	0.15728	0.19227	-2697.26955	-3305.82595	-2013.70371	0.00015

Table 13.31: 10-component Exponential mixture on limit order arrival BARC data.

Goodness-of-fit Testing:

Tables [13.32 - 13.36] (Tables [C.187 - C.216] in appendix) present Akaike Information Criterion and Bayesian Information Criterion results for each g-component exponential mixture. Testing indicated that 4-component exponential mixtures most consistently provided the most suitable balance between goodness-of-fit and complexity of model. These mixtures were followed by 6-component exponential mixtures, and then for smaller sample sizes 2-component mixtures, and larger sample sizes 10-component mixtures. Tables [13.37 - 13.41] present additional information criteria results comparing the best two performing mixtures from Section 13.3, to the 4 and 6-component exponential mixture. Results indicated that mixtures which included a Weibull component better described the full distribution of limit order inter-arrival times, supporting the work of Kizilersü et al. [19].

	AIC			**BIC**		
2-comp-Exponential	0.1960	0.3743	0.7211	0.7338	-	-
4-comp-Exponential	0.5579	-	-	0.2639	0.9810	-
6-comp-Exponential	0.2274	0.5633	-	0.0023	0.0190	0.9999
10-comp-Exponential	0.0188	0.0624	0.2789	0.0000	0.0000	0.0000

Table 13.32: BARC, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			**BIC**		
2-comp-Exponential	0.0513	0.1108	0.2975	0.3810	0.7938	-
4-comp-Exponential	0.5742	-	-	0.6013	-	-
6-comp-Exponential	0.3147	0.7253	-	0.0177	0.2062	0.9998
10-comp-Exponential	0.0597	0.1639	0.7025	0.0000	0.0000	0.0002

Table 13.33: BARC, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			**BIC**		
2-comp-Exponential	0.0070	0.0168	0.0644	0.1127	0.3618	0.8749
4-comp-Exponential	0.5232	-	-	0.8249	-	-
6-comp-Exponential	0.3567	0.7261	-	0.0625	0.6378	-
10-comp-Exponential	0.1131	0.2571	0.9356	0.0000	0.0004	0.1251

Table 13.34: BARC, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			**BIC**		
2-comp-Exponential	0.0002	0.0006	0.0023	0.0060	0.0409	0.2482
4-comp-Exponential	0.4123	-	-	0.8385	-	-
6-comp-Exponential	0.3973	0.6611	-	0.1553	0.9487	-
10-comp-Exponential	0.1901	0.3383	0.9977	0.0002	0.0103	0.7517

Table 13.35: BARC, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-Exponential	0.0001	0.0001	0.0002	0.0001	0.0013	0.0328
4-comp-Exponential	0.3284	0.5462	-	0.7511	-	-
6-comp-Exponential	0.4173	-	-	0.2455	0.9628	-
10-comp-Exponential	0.2541	0.4536	0.9997	0.0033	0.0358	0.9671

Table 13.36: BARC, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
4-comp-Exponential	0.2161	0.3806	0.7076	0.1710	0.3798	0.9810
6-comp-Exponential	0.0765	0.1430	0.2924	0.0010	0.0036	0.0190
Exponential/Exponential/Weibull	0.2635	0.4763	-	0.4518	-	-
Loglogistic/Exponential/Weibull	0.4438	-	-	0.3762	0.6166	-

Table 13.37: BARC, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
4-comp-Exponential	0.2048	0.3806	0.6418	0.1782	0.3158	0.9607
6-comp-Exponential	0.1027	0.1956	0.3582	0.0023	0.0056	0.0393
Exponential/Exponential/Weibull	0.2252	0.4237	-	0.3940	0.6786	-
Loglogistic/Exponential/Weibull	0.4673	-	-	0.4255	-	-

Table 13.38: BARC, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
4-comp-Exponential	0.1861	0.3676	0.5878	0.1784	0.3436	0.9256
6-comp-Exponential	0.1272	0.2474	0.4122	0.0058	0.0141	0.0744
Exponential/Exponential/Weibull	0.1963	0.3850	-	0.3406	0.6423	-
Loglogistic/Exponential/Weibull	0.4903	-	-	0.4753	-	-

Table 13.39: BARC, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
4-comp-Exponential	0.1534	0.3280	0.5062	0.1642	0.3555	0.8435
6-comp-Exponential	0.1496	0.3135	0.4937	0.0178	0.0448	0.1564
Exponential/Exponential/Weibull	0.1694	0.3584	-	0.2810	0.5996	-
Loglogistic/Exponential/Weibull	0.5275	-	-	0.5369	-	-

Table 13.40: BARC, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
4-comp-Exponential	0.1339	0.2857	0.4513	0.1491	0.3336	0.7532
6-comp-Exponential	0.1643	0.3669	-	0.0365	0.0881	0.2467
Exponential/Exponential/Weibull	0.1617	0.3473	0.5486	0.2555	0.5782	-
Loglogistic/Exponential/Weibull	0.5399	-	-	0.5587	-	-

Table 13.41: BARC, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

13.5 Censoring "zero inflated" Data

Although orders are recorded by the EOB at microsecond resolution, the data made available in this study has been rounded to millisecond resolution. Two consecutive orders recorded with time separations within [0, 0.5]ms are therefore released as simultaneous. Censoring data in this region can mitigate many of the difficulties related to rounding errors and "zero inflation". As previously mentioned, the ratio of bin width to time increases for smaller time differences, meaning binning has a significant effect on the reliability of the estimator for small time differences. This analysis considered a single censored region [0, 0.5]ms and multi-censored regions [0, 0.5, 1.5, 2.5, 10]ms, both in the small time scale region of the distribution.

Tabulated Estimates:
Tables [13.42 -13.45] (Tables [C.217 - C.240] in appendix) present parameter estimates for the following mixtures, sample sizes, and censor regions,

- 3-comp exp: censored on $[0, 0.5]$ms
- 4-comp exp: censored on $[0, 0.5]$ms
- 3-comp exp: censored on $[0, 0.5, 1.5, 2.5, 10]$ms
- 4-comp exp: censored on $[0, 0.5, 1.5, 2.5, 10]$ms

$$n = \{50, 100, 200, 500, 1000\}, \qquad (13.8)$$

for the case when "zero inflated" data is dealt with by censoring data in the short time scale region. Recall censoring acknowledges the presence of data within a certain region without the requirement of full specification. Table 13.46 (Tables [C.241 - C.246] in appendix) presents estimates for the following mixture, sample sizes, and censor region,

- exp/exp/Weibull: censored on $[0, 0.5]$ms

$$n = \{200, 500, 1000, 2000, 5000\}. \qquad (13.9)$$

Tabulated data once again includes the mean, standard deviation, median, and percentile parameter estimates, in addition to the mean and percentile log-likelihood values, and the percentage of non-converging intervals.

Convergence:
Convergence was strong for the case of censored mixtures. Convergence for 3-component exponential mixtures was near perfect $\sim 100\%$ for both single and multi-censor regions. 4-component exponential and exponential/exponential/Weibull mixtures displayed a similar strong convergence $\sim 98\%$.

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\lambda}_{3\ \text{ave}}$	$\pm\hat{\lambda}_{3\ \text{ave}}$	$\hat{\lambda}_{3\ \text{median}}$	$\hat{\lambda}_{3\ \text{lower}}$	$\hat{\lambda}_{3\ \text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.057738	0.907880	0.002703	0.001099	0.007626	0.443	2.041	0.115	0.035	0.346	10.855	6.995	16.508	2.293	16.985	0.223	0.120	0.212	0.221	0.109	0.208	0.556	0.137	0.569	-158.579	-206.070	-111.111	0.000
100	0.031003	0.675184	0.002423	0.001118	0.005360	0.241	1.200	0.108	0.036	0.275	11.002	7.315	17.259	2.540	17.714	0.220	0.101	0.212	0.223	0.091	0.213	0.557	0.120	0.568	-322.990	-404.408	-242.145	0.000
200	0.008333	0.302377	0.002264	0.001144	0.004378	0.163	0.657	0.105	0.037	0.237	11.163	7.638	18.002	2.786	18.455	0.219	0.085	0.213	0.224	0.077	0.216	0.558	0.106	0.567	-653.735	-797.110	-511.542	0.000
500	0.002511	0.001620	0.002136	0.001203	0.003718	0.126	0.144	0.104	0.040	0.210	11.357	8.059	6.276	3.117	19.440	0.219	0.070	0.214	0.224	0.063	0.216	0.557	0.092	0.566	-1650.158	-1964.768	-1336.372	0.000
1000	0.002359	0.001341	0.002075	0.001240	0.003428	0.120	0.080	0.104	0.042	0.200	11.578	8.385	5.497	3.325	20.186	0.219	0.062	0.214	0.225	0.056	0.217	0.556	0.084	0.563	-3314.489	-3878.578	-2726.988	0.000

Table 13.42: 3-component Exponential mixture on limit order arrival BARC data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\lambda}_{3\ \text{ave}}$	$\pm\hat{\lambda}_{3\ \text{ave}}$	$\hat{\lambda}_{3\ \text{median}}$	$\hat{\lambda}_{3\ \text{lower}}$	$\hat{\lambda}_{3\ \text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.061833	0.924435	0.002671	0.001087	0.007531	0.462	2.087	0.106	0.033	0.309	7.311	4.927	5.873	2.885	13.857	0.221	0.120	0.209	0.215	0.110	0.202	0.564	0.138	0.578	-150.609	-197.546	-103.298	0.000
100	0.033722	0.690553	0.002376	0.001095	0.005245	0.239	1.252	0.094	0.033	0.235	6.351	3.925	5.533	3.152	9.723	0.215	0.100	0.207	0.215	0.092	0.205	0.569	0.120	0.583	-306.340	-386.088	-226.650	0.000
200	0.009016	0.311741	0.002199	0.001112	0.004245	0.152	0.744	0.086	0.033	0.185	5.624	2.975	5.339	3.399	7.551	0.212	0.084	0.206	0.214	0.076	0.206	0.574	0.106	0.586	-618.777	-758.505	-480.285	0.000
500	0.002414	0.001550	0.002059	0.001160	0.003581	0.097	0.201	0.079	0.035	0.150	5.061	2.048	5.199	3.699	6.484	0.210	0.068	0.204	0.211	0.061	0.204	0.579	0.091	0.590	-1557.975	-1864.624	-1255.683	0.000
1000	0.002261	0.001276	0.001995	0.001187	0.003292	0.087	0.096	0.077	0.036	0.136	4.867	1.689	5.151	3.877	6.115	0.209	0.060	0.204	0.210	0.052	0.204	0.581	0.082	0.591	-3126.244	-3678.732	-2561.884	0.000

Table 13.43: 3-component Exponential mixture on limit order arrival BARC data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\lambda}_{3\ \text{ave}}$	$\pm\hat{\lambda}_{3\ \text{ave}}$	$\hat{\lambda}_{3\ \text{median}}$	$\hat{\lambda}_{3\ \text{lower}}$	$\hat{\lambda}_{3\ \text{upper}}$	$\hat{\lambda}_{4\ \text{ave}}$	$\pm\hat{\lambda}_{4\ \text{ave}}$	$\hat{\lambda}_{4\ \text{median}}$	$\hat{\lambda}_{4\ \text{lower}}$	$\hat{\lambda}_{4\ \text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	$\hat{\pi}_{4\ \text{ave}}$	$\pm\hat{\pi}_{4\ \text{ave}}$	$\hat{\pi}_{4\ \text{median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.057820	0.912494	0.002454	0.000976	0.007415	0.266	1.657	0.056	0.017	0.185	1.870	4.199	0.572	0.172	1.964	13.904	5.931	16.724	3.251	17.048	0.199	0.120	0.185	0.156	0.084	0.144	0.151	0.098	0.134	0.495	0.137	0.510	-157.550	-204.430	-108.389	0.011
100	0.030827	0.678999	0.002089	0.000913	0.004864	0.132	1.020	0.041	0.014	0.132	1.208	2.974	0.549	0.198	1.422	15.065	5.799	17.508	16.271	17.769	0.185	0.100	0.174	0.162	0.073	0.154	0.158	0.092	0.143	0.495	0.124	0.509	-320.255	-400.249	-236.464	0.012
200	0.007911	0.304341	0.001864	0.000858	0.003812	0.069	0.532	0.031	0.011	0.094	0.863	2.068	0.503	0.208	1.218	16.055	5.762	18.287	17.807	18.497	0.171	0.084	0.163	0.167	0.063	0.160	0.164	0.082	0.153	0.498	0.110	0.510	-647.063	-787.056	-499.866	0.013
500	0.001983	0.001429	0.001670	0.000804	0.003156	0.036	0.115	0.023	0.008	0.066	0.614	1.134	0.430	0.203	1.036	17.162	5.873	19.312	19.011	19.471	0.157	0.066	0.152	0.168	0.052	0.164	0.171	0.070	0.161	0.503	0.093	0.514	-1629.823	-1932.111	-1305.586	0.015
1000	0.001810	0.001184	0.001575	0.000784	0.002854	0.028	0.031	0.020	0.007	0.052	0.519	0.609	0.385	0.199	0.944	17.909	6.050	20.084	19.848	20.221	0.150	0.055	0.146	0.169	0.046	0.165	0.174	0.063	0.166	0.507	0.081	0.516	-3270.261	-3808.014	-2661.660	0.017

Table 13.44: 4-component Exponential mixture on limit order arrival BARC data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\lambda}_{3\ \text{ave}}$	$\pm\hat{\lambda}_{3\ \text{ave}}$	$\hat{\lambda}_{3\ \text{median}}$	$\hat{\lambda}_{3\ \text{lower}}$	$\hat{\lambda}_{3\ \text{upper}}$	$\hat{\lambda}_{4\ \text{ave}}$	$\pm\hat{\lambda}_{4\ \text{ave}}$	$\hat{\lambda}_{4\ \text{median}}$	$\hat{\lambda}_{4\ \text{lower}}$	$\hat{\lambda}_{4\ \text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	$\hat{\pi}_{4\ \text{ave}}$	$\pm\hat{\pi}_{4\ \text{ave}}$	$\hat{\pi}_{4\ \text{median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.062195	0.929835	0.002447	0.000973	0.007466	0.280	1.666	0.055	0.017	0.200	2.137	4.087	0.471	0.155	4.390	8.425	5.065	7.347	3.588	14.901	0.198	0.120	0.184	0.152	0.084	0.139	0.137	0.097	0.116	0.512	0.139	0.535	-149.627	-195.968	-100.356	0.013
100	0.033711	0.695196	0.002060	0.000897	0.004865	0.143	1.062	0.038	0.013	0.137	1.340	2.800	0.408	0.160	1.877	7.820	4.406	6.887	3.928	13.202	0.182	0.101	0.172	0.157	0.074	0.146	0.141	0.092	0.124	0.520	0.128	0.543	-305.191	-384.125	-221.498	0.015
200	0.008615	0.313846	0.001796	0.000818	0.003704	0.075	0.617	0.025	0.009	0.088	0.830	1.790	0.344	0.150	0.959	7.259	3.805	6.537	4.185	11.607	0.165	0.086	0.156	0.159	0.064	0.150	0.149	0.085	0.137	0.527	0.118	0.549	-618.163	-758.109	-471.960	0.015
500	0.001833	0.001344	0.001537	0.000740	0.002908	0.029	0.158	0.016	0.006	0.045	0.447	0.901	0.251	0.128	0.544	6.540	3.089	6.130	4.439	9.682	0.144	0.070	0.137	0.159	0.054	0.153	0.158	0.071	0.149	0.539	0.106	0.557	-1562.295	-1868.476	-1241.181	0.015
1000	0.001623	0.001060	0.001414	0.000718	0.002544	0.019	0.049	0.013	0.006	0.030	0.295	0.473	0.204	0.118	0.427	6.093	2.688	5.879	4.551	8.191	0.133	0.059	0.126	0.159	0.048	0.153	0.161	0.062	0.154	0.548	0.096	0.563	-3140.044	-3692.264	-2542.069	0.015

Table 13.45: 4-component Exponential mixture on limit order arrival BARC data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
200	2.104	4.856	0.533	0.250	1.493	17.320	1.576	18.149	15.462	18.438	236.706	193.446	182.743	88.358	390.093	0.634	1.098	0.604	0.524	0.721	0.199	0.118	0.171	0.448	0.164	0.487	0.353	0.118	0.344	-653.065	-790.550	-506.820	0.012
500	1.280	3.522	0.458	0.261	1.147	17.944	1.818	19.103	15.588	19.372	220.464	148.075	181.203	96.473	343.621	0.590	0.073	0.582	0.525	0.653	0.187	0.108	0.160	0.459	0.161	0.502	0.354	0.108	0.342	-1635.256	-1942.541	-1321.633	0.002
1000	0.906	2.548	0.429	0.271	1.042	18.585	1.966	19.821	15.993	20.078	213.917	127.830	182.244	102.021	324.518	0.577	0.055	0.572	0.527	0.625	0.182	0.102	0.157	0.463	0.159	0.507	0.355	0.102	0.342	-3277.639	-3827.197	-2693.234	0.000
2000	0.695	1.697	0.416	0.283	0.982	19.372	1.975	20.464	16.865	20.728	209.623	115.898	183.786	105.641	312.441	0.568	0.045	0.565	0.528	0.608	0.179	0.099	0.156	0.464	0.157	0.510	0.356	0.098	0.342	-6566.997	-7566.787	-5468.960	0.000
5000	0.567	0.819	0.401	0.293	0.942	20.281	1.811	21.086	18.334	21.416	205.947	104.801	185.190	108.373	302.192	0.561	0.036	0.559	0.527	0.594	0.177	0.095	0.156	0.466	0.155	0.511	0.357	0.095	0.342	-16441.322	-18602.094	-13869.131	0.000

Table 13.46: Exponential-Exponential-Weibull mixture on limit order arrival BARC data with censoring on region [0, 0.5].

Mixture Proportions:

Once again, mixture proportions displayed strong stability with respect to both varying sample size n and varying stock. For censored mixtures which include the Weibull component, the exponential components restricted density almost entirely to the $[0, 10]$ms region, allowing the Weibull component to independently manage the intermediate and tail region of the distribution.

Component Parameters:

Censored mixtures recovered most closely the left-truncated estimates of Kizilersü et al. [19]. Figure 13.15 displays the Weibull shape parameter estimates for all stocks, sample sizes, and mixed distributions, for the exponential/exponential/Weibull censored mixture with a single censor region [0, 0.5]ms. Median values are represented on the figure with 68% percent of interval estimates falling within the error bars. The black dotted line represents the weighted estimate, and the red dotted line the Euler-Mascheroni constant. This case yields estimates of the Weibull shape parameter $\hat{\beta}$ which follow the Euler-Mascheroni constant closest.

Goodness-of-fit Testing:

Tables [13.47 - 13.49] (Tables [C.247 - C.264] in appendix) present Akaike Information Criterion and Bayesian Information Criterion results for comparison of the 3-component exponential, 4-component exponential, and exponential/exponential/Weibull mixtures for common sample sizes n = (200, 500, 1000), and a single censor region [0, 0.5]ms. Figure 13.16 provides a histogram of the first preference results of the information criteria testing. Although not abundantly conclusive, the criteria tests support the assertion that mixtures including a Weibull component outperform those without one.

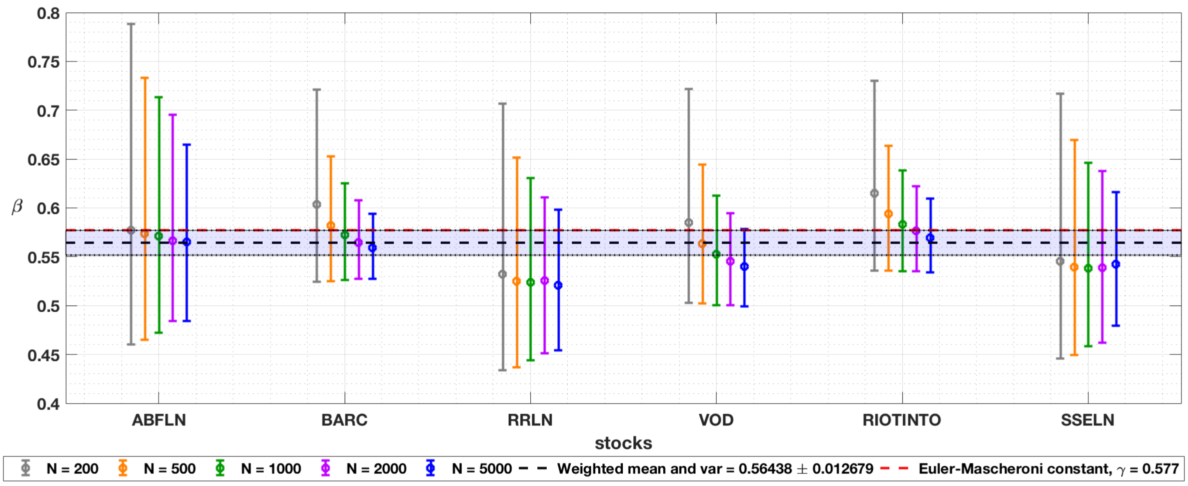


Figure 13.15: Exponential/Exponential/Weibull mixture: censored on $[0, 0.5]$ms. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

	AIC		**BIC**	
3-comp Exponential	0.1620	0.2421	0.5998	-
4-comp Exponential	0.4375	-	0.0064	0.0638
Exponential/Exponential/Weibull	0.4005	0.7579	0.3938	0.9361

Table 13.47: BARC, N = 200: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp Exponential	0.0344	0.0567	0.1872	0.3335
4-comp Exponential	0.4739	0.9433	0.0612	0.6665
Exponential/Exponential/Weibull	0.4917	-	0.7516	-

Table 13.48: BARC, N = 500: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp Exponential	0.0124	0.0283	0.0705	0.1203
4-comp Exponential	0.4061	0.9717	0.1054	0.8796
Exponential/Exponential/Weibull	0.5816	-	0.8241	-

Table 13.49: BARC, N = 1000: limit order arrival times, censored on region [0, 0.5].

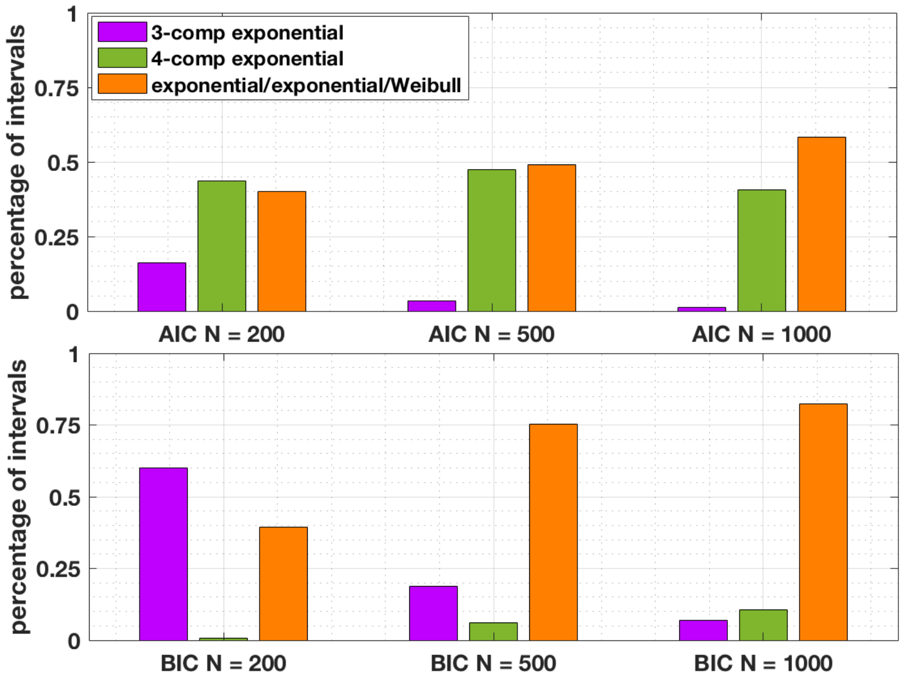


Figure 13.16: First preference information criterion results for censored mixed distributions.

13.6 Summary

Limit orders are only executed at a particular price, this is in contrast to market orders which are executed immediately at the best available offer. Many limit orders are placed with the sole intention to manipulate certain behaviours of the stock exchange mid-price, with a vast majority of these dummy orders placed by high-frequency trading algorithms (with no intention of any actual trading occurring). Consequently, a large proportion of limit order inter-arrival times are very small, making them difficult to model. The main outcomes regarding mixed distribution limit order analysis are summarised below:

- Mixed distributions allow for excess data in the short time scale region to be accounted for, without significant compromise to the estimation efforts in the intermediate and tail region of the distribution. A mixture model is therefore appropriate for description of the full distribution of limit order inter-arrival time data.

- A large degree of agreement exists between Weibull component parameter estimates and the left-truncated parameters estimates of Kizilersü et al. [19]. For mixtures which have additional components (non-Weibull) which minimise leakage into the intermediate and tail region of the distribution, the left-truncated estimates are largely recovered, even with a distinct difference in estimation methodology. This work supports the conclusion of Kizilersü et al. [19] that the intermediate and tail region of limit order inter-arrival times can be well described by a Weibull distribution. By inspection of interval estimates throughout a trading day, this work supports that the activity of a stock can be described by the inverse of the Weibull scale parameter α. The activity of a stock is highest at the opening of the market, decreases throughout a trading day, before increasing again before market closure. The work also supports the assertion that the Weibull shape parameter β is universal, corresponding to the maximum entropy of the system, defined by the Euler-Mascheroni constant.

- The estimated mixtures proportions displayed strong stability for varying sample size n and varying stock. The proportion of "zero inflated" data is also stable, comparably consistent for all stocks. It is comforting that when the uniform and exponential distribution were tasked with describing the excess data, their corresponding mixture proportion estimates were consistent with the proportion of data which was "zero inflated". This showed that both the uniform and exponential distribution are adequate for describing the excess data.

- Although estimates of loglogistic parameters displayed the largest variability of all component distributions, model selection indicated that mixture contributions of around 15% (i.e. $\pi_{\text{loglogistic}} \approx 0.15$) yielded useful additions to the transition region of the distribution, although this wasn't justified in the censored framework.

- Gamma distribution scale parameter $\theta \approx 1$ is universal for all stocks and sample sizes n.

- Censored mixtures offered the most statistically rigorous and hence most appropriate method to model the full distribution of limit order inter-arrival times. Strong convergence was evident, and the corresponding estimates agreed with the left-truncated analysis of Kizilersü et al. [19]. The weighted estimate for the Weibull component shape parameter was $\hat{\beta} = 0.564 \pm 0.013$.

13.7 Future Work

- Censored EM equations are required to be derived for the loglogistic and gamma distributions. The derivation of these equations and the numerical application of resulting equations is far from trivial, in fact it is one of the accomplishments of this thesis to present censored EM equations for both the exponential and Weibull distribution. Unfortunately time constraints related to the preparation of this thesis meant deriving censored EM equations for the loglogistic and gamma distribution was not possible. These derivations are currently been worked on and will be presented in a subsequent paper. Consequently it was not possible to include gamma/exponential/Weibull and loglogistic/exponential/Weibull mixtures within the censored framework.

- Improvements could have been made to the initialisation routine. As mentioned, initial conditions were chosen from repeated analysis of a four trading day subset of the full dataset, coupled with existing estimates from Kizilersü et al. [19]. Although full scale runs were undertaken with several different initial conditions, an initialisation routine which doesn't fix the values would improve the analysis.

- Because the large majority of limit orders are placed and then quickly cancelled, it is theorised that the distribution of limit order cancellation times is similar to that of arrival times. Empirical justification of this could be made.

- The distribution of limit order lifetimes is also of interest.

Chapter 14

Mid-Price Waiting Time Results

This chapter presents the relevant results and discussions of the mid-price waiting time analysis. A representative sample of estimates for **BARC** stock and sample sizes $\boldsymbol{n = \{50, 100, 200, 500, 1000\}}$ is presented in this chapter, however the full set of estimates are provided in Appendix D. This study had access to mid-price data for the period of 1$^{\text{st}}$ June 2010 until 30$^{\text{th}}$ September 2011 for the following stocks:

- **ABF:** Associated British Foods
- **BARC:** Barclays
- **RIO:** Rio Tinto
- **RR:** Rolls-Royce Holding
- **SSE:** Scottish and Southern Energy Company
- **VOD:** Vodafone Group

Section 12.3 outlined a procedure to describe the full distribution of mid-price waiting times by use of mixed distributions. Recall, waiting times are defined as the time differences between consecutive mid-price changes. It is important to note that many similarities exist between the nature of the mid-price waiting time distribution and that of the limit order inter-arrival times. Many of the resulting discussions therefore differ little from the limit-order analysis. Some of these similarities were discussed in Chapter 12, however others are discussed within this chapter. Although the important observations will be rementioned for both clarity and completeness, the focus of this chapter is largely presenting the results which differ from the previous analysis (i.e. the convergence results are not explicitly discussed in this chapter). Many of these unique results manifest from the fact that far less mid-price waiting time data exists within the short time scale region (including "zero inflated" data) which has a significant affect on the estimated mixture proportions. Additionally, at the time of thesis preparation, analysis of mid-price data was in its infancy. Section 14.7 outlines the significance of accurate modelling of mid-price information and the direction this work will take in the future.

14.1 Removal of "zero inflated" Data

<u>Tabulated Estimates:</u>
Tables [14.1 - 14.4] (Tables [D.1 - D.24] in appendix) present parameter estimates for the following mixtures and sample sizes,

$$\left.\begin{array}{l} \bullet\ \text{Exponential/Weibull} \\ \bullet\ \text{Gamma/Weibull} \\ \bullet\ \text{Loglogistic/Weibull} \\ \bullet\ \text{Weibull/Weibull} \end{array}\right\} \quad n = \{50, 100, 200, 500, 1000\}, \qquad (14.1)$$

for the case when "zero inflated" data is removed entirely from the dataset. Tabulated data includes the mean, standard deviation, median, and percentile parameter estimates, in addition to the mean and percentile log-likelihood values, and the percentage of non-converging intervals.

Mixture Proportions:
The mixture proportion estimates are stable with respect to a varying sample size n, but now exhibit a dependence on the activity of the stock. This is in clear contrast to the analysis of limit order inter-arrival times where mixture proportion estimates were stable for varying stocks. This is because the density of data in the short time scale, intermediate, and tail regions of the distribution of mid-price waiting times is not consistent for each stock (as was the case in the limit order analysis). Figure 12.2 presented data statistics for mid-price waiting times, displaying that more active stocks (BARC and RIOTINTO) exhibited larger densities of data in the short time scale region. This is reflected in the mixture proportion estimates, with the more active stocks having larger estimates for the first (non-Weibull) component.[1]

Component Parameters:
Convergence of the component parameter estimates is such that the first (non-Weibull) component described data in the short time scale region, whilst the Weibull component described both the intermediate and tail region data. This is consistent with the limit order analysis. The component parameter estimates remain dependent on stock, and the Weibull scale parameter α is once again related to the inverse of its activity. In the previous analysis, the Weibull shape parameter β corresponded to maximum entropy of the system for the Weibull distribution, consistent with the Euler-Mascheroni constant. For mid-price waiting times, estimates of the Weibull shape parameter β no longer follow the Euler-Mascheroni constant. Figures [14.1 - 14.4] display the Weibull shape parameter estimates for all stocks, sample sizes, and mixed distributions. Median values are represented on the figure with 68% percent of interval estimates falling within the error bars (defined by the lower and upper percentile estimates). The black dotted line represents a weighted estimate, and the red the Euler Mascheroni constant, displayed for reference. The Weibull shape parameter estimates are larger for more active stocks. Additionally, the apparent universality of the gamma scale parameter $\theta \approx 1$ is maintained.

Figure 14.5 provides an example (BARC: first $n = 1000$ interval of 1$^{\text{st}}$ June 2010) of the probability density function for the converged mixtures, noting the use of a logarithmic scale for the vertical axis. The excess data apparent in the short time scale region is well represented by the inset figure which gives the axis on a linear scale. For this particular interval, at a macro level all component parameter estimation was such that all mixtures converged to a similar density function. Figure 14.6 provides the corresponding cumulative distribution function, although the horizontal axis is now represented with a logarithmic scale.

Goodness-of-fit Testing:
Model selection was undertaken via information criteria testing. Tables [14.5 - 14.9] (Tables [D.25 - D.54] in appendix) present Akaike Information Criterion and Bayesian Information Criterion results for each mixed distribution. For this case, loglogistic/Weibull mixtures and exponential/Weibull mixtures most consistently described the mid-price waiting time data most appropriately, which is comparable to the analogous case for the limit order analysis. A gamma component doesn't yield a useful addition to the mixture.

[1] This is not exclusively true if the second (Weibull) component makes a contribution to the density in the short time scale region. Recall the discussion about component distribution leakage in Chapter 13.

n	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.209	0.148	0.173	0.083	0.333	-	-	-	-	-	1298.673	1410.190	891.982	326.634	2208.301	0.728	1.254	0.580	0.470	0.812	0.512	0.205	0.514	0.488	0.205	0.486	-285.026	-351.154	-215.103	0.005
100	0.201	0.125	0.176	0.095	0.301	-	-	-	-	-	1123.224	1043.512	841.438	360.321	1822.436	0.590	0.572	0.540	0.461	0.664	0.498	0.192	0.501	0.502	0.192	0.499	-573.084	-695.043	-448.242	0.001
200	0.197	0.108	0.178	0.105	0.281	-	-	-	-	-	1019.850	826.538	811.621	389.637	1601.874	0.536	0.166	0.519	0.457	0.601	0.489	0.181	0.495	0.511	0.181	0.505	-1150.728	-1377.497	-918.583	0.000
500	0.195	0.095	0.180	0.114	0.265	-	-	-	-	-	943.604	663.599	787.736	418.323	1434.817	0.511	0.096	0.503	0.456	0.562	0.482	0.171	0.493	0.518	0.171	0.507	-2883.337	-3412.542	-2335.236	0.000
1000	0.195	0.089	0.184	0.119	0.258	-	-	-	-	-	907.826	582.564	776.897	432.010	1338.499	0.501	0.061	0.496	0.456	0.544	0.480	0.166	0.495	0.520	0.166	0.505	-5762.342	-6779.756	-4696.868	0.000

Table 14.1: Exponential-Weibull mixture on BARC mid-price waiting time data: removal of zeros.

n	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.644	2.628	2.655	2.064	5.692	0.987	0.150	1.011	0.842	1.133	601.927	663.614	409.254	148.495	1048.880	0.517	0.697	0.462	0.391	0.570	0.347	0.206	0.309	0.653	0.206	0.691	-284.463	-351.050	-211.540	0.013
100	3.357	2.220	2.596	2.076	4.155	1.009	0.132	1.033	0.886	1.131	546.287	499.983	407.207	174.978	888.232	0.465	0.199	0.446	0.390	0.519	0.338	0.191	0.311	0.662	0.191	0.689	-574.617	-697.217	-447.950	0.002
200	3.154	1.906	2.570	2.108	3.484	1.024	0.116	1.046	0.921	1.130	518.753	416.792	407.355	196.548	816.776	0.447	0.104	0.437	0.390	0.495	0.335	0.180	0.318	0.665	0.180	0.682	-1154.659	-1383.010	-920.773	0.000
500	2.943	1.498	2.552	2.145	3.187	1.036	0.101	1.054	0.948	1.127	499.333	356.338	407.406	218.741	759.442	0.437	0.070	0.431	0.392	0.479	0.334	0.169	0.328	0.666	0.169	0.672	-2893.071	-3426.624	-2339.859	0.000
1000	2.844	1.251	2.543	2.175	3.069	1.041	0.093	1.057	0.963	1.125	488.085	321.208	406.118	230.109	722.751	0.432	0.055	0.427	0.393	0.471	0.336	0.163	0.334	0.664	0.163	0.666	-5781.282	-6808.659	-4710.950	0.000

Table 14.2: Gamma-Weibull mixture on BARC mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{\text{log ave}}$	$\pm\hat{\alpha}_{\text{log ave}}$	$\hat{\alpha}_{\text{log median}}$	$\hat{\alpha}_{\text{log lower}}$	$\hat{\alpha}_{\text{log upper}}$	$\hat{\beta}_{\text{log ave}}$	$\pm\hat{\beta}_{\text{log ave}}$	$\hat{\beta}_{\text{log median}}$	$\hat{\beta}_{\text{log lower}}$	$\hat{\beta}_{\text{log upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	7.262	9.960	4.085	1.881	11.608	7.969	21.420	1.533	1.071	2.877	1931.713	2548.161	1225.290	387.162	3428.635	0.909	1.477	0.649	0.494	1.112	0.564	0.213	0.584	0.436	0.213	0.416	-281.645	-348.237	-208.258	0.015
100	6.344	8.119	3.918	1.981	9.350	7.639	20.732	1.487	1.086	2.317	1646.598	1827.732	1145.952	419.491	2778.211	0.690	0.619	0.596	0.483	0.818	0.555	0.197	0.574	0.445	0.197	0.426	-564.980	-689.454	-435.829	0.005
200	5.736	6.825	3.804	2.085	8.245	7.223	19.762	1.471	1.103	2.070	1465.018	1503.321	1091.403	452.137	2377.733	0.611	0.417	0.569	0.480	0.703	0.547	0.184	0.569	0.453	0.184	0.431	-1133.934	-1366.886	-895.198	0.003
500	5.215	5.565	3.693	2.236	7.435	6.554	18.236	1.466	1.123	1.899	1319.608	1110.043	1053.919	496.226	2080.697	0.568	0.245	0.548	0.482	0.637	0.542	0.172	0.567	0.458	0.172	0.433	-2843.533	-3387.861	-2281.570	0.001
1000	4.914	4.796	3.611	2.331	6.982	6.007	17.046	1.470	1.143	1.821	1247.852	949.125	1027.004	520.286	1904.430	0.547	0.103	0.538	0.481	0.611	0.540	0.165	0.567	0.460	0.165	0.433	-5686.901	-6744.332	-4589.732	0.001

Table 14.3: Loglogistic-Weibull mixture on BARC mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{2\text{ave}}$	$\pm\hat{\alpha}_{2\text{ave}}$	$\hat{\alpha}_{2\text{median}}$	$\hat{\alpha}_{2\text{lower}}$	$\hat{\alpha}_{2\text{upper}}$	$\hat{\beta}_{2\text{ave}}$	$\pm\hat{\beta}_{2\text{ave}}$	$\hat{\beta}_{2\text{median}}$	$\hat{\beta}_{2\text{lower}}$	$\hat{\beta}_{2\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	9.870	17.220	6.058	3.076	15.366	1.334	1.242	1.107	0.845	1.785	1433.369	1725.250	936.861	321.628	2668.402	0.715	0.856	0.587	0.468	0.910	0.520	0.201	0.528	0.480	0.201	0.472	-283.712	-349.060	-206.915	0.028
100	8.950	13.579	5.914	3.195	13.312	1.204	0.626	1.064	0.840	1.533	1264.330	5610.384	880.480	344.896	2204.104	0.593	0.307	0.545	0.456	0.721	0.508	0.182	0.516	0.492	0.182	0.484	-571.349	-691.664	-435.954	0.021
200	8.186	9.760	5.813	3.339	12.105	1.148	0.462	1.044	0.846	1.391	1110.981	1015.369	842.385	363.406	1891.019	0.543	0.144	0.522	0.450	0.638	0.499	0.167	0.508	0.501	0.167	0.492	-1147.003	-1371.046	-897.021	0.020
500	7.462	7.832	5.667	3.531	10.823	1.118	0.415	1.031	0.863	1.270	1000.243	817.699	810.880	386.282	1637.783	0.514	0.077	0.503	0.448	0.587	0.490	0.154	0.503	0.510	0.154	0.497	-2873.816	-3396.472	-2286.987	0.018
1000	7.066	5.713	5.551	3.661	10.169	1.104	0.400	1.029	0.877	1.221	945.222	687.164	789.945	399.963	1505.934	0.502	0.061	0.494	0.447	0.566	0.487	0.147	0.502	0.513	0.147	0.498	-5742.118	-6757.131	-4616.632	0.019

Table 14.4: Weibull-Weibull mixture on BARC mid-price waiting time data: removal of zeros.

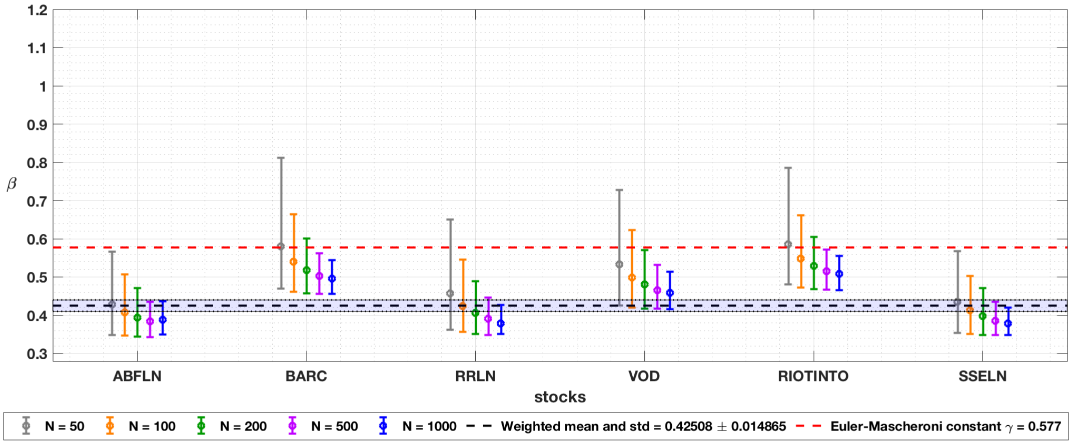


Figure 14.1: Exponential/Weibull mixture: removal of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

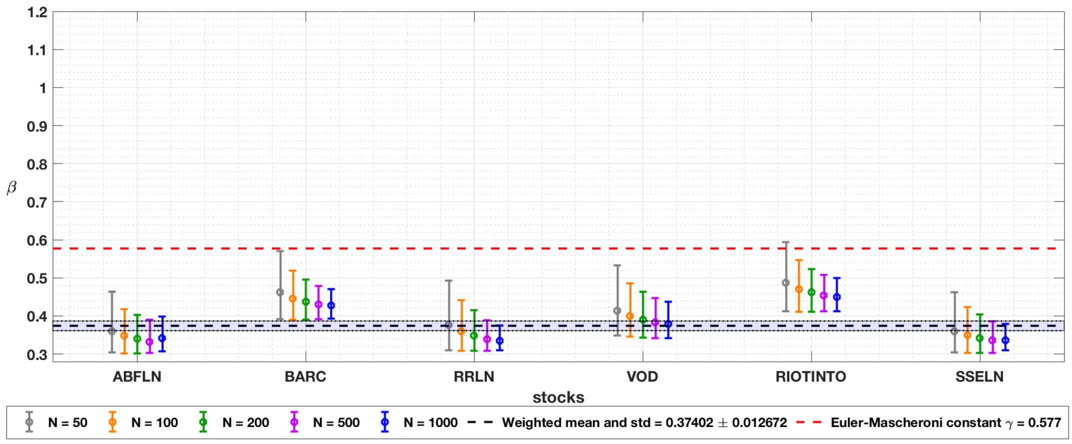


Figure 14.2: Gamma/Weibull mixture: removal of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

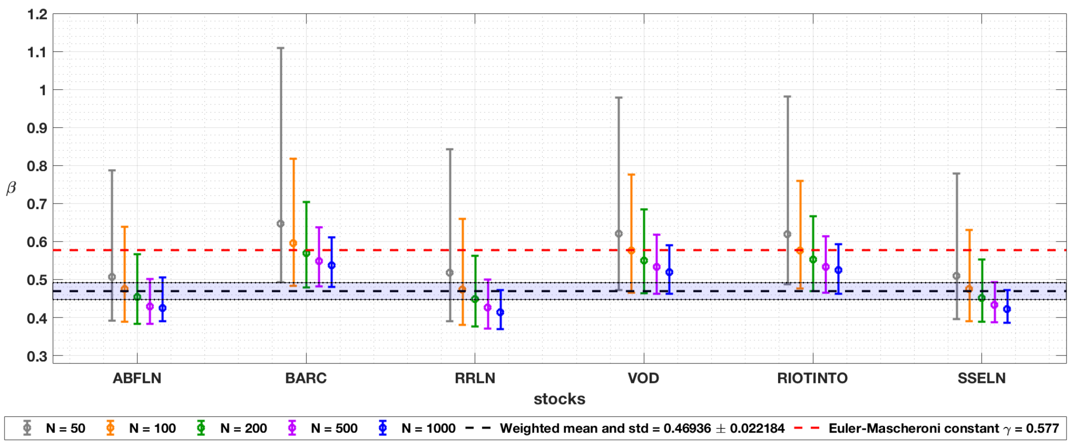


Figure 14.3: Loglogistic/Weibull mixture: removal of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

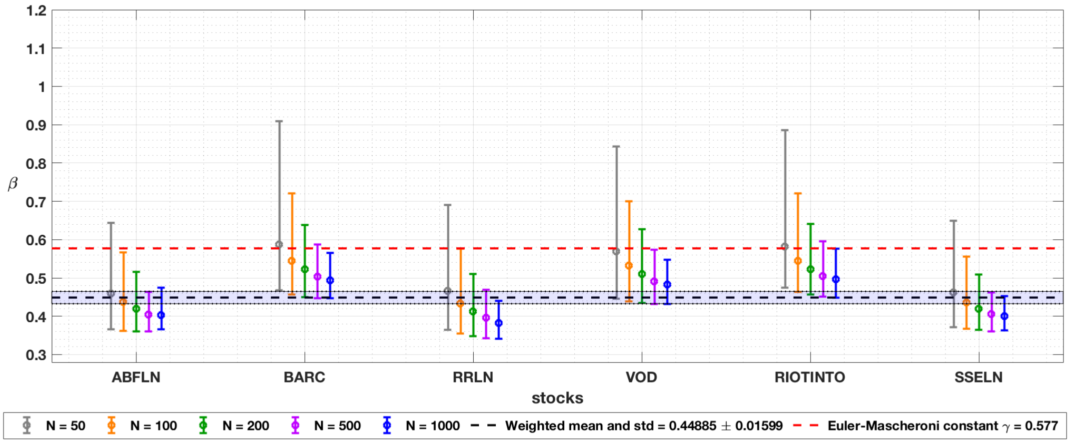


Figure 14.4: Weibull/Weibull mixture: removal of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

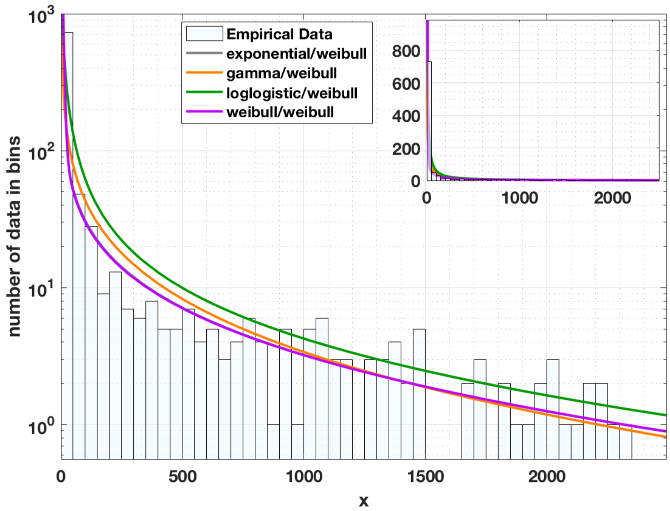


Figure 14.5: Probability density function for mixtures with BARC interval (first $n = 1000$ waiting times from 1st June 2010).

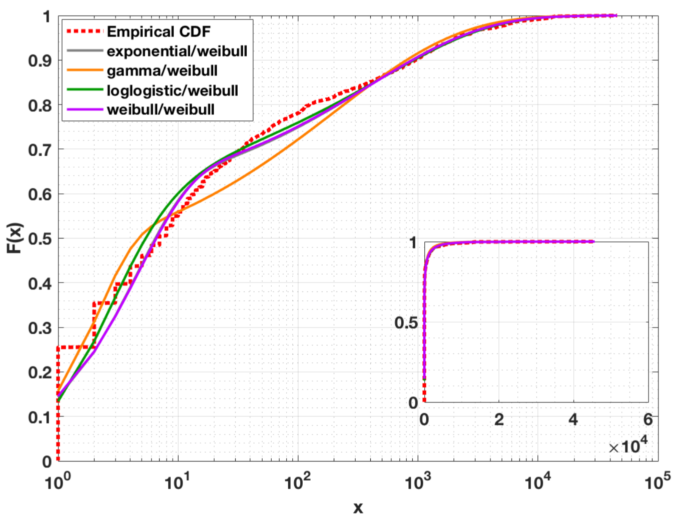


Figure 14.6: Cumulative distribution function for mixtures with BARC interval (first $n = 1000$ waiting times from 1st June 2010).

	AIC			BIC		
Exponential/Weibull	0.2091	0.5590	-	0.6888	-	-
Gamma/Weibull	0.0197	0.2139	0.2709	0.0084	0.0242	0.2709
Loglogistic/Weibull	0.7104	-	-	0.2867	0.8417	-
Weibull/Weibull	0.0605	0.2268	0.7266	0.0159	0.1337	0.7266

Table 14.5: BARC, N = 50: mid-price waiting times, removal of zeros.

	AIC			BIC		
Exponential/Weibull	0.0790	0.5097	-	0.5190	-	-
Gamma/Weibull	0.0126	0.2461	0.2778	0.0046	0.0136	0.2778
Loglogistic/Weibull	0.8790	-	-	0.4681	0.9323	-
Weibull/Weibull	0.0294	0.2441	0.7219	0.0084	0.0541	0.7219

Table 14.6: BARC, N = 100: mid-price waiting times, removal of zeros.

	AIC			BIC		
Exponential/Weibull	0.0169	0.4630	-	0.2754	0.7986	-
Gamma/Weibull	0.0091	0.2635	0.2790	0.0046	0.1437	0.2790
Loglogistic/Weibull	0.9586	-	-	0.7131	-	-
Weibull/Weibull	0.0154	0.2735	0.7210	0.0070	0.0577	0.7210

Table 14.7: BARC, N = 200: mid-price waiting times, removal of zeros.

	AIC			BIC		
Exponential/Weibull	0.0014	0.4035	-	0.0589	0.7297	-
Gamma/Weibull	0.0068	0.2610	0.2662	0.0061	0.1941	0.2662
Loglogistic/Weibull	0.9837	-	-	0.9287	-	-
Weibull/Weibull	0.0081	0.3355	0.7338	0.0062	0.0762	0.7338

Table 14.8: BARC, N = 500: mid-price waiting times, removal of zeros.

	AIC			BIC		
Exponential/Weibull	0.0003	0.3417	0.6884	0.0059	0.6857	-
Gamma/Weibull	0.0071	0.2581	0.3116	0.0069	0.2173	0.2603
Loglogistic/Weibull	0.9862	-	-	0.9814	-	-
Weibull/Weibull	0.0065	0.4001	-	0.0058	0.0970	0.7397

Table 14.9: BARC, N = 1000: mid-price waiting times, removal of zeros.

14.2 Distributing "zero inflated" Data Uniformly

Tabulated Estimates:
Tables [14.10 - 14.13] (Tables [D.55 - D.78] in appendix) present parameter estimates for the following mixtures and sample sizes,

$$\left.\begin{array}{l} \bullet \text{ Exponential/Uniform/Weibull} \\ \bullet \text{ Gamma/Uniform/Weibull} \\ \bullet \text{ Loglogistic/Uniform/Weibull} \\ \bullet \text{ Weibull/Uniform/Weibull} \end{array}\right\} \quad n = \{50, 100, 200, 500, 1000\}, \tag{14.2}$$

for the case when "zero inflated" data is distributed by a uniform density with parameter $\theta = 0.5$. Estimation of all parameters was undertaken from the data with the exception of θ, the parameter of the uniform density.[2] This was fixed at $\theta = 0.5$, consistent with the probability density which distributed the "zero inflated" data. Tabulated data includes the mean, standard deviation, median, and percentile parameter estimates, in addition to the mean and percentile log-likelihood values, and the percentage of non-converging intervals.

Mixture Proportions:
For this case, the mixture proportion estimates remain stock dependent, with the second mixture proportion $\hat{\pi}_2$ now not consistent with the proportion of data which is "zero-inflated". In fact, convergence is such that this mixture component makes a negligible contribution to the overall mixture, i.e. $\hat{\pi}_2 \sim 0.05$, with the remaining components sufficient to describe the excess data in the short time-scale region.

Component Parameters:
Figures [14.7 - 14.10] display the Weibull shape parameter estimates for all stocks, sample sizes, and mixed distributions. Median values are represented on the figure with 68% percent of interval estimates falling within the error bars. The black dotted line represents the weighted estimate, and the red line the Euler-Mascheroni constant.

Goodness-of-fit Testing:
Tables [14.14 - 14.18] (Tables [D.79 - D.108] in appendix) present Akaike Information Criterion and Bayesian Information Criterion results for each mixed distribution. Model selection results differ little from the case presented in Section 14.1, where "zero-inflated" data was removed entirely from the dataset. Once again, mixtures with loglogistic and exponential components outperformed those with gamma and Weibull components.

[2] Noting that the mixture proportion for the uniform density π_2 is estimated from the data.

n	θ	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	-	-	-	-	-	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	0.668	18.301	0.237	0.104	0.667	-	-	-	-	-	1066.908	1350.972	647.370	174.730	1915.062	0.794	3.475	0.524	0.411	0.797	0.465	0.209	0.460	0.059	0.098	0.021	0.476	0.207	0.475	-256.689	-329.702	-179.275	0.007
100	0.5	5.355	1137.031	0.245	0.121	0.602	-	-	-	-	-	874.609	965.466	587.679	194.352	1501.615	0.651	3.479	0.478	0.400	0.619	0.450	0.197	0.449	0.058	0.093	0.027	0.492	0.192	0.493	-516.779	-648.945	-381.923	0.001
200	0.5	0.594	15.715	0.252	0.136	0.553	-	-	-	-	-	753.588	741.380	545.268	213.810	1253.548	0.574	2.333	0.454	0.396	0.542	0.438	0.187	0.444	0.056	0.088	0.031	0.506	0.179	0.504	-1039.209	-1281.386	-794.019	0.000
500	0.5	0.787	28.976	0.259	0.151	0.510	-	-	-	-	-	664.160	576.623	516.475	228.235	1066.455	0.574	3.743	0.437	0.394	0.495	0.430	0.178	0.442	0.054	0.085	0.034	0.516	0.167	0.512	-2607.888	-3159.829	-2046.025	0.000
1000	0.5	0.474	4.178	0.263	0.157	0.484	-	-	-	-	-	624.201	486.003	508.337	239.752	995.066	0.572	5.714	0.429	0.394	0.477	0.427	0.173	0.445	0.052	0.082	0.034	0.521	0.160	0.513	-5220.799	-6279.564	-4160.430	0.000

Table 14.10: Exponential-Uniform-Weibull mixture on mid-price waiting time zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	3.089	2.626	2.302	1.216	4.975	1.002	0.107	1.006	0.907	1.106	487.846	629.450	296.003	83.054	855.540	0.579	2.720	0.423	0.354	0.540	0.314	0.194	0.277	0.064	0.101	0.028	0.622	0.208	0.661	-257.533	-331.158	-180.232	0.004
100	0.5	2.845	2.253	2.239	1.264	3.880	1.022	0.088	1.024	0.944	1.108	416.330	449.080	284.148	98.471	702.925	0.507	2.669	0.402	0.350	0.477	0.303	0.177	0.275	0.063	0.095	0.036	0.634	0.191	0.665	-519.709	-652.025	-384.877	0.001
200	0.5	2.654	1.951	2.203	1.322	3.301	1.033	0.077	1.036	0.966	1.109	380.046	362.717	278.450	112.451	619.166	0.469	1.700	0.391	0.349	0.448	0.296	0.163	0.278	0.063	0.090	0.042	0.641	0.177	0.663	-1045.216	-1287.610	-799.654	0.000
500	0.5	2.456	1.556	2.172	1.373	2.984	1.040	0.069	1.044	0.981	1.107	355.911	302.123	274.312	127.066	559.266	0.441	1.231	0.382	0.349	0.430	0.292	0.150	0.282	0.064	0.085	0.047	0.644	0.165	0.660	-2622.137	-3172.454	-2060.003	0.000
1000	0.5	2.365	1.351	2.163	1.402	2.852	1.043	0.066	1.048	0.986	1.105	344.809	272.392	272.352	133.155	533.782	0.449	1.685	0.379	0.350	0.423	0.291	0.143	0.286	0.064	0.082	0.050	0.644	0.158	0.654	-5248.344	-6308.594	-4193.548	0.000

Table 14.11: Gamma-Uniform-Weibull mixture mid-price waiting time zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	12.417	954.415	3.463	1.492	11.437	7.053	22.275	1.232	0.858	2.594	2059.773	2897.753	1237.903	312.218	3988.956	1.180	3.379	0.696	0.470	1.468	0.558	0.216	0.584	0.066	0.099	0.031	0.376	0.206	0.352	-256.977	-327.911	-173.202	0.032
100	0.5	6.589	198.075	3.272	1.559	8.523	6.695	21.729	1.153	0.849	1.882	1790.704	2187.552	1199.863	356.006	3177.630	0.862	3.086	0.619	0.454	0.993	0.561	0.205	0.589	0.063	0.095	0.033	0.375	0.191	0.355	-514.457	-645.535	-374.101	0.012
200	0.5	5.047	13.764	3.123	1.636	7.256	6.203	20.728	1.116	0.852	1.614	1585.038	1621.156	1157.972	395.023	2713.181	0.719	2.301	0.580	0.450	0.807	0.562	0.196	0.592	0.060	0.092	0.033	0.378	0.178	0.359	-1032.900	-1275.480	-781.112	0.006
500	0.5	4.344	5.204	3.013	1.730	6.392	5.771	19.675	1.095	0.859	1.431	1409.416	1235.047	1096.345	446.539	2308.285	0.622	1.259	0.549	0.450	0.690	0.563	0.186	0.595	0.056	0.088	0.032	0.381	0.161	0.364	-2590.525	-3144.710	-2016.644	0.003
1000	0.5	4.027	3.938	2.915	1.799	5.929	5.305	16.801	1.092	0.868	1.355	1325.966	1074.296	1073.247	476.881	2119.168	0.618	2.106	0.534	0.453	0.642	0.563	0.180	0.600	0.054	0.087	0.032	0.383	0.153	0.365	-5182.771	-6255.957	-4104.688	0.002

Table 14.12: Loglogistic-Uniform-Weibull mixture on mid-price waiting time zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	10.458	25.734	5.229	2.364	17.348	1.190	3.360	0.897	0.669	1.592	1553.749	2399.127	958.907	270.353	3166.958	0.904	2.131	0.610	0.448	1.176	0.530	0.203	0.541	0.057	0.094	0.019	0.413	0.199	0.401	-257.645	-327.626	-168.541	0.045
100	0.5	8.825	18.780	4.960	2.456	13.705	1.057	2.685	0.845	0.659	1.265	1350.772	3205.217	893.890	294.853	2556.384	0.691	1.617	0.548	0.432	0.839	0.525	0.189	0.537	0.053	0.089	0.019	0.422	0.182	0.414	-517.973	-645.537	-365.184	0.034
200	0.5	7.619	13.199	4.764	2.573	11.794	0.994	2.459	0.821	0.662	1.105	1204.569	7210.955	829.578	312.868	2145.219	0.594	1.230	0.514	0.425	0.699	0.520	0.177	0.534	0.049	0.086	0.019	0.431	0.166	0.423	-1039.932	-1274.968	-761.290	0.032
500	0.5	6.534	8.331	4.579	2.710	10.148	0.968	3.071	0.809	0.670	0.989	1010.956	1332.997	776.067	333.934	1768.964	0.569	2.350	0.489	0.423	0.609	0.516	0.165	0.534	0.044	0.083	0.017	0.440	0.150	0.433	-2604.041	-3141.768	-1963.244	0.032
1000	0.5	6.023	5.800	4.474	2.822	9.376	0.991	5.866	0.806	0.677	0.940	931.232	739.182	752.943	352.712	1608.174	0.532	1.066	0.478	0.423	0.570	0.514	0.159	0.536	0.042	0.081	0.016	0.444	0.142	0.436	-5208.420	-6250.628	-3999.503	0.032

Table 14.13: Weibull-Uniform-Weibull mixture on mid-price waiting time zero inflated (uniformly distributed) BARC data.

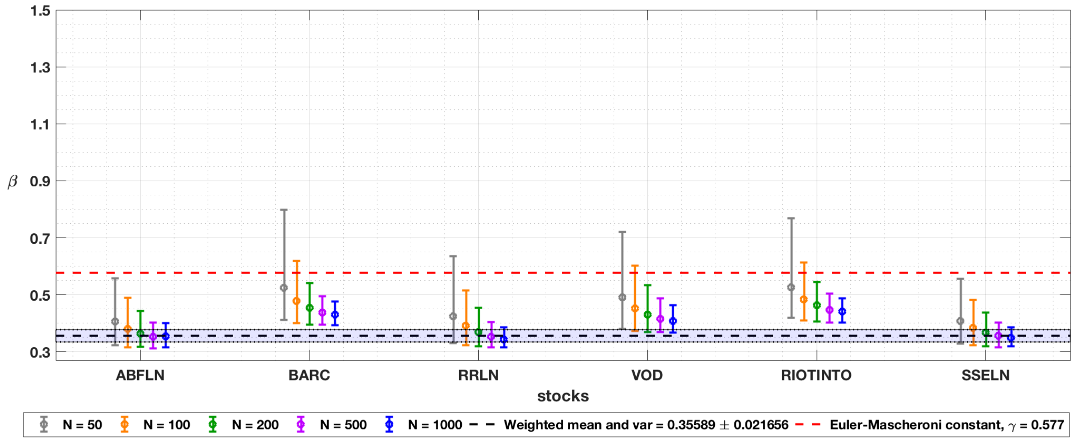


Figure 14.7: Exponential/Uniform/Weibull mixture: uniform distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

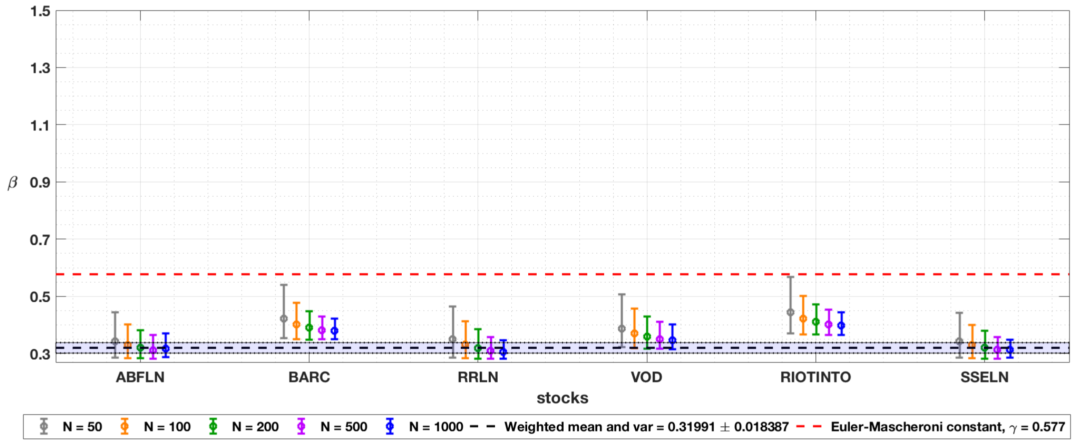


Figure 14.8: Gamma/Uniform/Weibull mixture: uniform distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

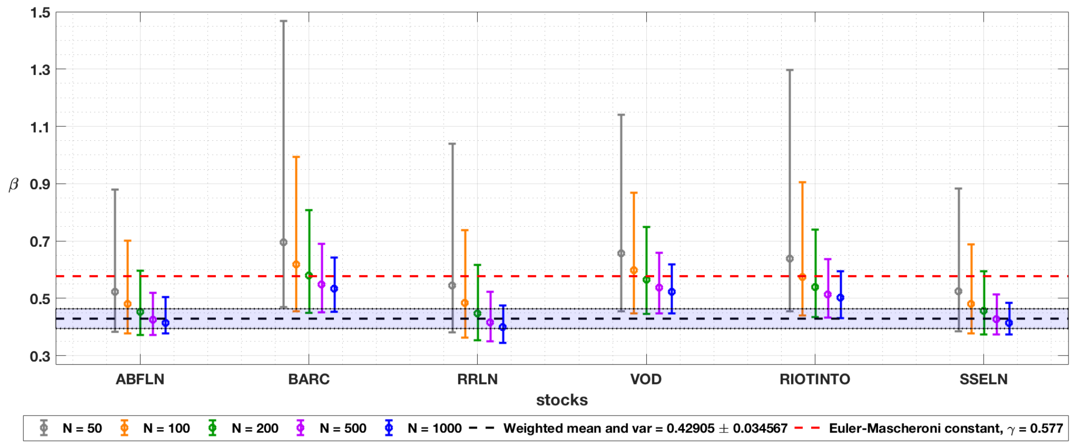


Figure 14.9: Loglogistic/Uniform/Weibull mixture: uniform distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

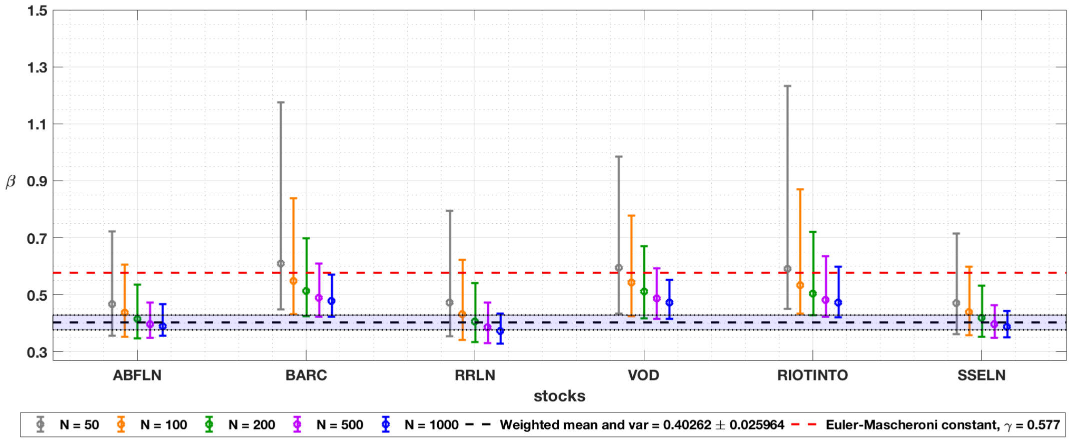


Figure 14.10: Weibull/Uniform/Weibull mixture: uniform distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

	AIC			BIC		
Exponential/Uniform/Weibull	0.3451	0.6437	-	0.8158	-	-
Gamma/Uniform/Weibull	0.0143	0.0597	0.1778	0.0042	0.0396	0.1778
Loglogistic/Uniform/Weibull	0.5120	-	-	0.1544	0.6655	-
Weibull/Uniform/Weibull	0.1270	0.2946	0.8199	0.0240	0.2931	0.8199

Table 14.14: BARC, N = 50: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
Exponential/Uniform/Weibull	0.1915	0.5559	-	0.7347	-	-
Gamma/Uniform/Weibull	0.0074	0.0585	0.1549	0.0015	0.0217	0.1549
Loglogistic/Uniform/Weibull	0.6900	-	-	0.2430	0.7744	-
Weibull/Uniform/Weibull	0.1108	0.3852	0.8447	0.0206	0.2036	0.8447

Table 14.15: BARC, N = 100: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
Exponential/Uniform/Weibull	0.0776	0.4311	0.9034	0.5237	-	-
Gamma/Uniform/Weibull	0.0051	0.0579	0.0964	0.0013	0.0138	0.1342
Loglogistic/Uniform/Weibull	0.8260	-	-	0.4519	0.8569	-
Weibull/Uniform/Weibull	0.0912	0.5109	-	0.0230	0.1293	0.8656

Table 14.16: BARC, N = 200: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
Exponential/Uniform/Weibull	0.0188	0.2424	0.9015	0.1702	0.7634	-
Gamma/Uniform/Weibull	0.0027	0.0543	0.0985	0.0012	0.0068	0.1054
Loglogistic/Uniform/Weibull	0.9248	-	-	0.8029	-	-
Weibull/Uniform/Weibull	0.0536	0.7033	-	0.0256	0.2298	0.8946

Table 14.17: BARC, N = 500: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
Exponential/Uniform/Weibull	0.0071	0.1374	0.8960	0.0521	0.5598	-
Gamma/Uniform/Weibull	0.0035	0.0610	0.1040	0.0007	0.0057	0.0984
Loglogistic/Uniform/Weibull	0.9582	-	-	0.9248	-	-
Weibull/Uniform/Weibull	0.0312	0.8016	-	0.0223	0.4345	0.9016

Table 14.18: BARC, N = 1000: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

14.3 Distributing "zero inflated" Data Exponentially

Tabulated Estimates:
Tables [14.19 - 14.22] (Tables [D.109 - D.132] in appendix) present parameter estimates for the following mixtures and sample sizes,

- Exponential/Exponential/Weibull
- Gamma/Exponential/Weibull
- Loglogistic/Exponential/Weibull
- Weibull/Exponential/Weibull

$$n = \{50, 100, 200, 500, 1000\}, \tag{14.3}$$

for the case when "zero inflated" data is now distributed by an exponential density. A rate parameter $\lambda = 8$ was chosen such that the distribution of density for the replaced "zero inflated" data was almost entirely restricted to the $[0, 0.5]$ms region, as was the case for the limit order analysis presented in Section 13.3. Tabulated data once again includes the mean, standard deviation, median, and percentile parameter estimates, in addition to the mean and percentile log-likelihood values, and the percentage of non-converging intervals.

Mixture Proportions:
The second mixture component $\hat{\pi}_2$ is stock dependent as well as being inconsistent with the proportion of "zero-inflated" data for each stock. Convergence is such that the second mixture component makes a negligible contribution to the overall mixture, as was seen in Section 14.2 when the "zero-inflated" data was distributed according to a uniform density. The additional components are therefore adequate to describe the excess data apparent in the short time scale region of the distribution of waiting times.

Component Parameters:
Figures [14.11 - 14.14] display the Weibull shape parameter estimates for all stocks, sample sizes, and mixed distributions. Median values are represented on the figure with 68% percent of interval estimates falling within the error bars. The black dotted line represents the weighted estimate, and the red line the Euler-Mascheroni constant.

n	$\hat{\lambda}_{1\,ave}$	$\pm\hat{\lambda}_{1\,ave}$	$\hat{\lambda}_{1\,median}$	$\hat{\lambda}_{1\,lower}$	$\hat{\lambda}_{1\,upper}$	-	-	-	-	-	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.374	0.857	0.219	0.091	0.588	-	-	-	-	-	33.407	1018.355	12.684	5.054	31.032	1131.429	1453.955	685.247	175.769	2034.953	0.744	1.901	0.536	0.409	0.834	0.446	0.201	0.438	0.086	0.099	0.054	0.468	0.207	0.466	-253.907	-327.764	-175.552	0.008
100	0.364	0.701	0.230	0.112	0.532	-	-	-	-	-	27.337	272.177	13.516	8.794	27.032	902.136	1035.732	591.019	184.072	1557.824	0.567	1.163	0.477	0.391	0.637	0.432	0.190	0.429	0.077	0.083	0.053	0.491	0.193	0.491	-511.374	-645.066	-374.990	0.001
200	0.360	0.671	0.239	0.131	0.493	-	-	-	-	-	22.030	95.681	13.693	9.971	22.629	751.406	767.850	531.656	198.212	1262.748	0.485	0.496	0.446	0.384	0.545	0.420	0.181	0.425	0.070	0.071	0.052	0.510	0.181	0.507	-1028.853	-1273.985	-781.586	0.000
500	0.346	0.547	0.245	0.146	0.454	-	-	-	-	-	16.580	22.836	13.719	10.874	19.336	645.935	582.543	491.908	211.364	1047.395	0.448	0.336	0.427	0.382	0.490	0.410	0.171	0.424	0.066	0.061	0.053	0.523	0.170	0.518	-2584.132	-3139.807	-2018.869	0.001
1000	0.339	0.501	0.248	0.153	0.433	-	-	-	-	-	14.913	6.817	13.770	11.153	17.771	601.781	484.002	484.648	221.318	969.823	0.430	0.111	0.419	0.381	0.469	0.407	0.166	0.425	0.065	0.057	0.053	0.529	0.163	0.519	-5174.694	-6244.863	-4098.721	0.001

Table 14.19: Exponential-Exponential-Weibull mixture on mid-price waiting time zero inflated (exponentially distributed) BARC data.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.591	3.482	2.413	1.396	6.299	0.989	0.096	0.985	0.908	1.084	21.208	230.455	11.336	1.607	23.989	615.582	965.189	318.908	81.651	1055.254	0.547	1.101	0.430	0.352	0.586	0.284	0.186	0.244	0.114	0.140	0.065	0.602	0.215	0.636	-253.992	-328.683	-175.592	0.005
100	3.329	3.308	2.343	1.464	4.954	1.011	0.076	1.006	0.943	1.089	16.879	65.775	11.723	7.525	20.203	468.237	632.470	280.804	91.379	774.924	0.449	0.815	0.397	0.343	0.486	0.274	0.171	0.244	0.097	0.122	0.058	0.629	0.194	0.657	-513.277	-647.371	-376.850	0.001
200	3.084	3.090	2.306	1.523	3.569	1.024	0.066	1.019	0.961	1.093	14.034	28.934	11.691	8.682	16.749	387.806	442.892	261.249	99.940	629.164	0.406	0.293	0.381	0.339	0.444	0.270	0.158	0.250	0.083	0.105	0.055	0.647	0.177	0.668	-1033.137	-1278.385	-785.337	0.000
500	2.728	2.381	2.275	1.582	3.064	1.032	0.058	1.030	0.974	1.094	12.227	5.299	11.590	9.633	14.441	336.006	312.566	247.711	111.041	529.576	0.384	0.114	0.370	0.337	0.419	0.270	0.146	0.260	0.071	0.080	0.053	0.659	0.162	0.675	-2594.002	-3149.342	-2025.888	0.000
1000	2.586	2.071	2.270	1.603	2.936	1.035	0.054	1.034	0.978	1.093	11.768	2.454	11.533	10.032	13.436	317.986	270.289	242.536	116.803	496.287	0.376	0.073	0.366	0.336	0.409	0.270	0.139	0.265	0.068	0.072	0.054	0.663	0.154	0.673	-5193.187	-6266.868	-4121.083	0.000

Table 14.20: Gamma-Exponential-Weibull mixture on mid-price waiting time zero inflated (exponentially distributed) BARC data.

n	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	16.679	1517.534	3.693	1.609	12.985	9.385	25.431	1.387	0.911	3.749	30.671	792.873	11.653	6.018	30.501	2012.603	2893.017	1203.321	286.730	3958.136	1.103	2.529	0.680	0.461	1.422	0.518	0.214	0.542	0.095	0.101	0.065	0.388	0.208	0.365	-253.595	-325.728	-167.883	0.034
100	23.162	2612.882	3.497	1.689	9.020	8.678	23.910	1.276	0.864	2.452	23.901	244.704	12.606	8.572	24.274	1738.435	2180.157	1144.371	303.633	3128.261	0.807	2.062	0.605	0.439	0.971	0.523	0.201	0.551	0.086	0.089	0.063	0.390	0.195	0.369	-507.722	-641.615	-364.884	0.012
200	5.788	84.496	3.369	1.790	7.510	7.968	21.857	1.234	0.848	1.970	19.682	156.879	12.996	9.741	20.142	1516.755	1619.710	1078.929	332.029	2630.856	0.658	1.562	0.564	0.430	0.793	0.522	0.188	0.553	0.083	0.082	0.063	0.395	0.182	0.376	-1019.666	-1267.975	-763.392	0.006
500	4.763	14.446	3.297	1.939	6.719	7.175	20.184	1.224	0.844	1.701	15.052	17.145	13.268	10.692	17.513	1325.946	1223.694	1011.806	367.932	2226.557	0.577	0.582	0.533	0.429	0.678	0.519	0.172	0.552	0.082	0.077	0.065	0.399	0.163	0.380	-2559.804	-3123.601	-1979.192	0.003
1000	4.274	4.128	3.220	2.020	6.281	6.684	18.833	1.226	0.848	1.612	14.207	13.571	13.451	11.177	16.472	1240.451	1079.265	972.460	395.594	2039.178	0.538	0.162	0.516	0.432	0.631	0.517	0.163	0.553	0.081	0.069	0.066	0.402	0.153	0.383	-5129.873	-6212.405	-4026.521	0.003

Table 14.21: Loglogistic-Exponential-Weibull mixture on mid-price waiting time zero inflated (exponentially distributed) BARC data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	12.242	30.057	5.678	2.580	21.892	1.268	1.873	0.945	0.662	1.875	36.300	1049.228	12.441	5.319	40.379	1608.040	2477.019	999.278	280.008	3298.617	0.906	1.760	0.623	0.448	1.227	0.504	0.204	0.514	0.088	0.108	0.052	0.408	0.199	0.395	-255.035	-325.565	-164.413	0.047
100	10.174	22.155	5.244	2.658	15.690	1.034	1.619	0.845	0.623	1.368	30.154	244.998	13.681	8.873	33.797	1409.066	3203.267	924.010	291.124	2662.765	0.681	1.062	0.556	0.425	0.873	0.513	0.188	0.526	0.070	0.085	0.043	0.417	0.183	0.407	-512.639	-641.838	-358.932	0.032
200	8.425	15.930	4.992	2.768	12.605	0.913	0.617	0.799	0.612	1.151	27.388	192.228	14.309	10.234	27.957	1262.539	7906.792	857.451	307.419	2240.207	0.579	0.575	0.517	0.415	0.725	0.516	0.172	0.530	0.058	0.069	0.039	0.426	0.168	0.418	-1029.384	-1267.894	-751.095	0.029
500	7.046	9.228	4.814	2.924	10.756	0.841	0.379	0.778	0.608	1.000	19.892	41.945	14.950	11.353	23.649	1056.311	1563.211	796.764	327.735	1858.706	0.518	0.241	0.489	0.412	0.628	0.514	0.153	0.530	0.051	0.055	0.037	0.435	0.150	0.428	-2582.110	-3123.164	-1939.894	0.030
1000	6.490	6.401	4.700	3.040	9.890	0.811	0.319	0.770	0.611	0.939	17.601	20.799	15.255	11.779	21.732	971.370	798.009	769.677	346.945	1669.095	0.495	0.108	0.477	0.413	0.585	0.513	0.145	0.531	0.049	0.050	0.037	0.438	0.140	0.431	-5168.000	-6210.824	-3964.365	0.029

Table 14.22: Weibull-Exponential-Weibull mixture on mid-price waiting time zero inflated (exponentially distributed) BARC data.

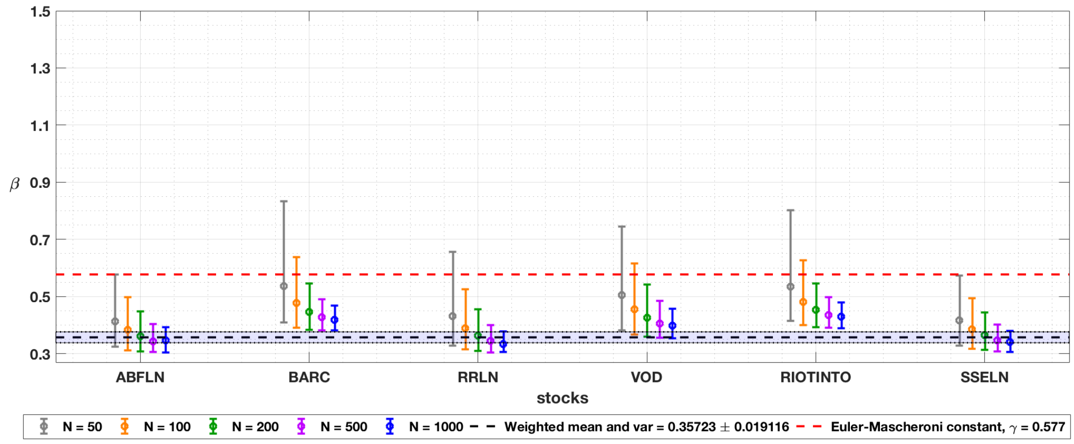


Figure 14.11: Exponential/Exponential/Weibull mixture: exponential distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

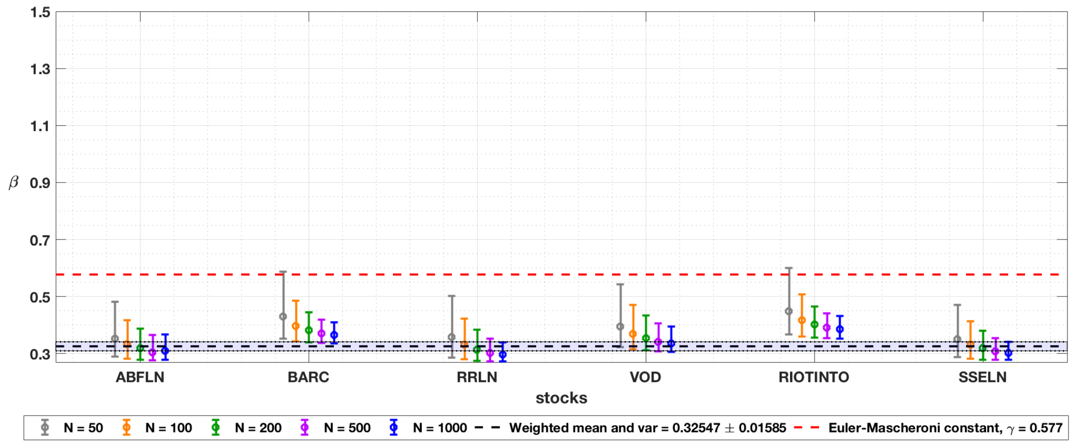


Figure 14.12: Gamma/Exponential/Weibull mixture: exponential distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

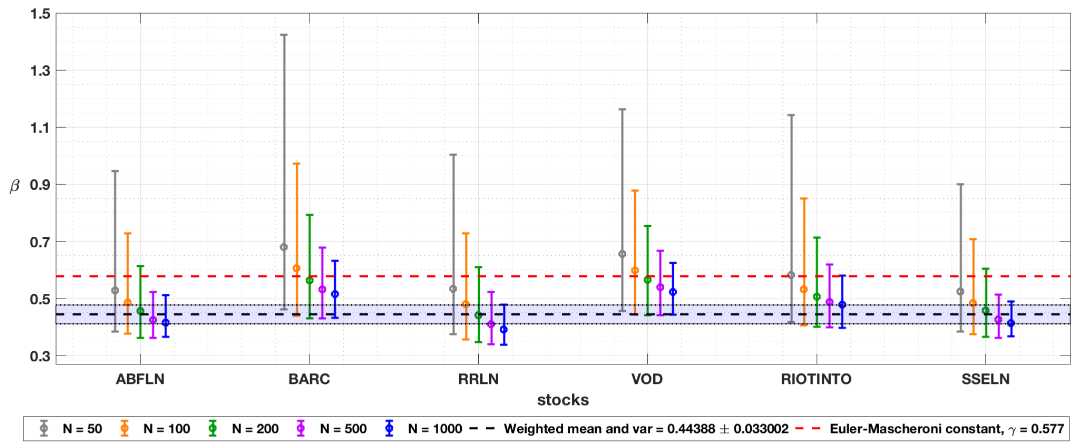


Figure 14.13: Loglogistic/Exponential/Weibull mixture: exponential distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

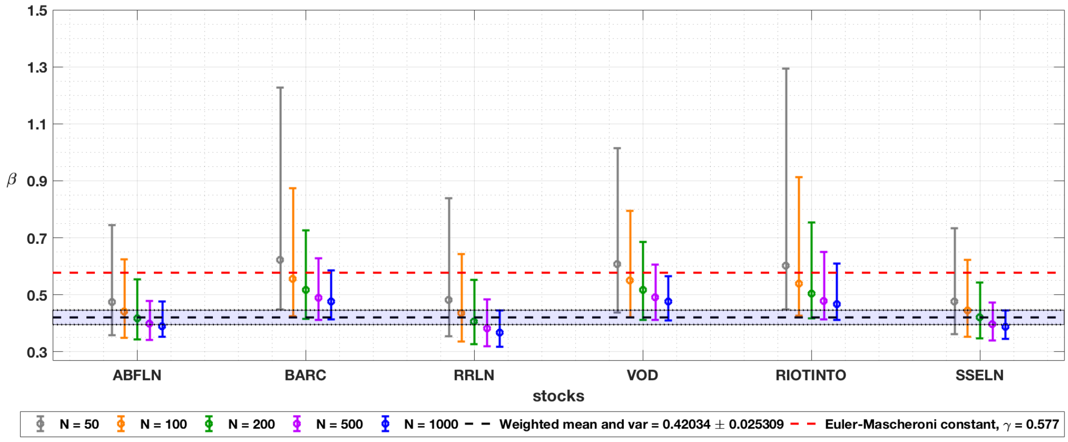


Figure 14.14: Weibull/Exponential/Weibull mixture: exponential distribution of zeros. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

Goodness-of-fit Testing:

Tables [14.23 - 14.27] (Tables [D.133 - D.162] in appendix) present Akaike Information Criterion and Bayesian Information Criterion results for each mixed distribution. Model selection results for this case indicate that loglogistic/exponential/Weibull mixtures most appropriately described the full distribution of mid-price waiting times. Mixtures including a gamma component once again performed poorly.

	AIC			BIC		
Exponential/Exponential/Weibull	0.3477	0.6011	-	0.7878	-	-
Gamma/Exponential/Weibull	0.0405	0.1030	0.2388	0.0121	0.0740	0.2388
Loglogistic/Exponential/Weibull	0.4811	-	-	0.1669	0.6450	-
Weibull/Exponential/Weibull	0.1286	0.2937	0.7583	0.0311	0.2786	0.7583

Table 14.23: BARC, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
Exponential/Exponential/Weibull	0.2016	0.5249	-	0.7294	-	-
Gamma/Exponential/Weibull	0.0223	0.1011	0.2022	0.0058	0.0382	0.2022
Loglogistic/Exponential/Weibull	0.6674	-	-	0.2393	0.7672	-
Weibull/Exponential/Weibull	0.1083	0.3737	0.7973	0.0252	0.1942	0.7973

Table 14.24: BARC, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
Exponential/Exponential/Weibull	0.0828	0.4088	0.8475	0.5457	-	-
Gamma/Exponential/Weibull	0.0097	0.0970	0.1524	0.0030	0.0163	0.1699
Loglogistic/Exponential/Weibull	0.8189	-	-	0.4264	0.8592	-
Weibull/Exponential/Weibull	0.0886	0.4943	-	0.0248	0.1244	0.8301

Table 14.25: BARC, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
Exponential/Exponential/Weibull	0.0188	0.2470	0.8535	0.2010	0.7554	-
Gamma/Exponential/Weibull	0.0042	0.0891	0.1464	0.0015	0.0180	0.1317
Loglogistic/Exponential/Weibull	0.9158	-	-	0.7672	-	-
Weibull/Exponential/Weibull	0.0613	0.6639	-	0.0303	0.2265	0.8683

Table 14.26: BARC, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
Exponential/Exponential/Weibull	0.0094	0.1500	0.8582	0.0707	0.5786	-
Gamma/Exponential/Weibull	0.0036	0.0818	0.1417	0.0007	0.0212	0.1130
Loglogistic/Exponential/Weibull	0.9472	-	-	0.8983	-	-
Weibull/Exponential/Weibull	0.0398	0.7682	-	0.0302	0.4002	0.8869

Table 14.27: BARC, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

14.4 g-component Exponential Mixture

Tabulated Estimates:
Tables [14.28 - 14.31] (Tables [D.163 - D.186] in appendix) present parameter estimates for the following g-component exponential mixtures and sample sizes,

- 2-component Exponential
- 4-component Exponential
- 6-component Exponential
- 10-component Exponential

$$n = \{50, 100, 200, 500, 1000\}, \tag{14.4}$$

where "zero inflated" data is once again distributed according to an exponential distribution with rate parameter $\lambda = 8$, as adopted by the previous case presented in Section 14.3. For mixtures with number of components $g = (2, 4)$ the mean, standard deviation, percentile, and median parameter estimates are tabulated. Due to space restrictions on the page, only the mean estimates are quoted for $g = (6, 10)$. Mean and percentile log-likelihood values, in addition to the percentage of non-converging intervals are provided for all g-component mixtures.

Mixture Proportions:
Mixture proportion estimates are once again stable with respect to varying sample sizes n, but remain stock dependent.

Component Parameters:
As mentioned in the limit order analysis, this thesis adopted the convention for parametrising the exponential distribution with rate parameter λ. Mixture components parametrised by a larger λ value therefore describe density in the short time scale region, whilst components with smaller λ values are responsible for density in the intermediate and tail region.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04393	0.54143	0.00063	0.50607	7.20561	0.19966	0.37808	0.17903	0.36499	0.62192	0.17903	0.63501	-262.02611	-338.10272	-183.96923	0.00000
100	0.03982	0.52761	0.00057	0.38837	1.24377	0.18557	0.37125	0.16065	0.36066	0.62875	0.16065	0.63934	-531.54253	-667.91673	-393.21979	0.00000
200	0.03863	0.52504	0.00054	0.33587	0.77718	0.17961	0.36823	0.14812	0.35783	0.63177	0.14812	0.64217	-1072.58913	-1323.07537	-819.41032	0.00000
500	0.03523	0.49708	0.00052	0.30555	0.68761	0.17692	0.36656	0.13707	0.35481	0.63344	0.13707	0.64519	-2698.88315	-3270.18187	-2117.82716	0.00000
1000	0.03038	0.45308	0.00051	0.29735	0.76597	0.17658	0.36571	0.13122	0.35265	0.63429	0.13122	0.64735	-5409.99732	-6520.32446	-4295.76057	0.00000

Table 14.28: 2-component Exponential mixture on mid-price waiting time BARC data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04194	0.52922	0.00044	0.08258	0.52905	0.02448	0.51238	0.65050	0.35749	47.40284	4245.93011	10.66680	0.25187	0.14828	0.23896	0.28195	0.12706	0.27206	0.36792	0.17968	0.35227	0.09826	0.10743	0.06733	-251.58843	-327.09943	-174.00351	0.00015
100	0.03834	0.51625	0.00039	0.07008	0.51553	0.01947	0.46222	0.59850	0.33011	30.48914	2195.16076	11.99219	0.24371	0.12976	0.23306	0.26950	0.11057	0.26048	0.38331	0.16619	0.37117	0.10348	0.11015	0.07114	-509.63260	-644.71783	-372.96931	0.00015
200	0.03710	0.51247	0.00036	0.06275	0.51178	0.01625	0.41737	0.57669	0.29993	15.73491	83.30883	12.68733	0.23700	0.11621	0.22649	0.25825	0.09705	0.25055	0.39470	0.15706	0.38678	0.11005	0.11829	0.07383	-1027.59484	-1274.86461	-778.25865	0.00005
500	0.03372	0.48266	0.00034	0.05394	0.48213	0.01366	0.37032	0.53881	0.26907	13.53844	12.35207	13.19343	0.23006	0.10293	0.22076	0.24712	0.08534	0.24021	0.41068	0.15154	0.40711	0.11214	0.12262	0.07554	-2583.86425	-3144.09582	-2017.00639	0.00012
1000	0.03113	0.46364	0.00033	0.04888	0.46306	0.01287	0.34902	0.52093	0.25717	13.50361	5.38433	13.55845	0.22760	0.09469	0.21867	0.24102	0.07856	0.23348	0.42485	0.14745	0.42932	0.10653	0.11614	0.07485	-5177.61116	-6254.19577	-4101.32274	0.00025

Table 14.29: 4-component Exponential mixture on mid-price waiting time BARC data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04208	0.04758	0.12453	0.56811	4.73889	80.17121	0.17071	0.18290	0.24623	0.29444	0.05544	0.05027	-250.52166	-325.67644	-173.01448	0.00122
100	0.03844	0.04315	0.11114	0.55482	4.34241	69.33903	0.16354	0.18483	0.24381	0.30134	0.05468	0.05179	-507.36231	-641.97684	-370.40144	0.00166
200	0.03705	0.04115	0.10149	0.53741	3.97277	62.31217	0.15580	0.18580	0.24030	0.30546	0.05827	0.05438	-1022.59061	-1268.75510	-772.38535	0.00148
500	0.03643	0.03966	0.09135	0.49759	3.38973	28.84779	0.14428	0.18526	0.23179	0.30297	0.07821	0.05749	-2570.48587	-3129.15968	-2005.04641	0.00043
1000	0.03293	0.03567	0.08193	0.46002	3.12864	18.34616	0.13603	0.18373	0.22193	0.29438	0.10478	0.05916	-5149.11728	-6223.15426	-4079.04646	0.00025

Table 14.30: 6-component Exponential mixture on mid-price waiting time BARC data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\lambda}_{7\text{ ave}}$	$\hat{\lambda}_{8\text{ ave}}$	$\hat{\lambda}_{9\text{ ave}}$	$\hat{\lambda}_{10\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	$\hat{\pi}_{7\text{ ave}}$	$\hat{\pi}_{8\text{ ave}}$	$\hat{\pi}_{9\text{ ave}}$	$\hat{\pi}_{10\text{ ave}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04255	0.04713	0.11433	0.31252	0.59605	0.77577	1.21738	10.48246	28.49003	95.24428	0.16865	0.17379	0.20799	0.14064	0.13522	0.05919	0.04110	0.01402	0.01925	0.04016	-250.45201	-325.44005	-170.88594	0.00668
100	0.03884	0.04279	0.10028	0.29273	0.58858	0.70859	0.91601	11.16070	22.82700	124.68247	0.15896	0.17687	0.20168	0.14802	0.13879	0.06137	0.04196	0.01341	0.01945	0.03948	-507.03864	-641.01691	-364.54120	0.01038
200	0.03839	0.04172	0.08964	0.26963	0.58651	0.67943	0.75340	12.05328	20.35473	72.75583	0.14706	0.17797	0.19398	0.15903	0.14347	0.06388	0.04327	0.01279	0.01940	0.03916	-1021.84425	-1267.12374	-764.98070	0.00713
500	0.03535	0.03777	0.07317	0.22650	0.58242	0.64906	0.66987	13.34231	17.74193	31.84452	0.12868	0.17633	0.17887	0.17858	0.15320	0.06815	0.04558	0.01210	0.01916	0.03935	-2568.52485	-3124.81569	-1997.14730	0.00235
1000	0.03102	0.03285	0.06200	0.19021	0.57420	0.63171	0.64223	13.92086	17.16302	21.81031	0.11356	0.17468	0.16538	0.19361	0.16234	0.07230	0.04790	0.01175	0.01888	0.03958	-5145.57982	-6215.90880	-4068.28048	0.00187

Table 14.31: 10-component Exponential mixture on mid-price waiting time BARC data.

Goodness-of-fit Testing:
Tables [14.32 - 14.36] (Tables [D.187 - D.216] in appendix) present Akaike Information Criterion and Bayesian Information Criterion results for each g-component exponential mixture. Testing indicated (for most sample sizes and stocks) that 4-component exponential mixtures most consistently provided the most suitable balance between goodness-of-fit and complexity of model. BIC, which imposes a stricter penalty on the complexity of the model, favoured 4-component mixtures even for larger sample sizes. As seen in the limit order analysis, Tables [14.37 - 14.41] present model selection results comparing the best two performing mixtures from Section 14.3 to the 4 and 6-component exponential mixture. Results indicated that the inclusion of a Weibull component yields better results.

	AIC			**BIC**		
2-comp-Exponential	0.1940	0.3806	0.7622	0.7860	-	-
4-comp-Exponential	0.6780	-	-	0.2136	0.9985	-
6-comp-Exponential	0.1267	0.6121	-	0.0004	0.0015	0.9988
10-comp-Exponential	0.0014	0.0073	0.2378	0.0000	0.0000	0.0010

Table 14.32: BARC, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			**BIC**		
2-comp-Exponential	0.0388	0.0862	0.3133	0.4547	0.8383	-
4-comp-Exponential	0.6720	-	-	0.5427	-	-
6-comp-Exponential	0.2829	0.8907	-	0.0026	0.1617	0.9983
10-comp-Exponential	0.0062	0.0231	0.6867	0.0000	0.0000	0.0014

Table 14.33: BARC, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			**BIC**		
2-comp-Exponential	0.0066	0.0081	0.0492	0.1366	0.4375	0.9151
4-comp-Exponential	0.4810	0.8572	-	0.8463	-	-
6-comp-Exponential	0.4871	-	-	0.0171	0.5625	-
10-comp-Exponential	0.0253	0.1346	0.9508	0.0000	0.0000	0.0849

Table 14.34: BARC, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			**BIC**		
2-comp-Exponential	0.0028	0.0035	0.0042	0.0073	0.0277	0.3406
4-comp-Exponential	0.1987	0.4725	0.9958	0.8383	-	-
6-comp-Exponential	0.6769	-	-	0.1544	0.9722	-
10-comp-Exponential	0.1216	0.5240	-	0.0000	0.0001	0.6594

Table 14.35: BARC, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-Exponential	0.0025	0.0027	0.0031	0.0037	0.0055	0.0242
4-comp-Exponential	0.0655	0.1824	0.9969	0.5133	-	-
6-comp-Exponential	0.6902	-	-	0.4829	0.9943	-
10-comp-Exponential	0.2418	0.8149	-	0.0000	0.0002	0.9758

Table 14.36: BARC, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
4-comp-Exponential	0.3177	0.4625	0.8572	0.1440	0.4005	0.9985
6-comp-Exponential	0.0418	0.0563	0.1428	0.0001	0.0003	0.0015
Exponential/Exponential/Weibull	0.2419	0.4812	-	0.6915	-	-
Loglogistic/Exponential/Weibull	0.3985	-	-	0.1644	0.5993	-

Table 14.37: BARC, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
4-comp-Exponential	0.2914	0.4568	-	0.1829	0.3552	0.9957
6-comp-Exponential	0.0964	0.1497	0.3358	0.0002	0.0004	0.0043
Exponential/Exponential/Weibull	0.1300	0.3936	0.6642	0.6115	-	-
Loglogistic/Exponential/Weibull	0.4822	-	-	0.2053	0.6444	-

Table 14.38: BARC, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
4-comp-Exponential	0.1962	0.3506	-	0.1861	0.2620	0.9809
6-comp-Exponential	0.1761	0.3290	0.5414	0.0007	0.0011	0.0191
Exponential/Exponential/Weibull	0.0535	0.3204	0.4586	0.4589	-	-
Loglogistic/Exponential/Weibull	0.5742	-	-	0.3543	0.7369	-

Table 14.39: BARC, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
4-comp-Exponential	0.0710	0.1708	0.4248	0.1110	0.2636	0.8445
6-comp-Exponential	0.2692	0.6082	-	0.0055	0.0139	0.1554
Exponential/Exponential/Weibull	0.0123	0.2210	0.5752	0.1777	0.7225	-
Loglogistic/Exponential/Weibull	0.6475	-	-	0.7058	-	-

Table 14.40: BARC, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
4-comp-Exponential	0.0238	0.0675	0.3418	0.0505	0.2107	0.5161
6-comp-Exponential	0.3030	0.7864	-	0.0192	0.1042	0.4837
Exponential/Exponential/Weibull	0.0052	0.1461	0.6582	0.0629	0.6851	-
Loglogistic/Exponential/Weibull	0.6679	-	-	0.8674	-	-

Table 14.41: BARC, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

14.5 Censoring "zero inflated" Data

Tabulated Estimates:
Tables [14.42 - 14.45] (Tables [D.217 - D.240] in appendix) present parameter estimates for the following mixtures, sample sizes, and censor regions,

- 3-comp exp: censored on $[0, 0.5]$ms
- 4-comp exp: censored on $[0, 0.5]$ms
- 3-comp exp: censored on $[0, 0.5, 1.5, 2.5, 10]$ms
- 4-comp exp: censored on $[0, 0.5, 1.5, 2.5, 10]$ms

$$n = \{50, 100, 200, 500, 1000\}, \qquad (14.5)$$

for the case when "zero inflated" data is dealt with by censoring data in the short time scale region. Table 14.46 (Tables [D.241 - D.246] in appendix) presents estimates for the following mixture, sample sizes, and censor region,

- exp/exp/Weibull: censored on $[0, 0.5]$ms

$$n = \{200, 500, 1000, 2000, 5000\}. \qquad (14.6)$$

Tabulated data once again includes the mean, standard deviation, median, and percentile parameter estimates, in addition to the mean and percentile log-likelihood values, and the percentage of non-converging intervals.

Mixture Proportions:
Once again, mixture proportions displayed strong stability with respect to varying sample size n, but are dependent on stock. For censored mixtures which include a Weibull component, the second component (exponential) makes a negligible contribution to the mixture. This is consistent with the behaviour of the uniform and exponential density from Section 14.2 and 14.3 respectively.

Component Parameters:
Figure 14.15 displays the Weibull shape parameter estimates for the exponential/exponential/Weibull mixture with a censor region [0, 0.5]ms, for all stocks and sample sizes. Median values are represented on the figure with 68% percent of interval estimates falling within the error bars. The black dotted line represents the weighted estimate, and the red dotted line the Euler-Mascheroni constant. Consistent with the previous observations in the mid-price analysis, the Weibull shape parameter doesn't follow the Euler-Mascheroni constant, as was the case with the limit order analysis where it corresponded to the maximum entropy of the system for the Weibull distribution.

Goodness-of-fit Testing:
Tables [14.47 - 14.49] (Tables [D.247 - D.264] in appendix) present Akaike Information Criterion and Bayesian Information Criterion results for comparison of the 3-component exponential, 4-component exponential, and exponential/exponential/Weibull mixtures, for common sample sizes n = (200, 500, 1000), and a single censor region [0, 0.5]ms. The analogous model selection for the limit order inter-arrival time analysis yielded mixtures including a Weibull component outperformed those without one. The results of this analysis are not definitive.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.096524	1.264367	0.000440	0.000179	0.001275	0.138	1.263	0.026	0.008	0.086	1.168	2.533	0.541	0.218	1.487	0.255	0.148	0.243	0.303	0.133	0.295	0.442	0.201	0.427	-259.258	-331.757	-184.068	0.000
100	0.097065	1.314782	0.000403	0.000181	0.000956	0.134	1.313	0.024	0.008	0.072	0.941	1.933	0.521	0.225	1.349	0.254	0.132	0.244	0.296	0.115	0.290	0.449	0.183	0.440	-524.785	-655.172	-392.193	0.000
200	0.099302	1.363776	0.000384	0.000184	0.000808	0.132	1.362	0.023	0.008	0.063	0.828	1.626	0.499	0.229	1.256	0.254	0.120	0.244	0.289	0.100	0.283	0.457	0.170	0.455	-1057.632	-1297.004	-814.541	0.001
500	0.098975	1.400181	0.000369	0.000190	0.000709	0.128	1.398	0.022	0.009	0.053	0.763	1.507	0.484	0.232	1.165	0.255	0.108	0.243	0.281	0.088	0.275	0.464	0.159	0.473	-2658.396	-3202.160	-2097.384	0.002
1000	0.096866	1.412820	0.000362	0.000193	0.000663	0.124	1.411	0.021	0.009	0.048	0.735	1.487	0.469	0.236	1.118	0.254	0.101	0.240	0.276	0.083	0.269	0.470	0.153	0.486	-5326.843	-6382.365	-4262.027	0.002

Table 14.42: 3-component Exponential mixture on mid-price waiting time BARC data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.097404	1.266605	0.000436	0.000178	0.001265	0.138	1.266	0.025	0.008	0.081	1.080	1.890	0.550	0.228	1.703	0.253	0.147	0.241	0.298	0.136	0.289	0.450	0.205	0.435	-245.651	-320.807	-166.787	0.001
100	0.097311	1.315710	0.000399	0.000180	0.000945	0.132	1.314	0.022	0.008	0.067	0.919	1.570	0.528	0.238	1.473	0.252	0.131	0.242	0.289	0.116	0.282	0.459	0.186	0.449	-495.760	-630.966	-356.475	0.001
200	0.099314	1.363923	0.000380	0.000184	0.000795	0.130	1.362	0.021	0.008	0.056	0.837	1.494	0.513	0.243	1.319	0.252	0.119	0.242	0.281	0.100	0.274	0.467	0.171	0.465	-996.280	-1245.270	-741.319	0.001
500	0.098948	1.400268	0.000365	0.000190	0.000691	0.125	1.399	0.020	0.009	0.046	0.773	1.484	0.500	0.249	1.068	0.251	0.106	0.241	0.271	0.087	0.264	0.477	0.158	0.483	-2497.546	-3069.221	-1913.870	0.002
1000	0.096855	1.412908	0.000359	0.000194	0.000640	0.121	1.411	0.019	0.010	0.041	0.738	1.477	0.492	0.256	0.924	0.251	0.099	0.239	0.266	0.081	0.257	0.483	0.150	0.495	-4996.085	-6105.190	-3877.189	0.002

Table 14.43: 3-component Exponential mixture on mid-price waiting time BARC data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\lambda}_{4\text{ lower}}$	$\hat{\lambda}_{4\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.097569	1.272208	0.000367	0.000146	0.001188	0.111	1.271	0.008	0.002	0.027	0.235	1.270	0.109	0.047	0.238	2.172	4.083	0.844	0.337	2.179	0.204	0.135	0.187	0.199	0.126	0.182	0.283	0.133	0.273	0.314	0.189	0.278	-257.698	-329.599	-178.563	0.013
100	0.098362	1.324506	0.000321	0.000139	0.000829	0.109	1.324	0.007	0.002	0.021	0.220	1.321	0.100	0.046	0.201	1.777	3.495	0.832	0.370	1.910	0.195	0.119	0.182	0.200	0.111	0.188	0.279	0.115	0.272	0.326	0.177	0.297	-521.325	-650.325	-380.375	0.015
200	0.100772	1.374796	0.000292	0.000133	0.000663	0.109	1.374	0.005	0.002	0.016	0.205	1.370	0.091	0.042	0.166	1.364	2.659	0.797	0.385	1.704	0.185	0.106	0.175	0.199	0.099	0.188	0.274	0.097	0.269	0.342	0.165	0.318	-1050.222	-1286.532	-790.314	0.017
500	0.100901	1.414675	0.000264	0.000127	0.000557	0.107	1.414	0.004	0.001	0.012	0.186	1.410	0.080	0.036	0.137	1.072	1.887	0.737	0.384	1.567	0.172	0.091	0.161	0.195	0.088	0.186	0.270	0.078	0.266	0.363	0.152	0.348	-2637.142	-3176.440	-2029.638	0.022
1000	0.098832	1.428019	0.000248	0.000125	0.000510	0.103	1.428	0.003	0.001	0.010	0.175	1.424	0.073	0.033	0.123	0.988	1.689	0.698	0.371	1.515	0.164	0.082	0.154	0.194	0.081	0.184	0.267	0.067	0.265	0.376	0.143	0.370	-5281.976	-6320.345	-4118.059	0.024

Table 14.44: 4-component Exponential mixture on mid-price waiting time BARC data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\lambda}_{4\text{ lower}}$	$\hat{\lambda}_{4\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.098552	1.274652	0.000365	0.000145	0.001192	0.112	1.274	0.008	0.002	0.028	0.247	1.274	0.113	0.048	0.267	1.740	2.692	0.842	0.332	2.718	0.204	0.135	0.186	0.200	0.127	0.184	0.280	0.132	0.270	0.316	0.189	0.279	-242.728	-318.812	-157.710	0.014
100	0.098560	1.325111	0.000320	0.000138	0.000825	0.109	1.324	0.007	0.002	0.021	0.229	1.322	0.102	0.045	0.223	1.467	2.085	0.830	0.365	2.381	0.195	0.119	0.181	0.201	0.111	0.189	0.275	0.114	0.268	0.330	0.178	0.299	-491.561	-628.714	-340.596	0.015
200	0.100617	1.373676	0.000290	0.000132	0.000658	0.109	1.373	0.005	0.001	0.017	0.212	1.370	0.090	0.040	0.181	1.274	1.731	0.789	0.381	2.117	0.185	0.106	0.174	0.199	0.099	0.189	0.269	0.098	0.262	0.347	0.170	0.323	-990.455	-1242.243	-715.112	0.015
500	0.100382	1.411255	0.000261	0.000124	0.000550	0.106	1.411	0.003	0.001	0.012	0.188	1.409	0.075	0.033	0.141	1.115	1.574	0.698	0.374	1.898	0.170	0.092	0.159	0.195	0.088	0.186	0.260	0.078	0.255	0.375	0.159	0.362	-2487.664	-3069.376	-1850.116	0.017
1000	0.098014	1.422243	0.000244	0.000122	0.000499	0.102	1.422	0.003	0.001	0.010	0.173	1.418	0.067	0.029	0.121	1.043	1.534	0.646	0.368	1.800	0.161	0.083	0.150	0.194	0.081	0.184	0.255	0.066	0.251	0.391	0.151	0.389	-4986.320	-6107.545	-3776.119	0.016

Table 14.45: 4-component Exponential mixture on mid-price waiting time BARC data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	0.461	1.457	0.248	0.136	0.896	16.495	1.293	16.411	15.307	18.301	701.670	746.733	481.730	175.491	1646.589	0.433	0.105	0.420	0.346	0.577	0.396	0.181	0.396	0.035	0.073	0.002	0.569	0.179	0.568	-1094.288	-1292.837	-810.566	0.082
500	0.366	0.310	0.263	0.154	0.779	17.160	1.295	17.190	15.825	18.831	568.021	565.488	408.848	161.485	1182.902	0.402	0.071	0.395	0.338	0.491	0.387	0.175	0.397	0.025	0.043	0.000	0.588	0.165	0.578	-2727.808	-3190.939	-2095.658	0.063
1000	0.453	1.235	0.274	0.162	0.715	17.652	1.340	17.761	16.208	19.161	500.074	462.360	372.491	152.476	955.482	0.386	0.064	0.383	0.334	0.454	0.384	0.172	0.399	0.025	0.064	0.000	0.591	0.166	0.580	-5395.337	-6347.324	-4258.861	0.043
2000	0.438	1.122	0.280	0.171	0.685	18.116	1.400	18.254	16.607	19.629	452.352	390.519	345.164	146.203	841.301	0.375	0.055	0.374	0.331	0.431	0.383	0.168	0.402	0.022	0.058	0.000	0.596	0.160	0.582	-10738.821	-12562.312	-8529.814	0.035

Table 14.46: Exponential-Exponential-Weibull mixture on mid-price waiting time BARC data with censoring on region [0, 0.5].

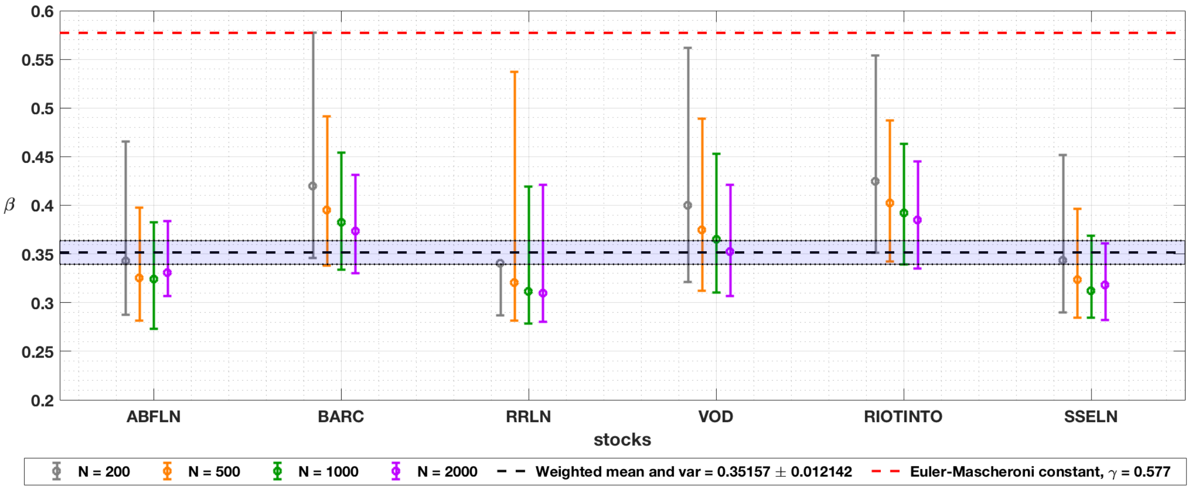


Figure 14.15: Exponential/Exponential/Weibull mixture: censored on $[0, 0.5]$ms. Median and percentile estimates $\hat{\beta}$ for each stock and sample size.

	AIC		BIC	
3-comp Exponential	0.2663	0.5078	0.7026	-
4-comp Exponential	0.5201	-	0.0559	0.4022
Exponential/Exponential/Weibull	0.2135	0.4921	0.2414	0.5965

Table 14.47: BARC, N = 200: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp Exponential	0.0816	0.3263	0.3245	0.4600
4-comp Exponential	0.7214	-	0.3394	-
Exponential/Exponential/Weibull	0.1968	0.6735	0.3359	0.5399

Table 14.48: BARC, N = 500: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp Exponential	0.0348	0.2236	0.1003	0.2914
4-comp Exponential	0.7643	-	0.5907	-
Exponential/Exponential/Weibull	0.2006	0.7762	0.3088	0.7083

Table 14.49: BARC, N = 1000: mid-price waiting times, censored on region [0, 0.5].

14.6 Summary

The output of the electronic order book is the variation in mid-price, with the waiting time defined as the time separation of consecutive mid-price changes. The main outcomes regarding mid-price waiting time analysis are summarised below:

- The analysis supported the hypothesis that mixed distributions are appropriate to model the distribution of mid-price waiting times.

- The proportion of "zero-inflated" data, and consequently the proportion of data in the other regions of the distribution, is not consistent for each stock. As a result, mixture proportion estimates display strong stability with respect to all sample sizes n, but vary with respect to the activity of a stock. When the uniform or exponential density was tasked with describing the "zero-inflated" data, convergence of the mixture proportions was such that these components made a negligible contribution to the overall mixture. The additional components in these mixtures were sufficient to describe the excess data in this region. This is a notable difference to the limit order analysis where these contributions were consistent with the proportion of "zero-inflated" data.

- The Weibull shape parameter β no longer corresponds to the Euler-Mascheroni constant, and has been reduced, consistent with the findings of the unpublished work of Kizilersü [80]. Further analysis is required to be undertaken to explore the potential universality of the shape parameter for the mid-price waiting time distribution.

- Model selection via information criteria testing yielded far less definitive results than that of the previous analysis. Mixtures including a Weibull component appear to outperform those without one, although further analysis is required. The inclusion of a loglogistic component proved useful, but further analysis is also required.

- Censored mixtures offered the most statistically rigorous and hence most appropriate method to model the full distribution of mid-price waiting times, with strong convergence evident for all mixtures. For censored mixtures including the Weibull component, the weighted estimate of the shape parameter was $\hat{\beta} = 0.352 \pm 0.012$.

14.7 Future Work

- Recall that the price of a particular stock at time t is denoted as S_t. Beginning with $t_0^{\mathrm{MP}} = 0$ and $S_{t_0^{\mathrm{MP}}} > 0$, the stochastic process which describes the stock price is given by the expression,

$$S_t = S_{t_0^{\mathrm{MP}}} + X_{t_1^{\mathrm{MP}}} + X_{t_1^{\mathrm{MP}}} + ... + X_{t_N^{\mathrm{MP}}} \ , \tag{14.7}$$

  where $X_{t_i^{\mathrm{MP}}}$ is the i^{th} incremental change in mid-price at time t_i^{MP}. The sequence of times at which the stock price changes, $t_1^{\mathrm{MP}}, t_2^{\mathrm{MP}}, ..., t_N^{\mathrm{MP}}$, correspond to the mid-price waiting times. The number N price changes from $t_0^{\mathrm{MP}} = 0$ up until $t^{\mathrm{MP}} > t_0^{\mathrm{MP}} = 0$ is a function of t^{MP}, where $N(t^{\mathrm{MP}})$ is the corresponding count process associated with the mid-price waiting time distribution. What the aforementioned analysis doesn't consider is the price changes themselves. There are two additional variables of interest here, the first being the direction (up or down) of the price movement denoted $Y_i = \mathrm{sgn}(X_{t_i^{\mathrm{MP}}})$, and the second being the magnitude of the price movement $Z_i = |X_{t_i^{\mathrm{MP}}}|$. This work does not analyse the potential correlation between the distribution of mid-price waiting times and the additional variables described.

- As discussed within the limit order inter-arrival time analysis, improvements could be made to the initialisation routine. Bias may have been introduced into the estimates from use of fixed initial values.

- Censored EM equations are required to be derived for loglogistic and gamma component distributions. Censored mixtures which include these mixture components were therefore unable to be provided in this analysis.

Chapter 15

Conclusion

An ever increasing need to accurately collect and analyse data is present in the modern world. It has been said that the world's most valuable resource is no longer oil, but data [81]. Data is such a valuable resource to corporate entities because it allows them to offer targeted services to users, ultimately increasing performance. Corporate giants such as Amazon and Google are examples of companies which collect an enormous amount of data from their users, leveraging modern data analysis tools such as machine learning for significant financial gain. Data analysis drives innovation and in recent years has become a key component of much of the progress made in a diverse range of disciplines. Data science skills are therefore heavily sought after in both industry and the sciences, in fact, Harvard Business Review ranked a 'data scientist' as the sexiest job of the 21st century [82].

Probability distributions are employed by statistical models in order to describe data resulting from a certain system. Modelling enhances understanding of certain behaviours, ultimately allowing predictions to be made of the future state of the system. This thesis was concerned with finite mixture models which are predominately used because they can describe quite complex distributions of data, often in situations where single parametric distributions are unable to provide satisfactory results. The construction of such models requires parameters of probability distributions to be estimated from the data. Finding maximum likelihood parameter estimates in mixture models is a typical example of a problem which can be simplified by formulation as an incomplete data problem, allowing the Expectation Maximisation (EM) algorithm to be used. The incompleteness in this context manifests from the fact that the corresponding component for each individual datum is unknown. The EM algorithm is ubiquitous for parameter estimation in mixture modelling. This thesis provided a thorough analysis of mixture modelling and estimation via the EM algorithm. EM equations were derived for normal, Weibull, exponential, gamma, loglogistic, and uniform component distributions. Additionally, a new censored EM framework was presented based on the prior considerations of Chauveau [17], who provided a mathematical proof of the convergence of the EM algorithm in this case. Censored EM equations were derived for an exponential component, and for the first time for a Weibull component.

Goodness-of-fit testing refers to the process of assessing whether a statistical model appropriately describes a given dataset. This is usually quantified by a measure of the discrepancy between the observed values and the values predicted by the candidate model. The first goodness-of-fit framework considered in this thesis was formal hypothesis based testing which is the standard frequentist approach to the task. The validity of a null hypothesis can be determined by comparison of a test-statistic to a previously determined critical value. If the test statistic is smaller than the critical value, the null hypothesis cannot be rejected. This work was concerned with a null hypothesis, H_0, that a given set of n observations $\boldsymbol{x} = \{x_1, x_2, ..., x_n\}$ arose from a population which was distributed according to a cumulative distribution function $F(\boldsymbol{x} \,|\, \hat{\boldsymbol{\psi}})$. The dependencies of critical values for Kolmogorov-Smirnov, Kuiper, Cramér-von Mises, and Anderson Darling goodness-of-fit tests for two-component normal and Weibull mixture models was studied. Our results are consistent with single distribution analysis, namely the critical values depend on sample size n, significance level α, and truncation level τ. Moreover, critical values for the case of mixed distributions also exhibit an additional dependence on the parameters of the component distributions and the mixture proportions. For most non-trivial examples, parameter dependent critical values are largely inadequate for use in goodness-of-fit testing. This thesis therefore considered a second goodness-of-fit framework, namely model selection via information criteria test. This approach compares the relative quality of candidate models via comparison of log-likelihood values, but imposes a penalty consistent with the complexity of each model. A thorough analysis of information criteria tests was presented for Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). Goodness-of-fit testing in this manner was found to be more appropriate for mixture modelling. The major focus of this work was the application of the aforementioned modelling and goodness-of-fit methodologies to financial tick-by-tick data.

Financial institutions are large exponents of statistical analysis. Understanding the stochastic nature of the stock market is important to traders, not just because of the direct prospect of financial gain, but also because of a desire to assess the stability of the market as a whole. In recent years, high frequency trading has become prevalent and institutions are now leveraging significant computing resources in order to partake in it. Computational and statistical methodologies underpin much of the work undertaken within this domain. The culmination of this thesis was an analysis of tick-by-tick data obtained from the London Stock Exchange (LSE). Mixture models were employed in this work because they are capable of disentangling the highly complex dynamical behaviour of high frequency trading from the more traditional forms. An attempt to describe the full distribution of both *limit order inter-arrival times* and *mid-price waiting times* by use of mixed distributions was made. With respect to these distributions, high frequency trading corresponds to excess data in the short time scale region, whilst the remainder of the distribution in the intermediate and tail region is slowly decaying and well behaved.

Limit orders are a specific type of order, defined by the intention to buy or sell a quantity of stock at a specific price. If the market never reaches the limit price, the transaction is not executed. Limit orders arrive to the electronic order book (EOB) and are recorded with an associated timestamp. The time separation of the arrival of consecutive limit orders are known as limit order inter-arrival times. Many limit orders are placed by algorithmic traders with no intention for any actual trading to occur, and sole intention to manipulate behaviours in the market. A large proportion of limit order inter-arrival times are therefore very small. The outcomes of the analysis of limit order inter-arrival times can be summarised as follows:

- Mixed distributions deal with the excess data in the short time scale region, without significant compromise to the estimation efforts in the intermediate and tail region of the distribution, which is well described by a Weibull mixture component.

- The Weibull mixture component is parametrised by a scale parameter α that relates to the inverse activity of the stock, and a shape parameter β related to maximum entropy of the system for the Weibull distribution, consistent with the Euler-Mascheroni constant. This finding supports the work of Kizilersü et al. [19] and Guscott [20].

- Because of the resolution of the available data, a large percentage (around 50%) of data is in fact "zero-inflated" and several methods were proposed to deal with this. Mixture proportion estimates for both the uniform and exponential distributions displayed consistency with the proportion of "zero inflated" data. Mixture proportion estimates were also stable for varying sample size n and varying stock.

- Censored mixtures offered the most statistically rigorous method to model the full distribution of limit order inter-arrival times, whilst dealing with the problem of zero inflation. Convergence was strong for censored mixtures and the corresponding estimates agreed most closely with the left-truncated estimates of Kizilersü et al. [19]. The weighted estimate of the Weibull component shape parameter was $\hat{\beta} = 0.564 \pm 0.013$.

- Although the loglogistic component displayed the largest variability in parameter estimates, model selection indicated that mixture contributions of around 15% (i.e. $\pi_{\text{loglogistic}} \approx 0.15$) yielded useful additions to the transition region of the distribution, although this was not verified in the censored framework.

The mid-price is determined as the arithmetic mean of "best-bid" and "best-ask" (determined by the nature of the limit orders) rounded to the nearest valid tradable price. The mid-price waiting time is defined as the time difference between consecutive price changes. Many of the outcomes of the mid-price waiting time analysis were similar to that of limit order inter-arrival times, although some clear distinctions are also evident:

- For mid-price waiting times, a far smaller percentage of data was evident in the short time scale region (including "zero inflation"). Mixture proportions remained stable for vary sample size n, but now displayed a dependence on the activity of the stock.

- Censored mixtures once again offered the most statistically rigorous method to describe the full distribution of mid-price waiting times, whilst dealing with the problem of zero inflation. For the Weibull mixture component the shape parameter β was no longer consistent with the Euler-Mascheroni constant, with weighted estimate $\hat{\beta} = 0.352 \pm 0.012$. This agrees with the unpublished findings of Kizilersü [80].

- Although model selection via information criteria testing yielded far less definitive results than that of the limit order analysis, mixtures including a Weibull component appeared to outperform those without one. The inclusion of a loglogistic component also proved useful, but further analysis is required.

Our application of the EM algorithm to stock market time stamp data, answered an open question posed by Scalas [79]. He noted that it is difficult to model, by use of probability distributions, the time information of real-life stock market data (with the Weibull and Mittag-Leffler distributions considered as his candidates), especially when considering "zero inflated" data which is evident in the distribution. Whereas Kizilersü et al. [19] handled the "zero inflation" with truncation, Scalas introduced the idea to model the data as a superposition of independent Poisson processes, where the times differences are exponentially distributed. This study demonstrated that the EM algorithm can be employed to estimate the rate parameters and mixture proportions of this model, and has shown that $g = 4$ exponentials (where censoring was considered to deal with problems related to rounding and "zero inflation") is sufficient to provide a satisfactory solution to the original problem posed by Scalas.

This thesis contributes to an effort to understand the temporal nature of tick-by-tick data, such that ultimately the full behaviour of the electronic order book can be modelled.

Appendix A

Critical Values: Two-component Normal Mixture

Section 5.2 considered hypothesis testing for normal mixture models. Critical values were calculated for various two-component normal mixture models defined by,

- homoscedastic: $\boldsymbol{\mu} = \Big\{0,\ [0.5, 1, 1.5, 3, 5]\Big\}, \quad \boldsymbol{\sigma} = \Big\{1, 1\Big\}$
- heteroscedastic: $\boldsymbol{\mu} = \Big\{0,\ 3\Big\}, \quad \boldsymbol{\sigma} = \Big\{1,\ [1, 3, 5, 7, 9]\Big\}$

with the following mixture proportions and sample sizes,

$$\pi_1 = \Big\{0,\ 0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9,\ 1\Big\}, \tag{A.1}$$

$$n = \Big\{50,\ 100,\ 200,\ 500,\ 1000\Big\}, \tag{A.2}$$

for Kolmogorov-Smirnov, Kuiper, Anderson-Darling, and Cramér-von Mises goodness-of-fit tests. The full set of critical values for the two cases considered are presented in this appendix.

A.1 Case I: Parameters/Mixture Proportions Known *a Priori*

- Kolmogorov- Smirnov: Tables [A.1 - A.10]
- Kuiper: Tables [A.11 - A.20]
- Anderson-Darling: Tables [A.21 - A.30]
- Cramér-von Mises: Tables [A.31 - A.40]

A.2 Case II: Component Parameters Known *a Priori*, Mixture Proportions Requiring Estimation

- Kolmogorov- Smirnov: Tables [A.41 - A.50]
- Kuiper: Tables [A.51 - A.60]
- Anderson-Darling: Tables [A.61 - A.70]
- Cramér-von Mises: Tables [A.71 - A.80]

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.3329(17)	1.3243(16)	1.3159(15)	1.3135(17)	1.3164(16)	1.3255(15)	1.3313(15)
$\Delta = 1.0$	1.3326(17)	1.3083(16)	1.2730(15)	1.2595(13)	1.2734(15)	1.3082(15)	1.3321(17)
$\Delta = 1.5$	1.3321(14)	1.2918(15)	1.2150(13)	1.1864(14)	1.2168(15)	1.2903(15)	1.3327(16)
$\Delta = 3.0$	1.3320(14)	1.2633(16)	1.1155(11)	1.0260(10)	1.1159(13)	1.2654(17)	1.3309(15)
$\Delta = 5.0$	1.3318(14)	1.2611(16)	1.1120(12)	1.0158(10)	1.1101(14)	1.2614(13)	1.3320(15)

Table A.1: **homoscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.3408(15)	1.3329(13)	1.3243(16)	1.3212(16)	1.3248(15)	1.3343(15)	1.3408(16)
$\Delta = 1.0$	1.3403(16)	1.3175(14)	1.2822(14)	1.2685(14)	1.2814(14)	1.3167(15)	1.3423(17)
$\Delta = 1.5$	1.3400(16)	1.2978(15)	1.2245(13)	1.1948(14)	1.2241(13)	1.2975(14)	1.3397(18)
$\Delta = 3.0$	1.3391(17)	1.2730(16)	1.1251(12)	1.0349(10)	1.1238(13)	1.2705(15)	1.3401(16)
$\Delta = 5.0$	1.3392(15)	1.2707(15)	1.1190(13)	1.0258(11)	1.1189(14)	1.2700(16)	1.3393(16)

Table A.2: **homoscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.3461(16)	1.3396(14)	1.3292(15)	1.3276(15)	1.3306(15)	1.3396(14)	1.3457(15)
$\Delta = 1.0$	1.3441(15)	1.3228(14)	1.2856(16)	1.2731(13)	1.2880(14)	1.3217(15)	1.3440(17)
$\Delta = 1.5$	1.3441(15)	1.3024(14)	1.2290(13)	1.1990(12)	1.2309(12)	1.3047(15)	1.3453(14)
$\Delta = 3.0$	1.3452(17)	1.2773(15)	1.1293(13)	1.0422(11)	1.1288(14)	1.2787(16)	1.3446(15)
$\Delta = 5.0$	1.3448(16)	1.2757(18)	1.1242(13)	1.0316(10)	1.1251(12)	1.2755(17)	1.3464(15)

Table A.3: **homoscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.3513(16)	1.3434(18)	1.3336(16)	1.3307(15)	1.3344(16)	1.3439(15)	1.3505(17)
$\Delta = 1.0$	1.3502(16)	1.3272(15)	1.2912(13)	1.2785(15)	1.2910(14)	1.3272(15)	1.3490(16)
$\Delta = 1.5$	1.3509(16)	1.3066(14)	1.2355(15)	1.2051(14)	1.2363(15)	1.3071(14)	1.3502(15)
$\Delta = 3.0$	1.3499(18)	1.2820(16)	1.1338(13)	1.0460(10)	1.1339(13)	1.2807(15)	1.3499(16)
$\Delta = 5.0$	1.3503(16)	1.2809(14)	1.1292(12)	1.0371(10)	1.1297(12)	1.2806(15)	1.3512(14)

Table A.4: **homoscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.3521(18)	1.3471(18)	1.3353(16)	1.3336(15)	1.3376(15)	1.3458(15)	1.3531(17)
$\Delta = 1.0$	1.3525(17)	1.3296(15)	1.2928(16)	1.2813(14)	1.2955(15)	1.3297(16)	1.3531(15)
$\Delta = 1.5$	1.3519(16)	1.3110(16)	1.2371(15)	1.2072(12)	1.2366(13)	1.3085(14)	1.3518(16)
$\Delta = 3.0$	1.3525(14)	1.2837(15)	1.1371(15)	1.0489(10)	1.1366(13)	1.2840(16)	1.3520(15)
$\Delta = 5.0$	1.3533(17)	1.2833(15)	1.1318(14)	1.0399(11)	1.1324(15)	1.2824(16)	1.3537(17)

Table A.5: **homoscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3320(14)	1.2633(16)	1.1155(11)	1.0260(10)	1.1159(13)	1.2654(17)	1.3309(15)
$\sigma = 3$	1.3325(15)	1.2796(14)	1.2025(14)	1.1898(14)	1.2360(16)	1.2997(15)	1.3312(16)
$\sigma = 5$	1.3307(17)	1.2934(16)	1.2477(15)	1.2466(15)	1.2762(18)	1.3150(15)	1.3316(17)
$\sigma = 7$	1.3322(13)	1.2980(16)	1.2612(16)	1.2646(15)	1.2893(14)	1.3199(15)	1.3313(16)
$\sigma = 9$	1.3329(17)	1.2990(16)	1.2663(15)	1.2726(16)	1.2960(18)	1.3206(16)	1.3318(15)

Table A.6: **heteroscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3391(17)	1.2730(16)	1.1251(12)	1.0349(10)	1.1238(13)	1.2705(15)	1.3401(16)
$\sigma = 3$	1.3402(17)	1.2880(17)	1.2123(12)	1.1983(13)	1.2440(14)	1.3070(17)	1.3399(15)
$\sigma = 5$	1.3397(15)	1.3025(17)	1.2571(14)	1.2540(14)	1.2842(15)	1.3212(16)	1.3416(15)
$\sigma = 7$	1.3395(16)	1.3063(15)	1.2684(14)	1.2742(15)	1.2980(14)	1.3262(16)	1.3387(16)
$\sigma = 9$	1.3396(16)	1.3083(16)	1.2747(15)	1.2802(16)	1.3034(14)	1.3290(13)	1.3408(15)

Table A.7: **heteroscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3452(17)	1.2773(15)	1.1293(13)	1.0422(11)	1.1288(14)	1.2787(16)	1.3446(15)
$\sigma = 3$	1.3453(16)	1.2927(18)	1.2173(13)	1.2026(14)	1.2480(15)	1.3143(16)	1.3459(16)
$\sigma = 5$	1.3456(16)	1.3089(16)	1.2626(16)	1.2599(16)	1.2907(15)	1.3279(15)	1.3464(15)
$\sigma = 7$	1.3446(14)	1.3119(18)	1.2749(15)	1.2785(16)	1.3033(15)	1.3320(17)	1.3460(16)
$\sigma = 9$	1.3453(16)	1.3123(17)	1.2791(15)	1.2860(16)	1.3109(15)	1.3342(16)	1.3460(15)

Table A.8: **heteroscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3499(18)	1.2820(16)	1.1338(13)	1.0460(10)	1.1339(13)	1.2807(15)	1.3499(16)
$\sigma = 3$	1.3495(15)	1.2970(15)	1.2214(14)	1.2083(14)	1.2529(15)	1.3172(16)	1.3498(14)
$\sigma = 5$	1.3497(14)	1.3114(15)	1.2672(13)	1.2665(15)	1.2953(16)	1.3326(15)	1.3511(17)
$\sigma = 7$	1.3496(17)	1.3180(18)	1.2810(15)	1.2836(14)	1.3075(18)	1.3363(16)	1.3485(16)
$\sigma = 9$	1.3492(14)	1.3176(16)	1.2842(14)	1.2890(16)	1.3138(15)	1.3385(18)	1.3511(15)

Table A.9: **heteroscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3525(14)	1.2837(15)	1.1371(15)	1.0489(10)	1.1366(13)	1.2840(16)	1.3520(15)
$\sigma = 3$	1.3520(17)	1.2998(14)	1.2248(12)	1.2102(14)	1.2553(16)	1.3189(15)	1.3522(15)
$\sigma = 5$	1.3534(15)	1.3151(14)	1.2697(15)	1.2674(17)	1.2966(17)	1.3358(17)	1.3518(14)
$\sigma = 7$	1.3521(18)	1.3200(16)	1.2824(15)	1.2855(15)	1.3120(16)	1.3398(16)	1.3512(14)
$\sigma = 9$	1.3530(14)	1.3188(17)	1.2880(13)	1.2919(17)	1.3174(17)	1.3408(14)	1.3528(15)

Table A.10: **heteroscedastic** Kolomogorov-Smirnov critical values, Case I: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.6934(17)	1.6896(16)	1.6873(13)	1.6873(15)	1.6874(15)	1.6914(14)	1.6940(16)
$\Delta = 1.0$	1.6931(15)	1.6827(14)	1.6701(16)	1.6676(13)	1.6722(15)	1.6824(15)	1.6924(14)
$\Delta = 1.5$	1.6933(16)	1.6735(15)	1.6453(13)	1.6370(14)	1.6465(14)	1.6735(15)	1.6937(15)
$\Delta = 3.0$	1.6937(14)	1.6355(13)	1.5735(13)	1.5578(13)	1.5750(12)	1.6393(15)	1.6930(14)
$\Delta = 5.0$	1.6925(15)	1.6267(13)	1.5594(15)	1.5449(14)	1.5590(13)	1.6281(16)	1.6936(14)

Table A.11: **homoscedastic** Kuiper critical values, Case I: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.7110(15)	1.7087(14)	1.7050(15)	1.7040(14)	1.7051(14)	1.7084(15)	1.7117(14)
$\Delta = 1.0$	1.7105(15)	1.7025(16)	1.6891(15)	1.6839(13)	1.6880(15)	1.7018(16)	1.7116(14)
$\Delta = 1.5$	1.7114(16)	1.6895(15)	1.6635(16)	1.6524(14)	1.6626(12)	1.6901(14)	1.7106(16)
$\Delta = 3.0$	1.7102(15)	1.6551(13)	1.5913(12)	1.5730(16)	1.5910(15)	1.6546(15)	1.7103(13)
$\Delta = 5.0$	1.7109(16)	1.6447(14)	1.5765(14)	1.5607(16)	1.5758(14)	1.6453(15)	1.7101(15)

Table A.12: **homoscedastic** Kuiper critical values, Case I: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.7236(15)	1.7215(14)	1.7160(13)	1.7150(15)	1.7166(15)	1.7206(13)	1.7216(14)
$\Delta = 1.0$	1.7207(16)	1.7123(15)	1.6995(16)	1.6944(12)	1.6998(15)	1.7122(14)	1.7221(15)
$\Delta = 1.5$	1.7220(15)	1.7008(15)	1.6734(15)	1.6653(15)	1.6755(14)	1.7020(15)	1.7221(17)
$\Delta = 3.0$	1.7230(16)	1.6668(14)	1.6024(15)	1.5849(16)	1.6018(14)	1.6663(15)	1.7222(14)
$\Delta = 5.0$	1.7220(14)	1.6562(14)	1.5861(13)	1.5719(15)	1.5860(13)	1.6563(15)	1.7226(16)

Table A.13: **homoscedastic** Kuiper critical values, Case I: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.7325(14)	1.7293(15)	1.7256(14)	1.7248(13)	1.7262(16)	1.7295(14)	1.7309(18)
$\Delta = 1.0$	1.7308(16)	1.7220(15)	1.7081(16)	1.7056(15)	1.7091(13)	1.7212(15)	1.7307(16)
$\Delta = 1.5$	1.7320(16)	1.7106(15)	1.6842(15)	1.6755(14)	1.6849(14)	1.7099(14)	1.7316(16)
$\Delta = 3.0$	1.7327(16)	1.6747(14)	1.6099(13)	1.5945(14)	1.6118(15)	1.6746(13)	1.7326(17)
$\Delta = 5.0$	1.7306(16)	1.6664(15)	1.5951(14)	1.5827(13)	1.5966(14)	1.6665(15)	1.7323(13)

Table A.14: **homoscedastic** Kuiper critical values, Case I: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.7384(15)	1.7350(15)	1.7298(15)	1.7303(16)	1.7310(14)	1.7339(13)	1.7376(15)
$\Delta = 1.0$	1.7364(14)	1.7272(16)	1.7127(13)	1.7086(15)	1.7141(15)	1.7268(14)	1.7370(15)
$\Delta = 1.5$	1.7367(15)	1.7165(16)	1.6881(15)	1.6791(14)	1.6876(14)	1.7135(15)	1.7352(14)
$\Delta = 3.0$	1.7362(12)	1.6797(15)	1.6156(16)	1.5984(15)	1.6161(15)	1.6802(15)	1.7367(15)
$\Delta = 5.0$	1.7363(16)	1.6700(15)	1.6001(15)	1.5867(13)	1.6011(13)	1.6697(16)	1.7367(12)

Table A.15: **homoscedastic** Kuiper critical values, Case I: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.6937(14)	1.6355(13)	1.5735(13)	1.5578(13)	1.5750(12)	1.6393(15)	1.6930(14)
$\sigma = 3$	1.6922(15)	1.6596(15)	1.6143(14)	1.6012(14)	1.6167(16)	1.6571(14)	1.6920(13)
$\sigma = 5$	1.6928(17)	1.6531(15)	1.6015(14)	1.5896(14)	1.6040(14)	1.6539(14)	1.6918(15)
$\sigma = 7$	1.6925(16)	1.6479(13)	1.5926(13)	1.5767(14)	1.5919(13)	1.6498(15)	1.6917(15)
$\sigma = 9$	1.6928(15)	1.6445(15)	1.5861(13)	1.5695(14)	1.5856(14)	1.6432(15)	1.6938(14)

Table A.16: **heteroscedastic** Kuiper critical values, Case I: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3$, $\sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.7102(15)	1.6551(13)	1.5913(12)	1.5730(16)	1.5910(15)	1.6546(15)	1.7103(13)
$\sigma = 3$	1.7109(14)	1.6757(16)	1.6301(14)	1.6176(15)	1.6328(14)	1.6758(16)	1.7115(16)
$\sigma = 5$	1.7112(15)	1.6706(13)	1.6213(16)	1.6056(14)	1.6199(13)	1.6695(14)	1.7133(15)
$\sigma = 7$	1.7111(15)	1.6663(13)	1.6098(15)	1.5949(12)	1.6102(14)	1.6650(12)	1.7096(15)
$\sigma = 9$	1.7097(17)	1.6644(17)	1.6041(16)	1.5870(14)	1.6036(12)	1.6614(15)	1.7108(15)

Table A.17: **heteroscedastic** Kuiper critical values, Case I: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3$, $\sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.7230(16)	1.6668(14)	1.6024(15)	1.5849(16)	1.6018(14)	1.6663(15)	1.7222(14)
$\sigma = 3$	1.7222(15)	1.6875(14)	1.6420(13)	1.6278(14)	1.6421(12)	1.6869(15)	1.7225(15)
$\sigma = 5$	1.7216(13)	1.6828(15)	1.6329(16)	1.6163(14)	1.6328(13)	1.6821(14)	1.7238(15)
$\sigma = 7$	1.7226(14)	1.6765(15)	1.6222(15)	1.6043(13)	1.6222(14)	1.6759(14)	1.7239(16)
$\sigma = 9$	1.7211(13)	1.6748(14)	1.6142(14)	1.5984(14)	1.6156(14)	1.6725(14)	1.7225(14)

Table A.18: **heteroscedastic** Kuiper critical values, Case I: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3$, $\sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.7327(16)	1.6747(14)	1.6099(13)	1.5945(14)	1.6118(15)	1.6746(13)	1.7326(17)
$\sigma = 3$	1.7313(15)	1.6963(16)	1.6509(14)	1.6379(15)	1.6524(14)	1.6952(13)	1.7318(16)
$\sigma = 5$	1.7302(15)	1.6915(14)	1.6423(15)	1.6273(16)	1.6422(16)	1.6912(15)	1.7319(15)
$\sigma = 7$	1.7323(15)	1.6883(16)	1.6333(14)	1.6172(14)	1.6319(14)	1.6848(13)	1.7307(15)
$\sigma = 9$	1.7313(15)	1.6837(14)	1.6261(13)	1.6074(15)	1.6259(13)	1.6828(15)	1.7311(15)

Table A.19: **heteroscedastic** Kuiper critical values, Case I: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3$, $\sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.7362(12)	1.6797(15)	1.6156(16)	1.5984(15)	1.6161(15)	1.6802(15)	1.7367(15)
$\sigma = 3$	1.7369(16)	1.7019(15)	1.6562(12)	1.6421(15)	1.6568(15)	1.6997(15)	1.7373(14)
$\sigma = 5$	1.7380(16)	1.6970(15)	1.6472(13)	1.6307(15)	1.6463(15)	1.6958(16)	1.7347(15)
$\sigma = 7$	1.7369(15)	1.6914(16)	1.6369(16)	1.6209(13)	1.6361(14)	1.6921(16)	1.7349(15)
$\sigma = 9$	1.7361(15)	1.6893(15)	1.6298(14)	1.6134(16)	1.6302(14)	1.6875(14)	1.7366(16)

Table A.20: **heteroscedastic** Kuiper critical values, Case I: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3$, $\sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	2.5008(82)	2.4549(79)	2.4064(70)	2.3933(77)	2.4079(76)	2.4608(69)	2.4924(72)
$\Delta = 1.0$	2.4973(76)	2.3558(58)	2.1831(60)	2.1293(55)	2.1848(58)	2.3590(69)	2.4975(77)
$\Delta = 1.5$	2.4977(73)	2.2359(59)	1.9201(52)	1.8295(46)	1.9277(50)	2.2309(65)	2.4996(76)
$\Delta = 3.0$	2.5025(73)	2.0014(56)	1.5031(34)	1.3554(31)	1.5052(37)	2.0046(64)	2.4909(69)
$\Delta = 5.0$	2.4936(71)	1.9657(58)	1.4624(35)	1.3052(29)	1.4604(36)	1.9624(54)	2.4959(71)

Table A.21: **homoscedastic** Anderson-Darling critical values, Case I: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5]$, $\sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	2.5004(78)	2.4528(71)	2.4085(64)	2.3910(68)	2.4063(60)	2.4622(76)	2.4921(74)
$\Delta = 1.0$	2.4967(77)	2.3576(52)	2.1810(58)	2.1281(60)	2.1859(60)	2.3514(66)	2.5028(69)
$\Delta = 1.5$	2.4975(70)	2.2295(64)	1.9245(48)	1.8307(48)	1.9219(49)	2.2280(66)	2.4922(73)
$\Delta = 3.0$	2.4915(66)	2.0058(61)	1.5057(38)	1.3535(29)	1.5042(39)	1.9951(59)	2.4955(70)
$\Delta = 5.0$	2.4882(79)	1.9632(53)	1.4609(37)	1.3041(30)	1.4608(38)	1.9607(62)	2.4885(69)

Table A.22: **homoscedastic** Anderson-Darling critical values, Case I: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5]$, $\sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	2.4952(73)	2.4608(63)	2.4101(69)	2.3936(61)	2.4082(64)	2.4583(59)	2.4892(68)
$\Delta = 1.0$	2.4895(69)	2.3518(52)	2.1866(62)	2.1268(52)	2.1852(58)	2.3507(68)	2.4885(69)
$\Delta = 1.5$	2.4871(72)	2.2284(64)	1.9218(50)	1.8236(43)	1.9269(46)	2.2363(62)	2.4895(59)
$\Delta = 3.0$	2.4933(70)	1.9983(52)	1.5030(37)	1.3537(31)	1.4996(37)	2.0019(61)	2.4899(62)
$\Delta = 5.0$	2.4876(66)	1.9645(58)	1.4603(37)	1.3037(27)	1.4577(37)	1.9687(62)	2.4935(76)

Table A.23: **homoscedastic** Anderson-Darling critical values, Case I: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5]$, $\sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	2.4938(68)	2.4572(70)	2.4029(67)	2.3849(68)	2.4042(73)	2.4564(67)	2.4979(70)
$\Delta = 1.0$	2.4932(78)	2.3548(71)	2.1776(64)	2.1279(51)	2.1795(52)	2.3540(70)	2.4885(68)
$\Delta = 1.5$	2.4940(65)	2.2256(62)	1.9239(52)	1.8287(43)	1.9199(50)	2.2211(61)	2.4921(74)
$\Delta = 3.0$	2.4937(74)	1.9969(61)	1.5010(40)	1.3525(29)	1.5043(35)	1.9959(56)	2.4923(73)
$\Delta = 5.0$	2.4961(73)	1.9638(56)	1.4572(36)	1.3038(31)	1.4604(36)	1.9676(60)	2.4963(67)

Table A.24: **homoscedastic** Anderson-Darling critical values, Case I: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5]$, $\sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	2.4878(76)	2.4592(73)	2.3966(64)	2.3898(68)	2.4036(65)	2.4510(64)	2.4915(80)
$\Delta = 1.0$	2.4946(69)	2.3498(66)	2.1821(62)	2.1271(47)	2.1914(63)	2.3572(72)	2.4944(68)
$\Delta = 1.5$	2.4892(71)	2.2304(65)	1.9187(54)	1.8263(45)	1.9229(51)	2.2251(61)	2.4880(75)
$\Delta = 3.0$	2.4966(63)	1.9988(57)	1.5044(41)	1.3511(27)	1.5035(39)	1.9952(58)	2.4924(68)
$\Delta = 5.0$	2.4936(66)	1.9676(55)	1.4592(41)	1.3035(32)	1.4588(35)	1.9622(58)	2.4953(75)

Table A.25: **homoscedastic** Anderson-Darling critical values, Case I: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	2.5025(73)	2.0014(56)	1.5031(34)	1.3554(31)	1.5052(37)	2.0046(64)	2.4909(69)
$\sigma = 3$	2.4965(77)	2.2763(66)	2.0060(52)	1.9081(51)	1.9709(67)	2.2298(68)	2.4937(68)
$\sigma = 5$	2.4934(74)	2.3814(69)	2.2250(66)	2.1565(60)	2.1750(60)	2.3201(62)	2.4925(80)
$\sigma = 7$	2.4942(70)	2.4114(70)	2.2896(72)	2.2403(58)	2.2452(62)	2.3525(62)	2.4916(65)
$\sigma = 9$	2.4955(75)	2.4212(79)	2.3283(78)	2.2840(70)	2.2720(73)	2.3533(70)	2.4994(73)

Table A.26: **heteroscedastic** Anderson-Darling critical values, Case I: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	2.4915(66)	2.0058(61)	1.5057(38)	1.3535(29)	1.5042(39)	1.9951(59)	2.4955(70)
$\sigma = 3$	2.4925(72)	2.2776(69)	2.0057(54)	1.9057(48)	1.9725(54)	2.2233(66)	2.4967(64)
$\sigma = 5$	2.4911(66)	2.3791(81)	2.2240(60)	2.1521(57)	2.1804(59)	2.3254(65)	2.4956(70)
$\sigma = 7$	2.4931(75)	2.4062(62)	2.2953(63)	2.2454(64)	2.2515(60)	2.3551(69)	2.4874(71)
$\sigma = 9$	2.4879(68)	2.4233(75)	2.3309(71)	2.2720(59)	2.2793(59)	2.3627(63)	2.4998(73)

Table A.27: **heteroscedastic** Anderson-Darling critical values, Case I: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	2.4933(70)	1.9983(52)	1.5030(37)	1.3537(31)	1.4996(37)	2.0019(61)	2.4899(62)
$\sigma = 3$	2.4899(79)	2.2803(78)	2.0045(44)	1.9002(45)	1.9677(60)	2.2334(70)	2.4959(61)
$\sigma = 5$	2.4940(68)	2.3806(72)	2.2266(65)	2.1586(68)	2.1853(65)	2.3262(67)	2.4976(69)
$\sigma = 7$	2.4888(71)	2.4117(77)	2.2943(62)	2.2424(70)	2.2510(70)	2.3517(62)	2.4949(63)
$\sigma = 9$	2.4952(75)	2.4245(69)	2.3300(70)	2.2829(66)	2.2891(69)	2.3656(77)	2.4947(67)

Table A.28: **heteroscedastic** Anderson-Darling critical values, Case I: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	2.4937(74)	1.9969(61)	1.5010(40)	1.3525(29)	1.5043(35)	1.9959(56)	2.4923(73)
$\sigma = 3$	2.4901(69)	2.2713(67)	1.9979(53)	1.9074(49)	1.9709(50)	2.2249(65)	2.4934(65)
$\sigma = 5$	2.4929(71)	2.3738(73)	2.2244(60)	2.1617(68)	2.1877(70)	2.3260(61)	2.4941(71)
$\sigma = 7$	2.4893(82)	2.4114(67)	2.2973(71)	2.2418(71)	2.2506(78)	2.3544(74)	2.4919(74)
$\sigma = 9$	2.4945(66)	2.4258(71)	2.3268(65)	2.2742(67)	2.2879(69)	2.3647(68)	2.4971(80)

Table A.29: **heteroscedastic** Anderson-Darling critical values, Case I: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	2.4966(63)	1.9988(57)	1.5044(41)	1.3511(27)	1.5035(39)	1.9952(58)	2.4924(68)
$\sigma = 3$	2.4906(73)	2.2742(65)	1.9995(56)	1.9073(49)	1.9725(60)	2.2252(64)	2.4951(77)
$\sigma = 5$	2.4916(67)	2.3752(73)	2.2232(68)	2.1622(67)	2.1804(65)	2.3320(65)	2.4896(67)
$\sigma = 7$	2.4854(81)	2.4073(71)	2.2957(70)	2.2437(69)	2.2616(70)	2.3575(72)	2.4880(67)
$\sigma = 9$	2.4919(69)	2.4228(71)	2.3304(64)	2.2788(73)	2.2881(76)	2.3683(69)	2.4915(71)

Table A.30: **heteroscedastic** Anderson-Darling critical values, Case I: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	0.4609(15)	0.4519(15)	0.4428(15)	0.4394(15)	0.4428(15)	0.4523(15)	0.4589(15)
$\Delta = 1.0$	0.4601(14)	0.4339(12)	0.3978(13)	0.3856(11)	0.3980(11)	0.4348(14)	0.4600(16)
$\Delta = 1.5$	0.4601(15)	0.4137(13)	0.3429(10)	0.3191(09)	0.3442(11)	0.4124(12)	0.4609(16)
$\Delta = 3.0$	0.4611(14)	0.3785(11)	0.2515(07)	0.1992(05)	0.2520(06)	0.3790(13)	0.4586(13)
$\Delta = 5.0$	0.4596(15)	0.3740(12)	0.2418(07)	0.1853(04)	0.2413(07)	0.3735(10)	0.4597(14)

Table A.31: **homoscedastic** Cramér-von Mises critical values, Case I: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	0.4617(16)	0.4525(13)	0.4438(14)	0.4402(14)	0.4432(12)	0.4545(14)	0.4608(15)
$\Delta = 1.0$	0.4601(15)	0.4360(11)	0.3985(11)	0.3858(12)	0.3989(12)	0.4340(14)	0.4623(15)
$\Delta = 1.5$	0.4612(14)	0.4137(13)	0.3445(10)	0.3197(09)	0.3442(09)	0.4131(13)	0.4606(15)
$\Delta = 3.0$	0.4610(15)	0.3799(13)	0.2524(07)	0.1998(04)	0.2522(07)	0.3783(12)	0.4609(12)
$\Delta = 5.0$	0.4599(15)	0.3744(12)	0.2423(07)	0.1860(05)	0.2421(07)	0.3739(12)	0.4596(14)

Table A.32: **homoscedastic** Cramér-von Mises critical values, Case I: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	0.4615(14)	0.4543(12)	0.4437(14)	0.4413(12)	0.4444(13)	0.4543(12)	0.4605(14)
$\Delta = 1.0$	0.4601(14)	0.4344(11)	0.3989(14)	0.3860(10)	0.3995(11)	0.4346(15)	0.4600(15)
$\Delta = 1.5$	0.4596(14)	0.4125(13)	0.3440(10)	0.3191(08)	0.3452(10)	0.4150(13)	0.4603(14)
$\Delta = 3.0$	0.4609(15)	0.3793(11)	0.2524(07)	0.2002(05)	0.2521(06)	0.3801(12)	0.4600(12)
$\Delta = 5.0$	0.4599(13)	0.3751(12)	0.2421(07)	0.1866(04)	0.2422(08)	0.3751(13)	0.4611(15)

Table A.33: **homoscedastic** Cramér-von Mises critical values, Case I: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	0.4614(14)	0.4542(14)	0.4435(15)	0.4400(13)	0.4437(15)	0.4542(13)	0.4615(14)
$\Delta = 1.0$	0.4611(16)	0.4350(14)	0.3983(13)	0.3867(10)	0.3986(11)	0.4351(14)	0.4600(15)
$\Delta = 1.5$	0.4617(14)	0.4134(12)	0.3449(10)	0.3203(08)	0.3449(10)	0.4129(12)	0.4615(15)
$\Delta = 3.0$	0.4614(15)	0.3790(12)	0.2521(07)	0.2001(05)	0.2525(07)	0.3780(12)	0.4609(14)
$\Delta = 5.0$	0.4623(15)	0.3752(11)	0.2422(07)	0.1869(05)	0.2425(07)	0.3756(12)	0.4623(13)

Table A.34: **homoscedastic** Cramér-von Mises critical values, Case I: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	0.4608(16)	0.4555(16)	0.4423(13)	0.4407(15)	0.4444(13)	0.4533(14)	0.4613(16)
$\Delta = 1.0$	0.4613(14)	0.4345(14)	0.3987(14)	0.3866(10)	0.4007(12)	0.4358(16)	0.4613(13)
$\Delta = 1.5$	0.4609(15)	0.4145(14)	0.3445(12)	0.3195(09)	0.3451(10)	0.4128(14)	0.4606(15)
$\Delta = 3.0$	0.4626(13)	0.3798(12)	0.2529(08)	0.2003(05)	0.2528(07)	0.3787(13)	0.4609(15)
$\Delta = 5.0$	0.4615(15)	0.3761(11)	0.2425(07)	0.1870(05)	0.2430(08)	0.3747(12)	0.4615(15)

Table A.35: **homoscedastic** Cramér-von Mises critical values, Case I: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	0.4611(14)	0.3785(11)	0.2515(07)	0.1992(05)	0.2520(06)	0.3790(13)	0.4586(13)
$\sigma = 3$	0.4600(16)	0.4095(12)	0.3438(10)	0.3287(11)	0.3595(12)	0.4207(13)	0.4596(15)
$\sigma = 5$	0.4595(14)	0.4308(14)	0.3950(12)	0.3879(11)	0.4039(13)	0.4384(12)	0.4589(15)
$\sigma = 7$	0.4597(15)	0.4399(14)	0.4146(14)	0.4106(12)	0.4197(13)	0.4455(13)	0.4592(14)
$\sigma = 9$	0.4602(17)	0.4432(15)	0.4249(16)	0.4220(14)	0.4275(16)	0.4467(14)	0.4600(16)

Table A.36: **heteroscedastic** Cramér-von Mises critical values, Case I: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	0.4610(15)	0.3799(13)	0.2524(07)	0.1998(04)	0.2522(07)	0.3783(12)	0.4609(12)
$\sigma = 3$	0.4599(15)	0.4113(13)	0.3455(10)	0.3292(09)	0.3596(12)	0.4201(14)	0.4601(14)
$\sigma = 5$	0.4598(14)	0.4317(14)	0.3961(13)	0.3884(12)	0.4055(11)	0.4389(14)	0.4607(15)
$\sigma = 7$	0.4604(16)	0.4395(13)	0.4154(12)	0.4121(12)	0.4219(12)	0.4456(13)	0.4589(13)
$\sigma = 9$	0.4597(13)	0.4445(15)	0.4264(14)	0.4203(12)	0.4290(12)	0.4480(12)	0.4617(15)

Table A.37: **heteroscedastic** Cramér-von Mises critical values, Case I: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	0.4609(15)	0.3793(11)	0.2524(07)	0.2002(05)	0.2521(06)	0.3801(12)	0.4600(12)
$\sigma = 3$	0.4605(15)	0.4116(15)	0.3458(09)	0.3284(09)	0.3588(13)	0.4224(14)	0.4612(12)
$\sigma = 5$	0.4611(14)	0.4325(15)	0.3963(12)	0.3891(13)	0.4060(13)	0.4401(13)	0.4620(14)
$\sigma = 7$	0.4598(14)	0.4412(16)	0.4163(13)	0.4118(13)	0.4215(14)	0.4456(13)	0.4612(14)
$\sigma = 9$	0.4610(15)	0.4457(15)	0.4263(14)	0.4223(13)	0.4311(14)	0.4484(16)	0.4613(13)

Table A.38: **heteroscedastic** Cramér-von Mises critical values, Case I: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	0.4614(15)	0.3790(12)	0.2521(07)	0.2001(05)	0.2525(07)	0.3780(12)	0.4609(14)
$\sigma = 3$	0.4610(15)	0.4108(14)	0.3443(10)	0.3302(10)	0.3594(11)	0.4214(14)	0.4610(14)
$\sigma = 5$	0.4615(15)	0.4316(15)	0.3971(11)	0.3901(13)	0.4063(13)	0.4404(13)	0.4615(14)
$\sigma = 7$	0.4606(17)	0.4410(15)	0.4167(14)	0.4116(14)	0.4222(15)	0.4451(14)	0.4604(14)
$\sigma = 9$	0.4613(14)	0.4456(15)	0.4259(14)	0.4205(13)	0.4304(13)	0.4483(14)	0.4616(15)

Table A.39: **heteroscedastic** Cramér-von Mises critical values, Case I: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	0.4626(13)	0.3798(12)	0.2529(08)	0.2003(05)	0.2528(07)	0.3787(13)	0.4609(15)
$\sigma = 3$	0.4611(15)	0.4115(13)	0.3454(10)	0.3296(09)	0.3598(13)	0.4208(13)	0.4618(16)
$\sigma = 5$	0.4614(14)	0.4316(14)	0.3970(13)	0.3901(14)	0.4052(14)	0.4409(14)	0.4603(12)
$\sigma = 7$	0.4602(17)	0.4411(13)	0.4172(14)	0.4116(15)	0.4239(15)	0.4461(14)	0.4607(13)
$\sigma = 9$	0.4617(14)	0.4451(15)	0.4271(14)	0.4222(15)	0.4309(15)	0.4487(14)	0.4611(14)

Table A.40: **heteroscedastic** Cramér-von Mises critical values, Case I: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.2094(16)	1.1078(14)	0.9920(10)	0.9614(10)	0.9921(10)	1.1092(16)	1.2082(16)
$\Delta = 1.0$	1.2361(15)	1.1119(15)	0.9827(10)	0.9248(10)	0.9826(12)	1.1123(14)	1.2366(15)
$\Delta = 1.5$	1.2725(16)	1.1630(13)	0.9997(11)	0.9047(08)	1.0003(11)	1.1633(13)	1.2729(14)
$\Delta = 3.0$	1.3250(15)	1.2406(14)	1.0722(13)	0.9661(09)	1.0722(12)	1.2401(14)	1.3259(15)
$\Delta = 5.0$	1.3327(16)	1.2594(15)	1.1078(12)	1.0115(09)	1.1070(12)	1.2609(15)	1.3315(15)

Table A.41: **homoscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.2177(16)	1.0900(12)	0.9746(09)	0.9498(09)	0.9735(09)	1.0897(14)	1.2207(15)
$\Delta = 1.0$	1.2512(16)	1.1118(14)	0.9786(10)	0.9169(09)	0.9795(10)	1.1113(13)	1.2518(15)
$\Delta = 1.5$	1.2879(17)	1.1705(12)	1.0000(11)	0.9016(08)	1.0011(12)	1.1720(12)	1.2881(16)
$\Delta = 3.0$	1.3348(16)	1.2488(15)	1.0810(14)	0.9732(09)	1.0805(13)	1.2484(13)	1.3340(14)
$\Delta = 5.0$	1.3391(16)	1.2676(14)	1.1146(14)	1.0210(09)	1.1152(13)	1.2681(14)	1.3399(16)

Table A.42: **homoscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.2262(17)	1.0673(13)	0.9663(08)	0.9458(09)	0.9668(09)	1.0672(12)	1.2270(14)
$\Delta = 1.0$	1.2618(16)	1.1115(12)	0.9782(09)	0.9141(09)	0.9784(10)	1.1111(13)	1.2627(17)
$\Delta = 1.5$	1.3000(14)	1.1745(14)	1.0030(11)	0.9003(08)	1.0038(10)	1.1732(15)	1.3005(17)
$\Delta = 3.0$	1.3426(16)	1.2526(16)	1.0848(12)	0.9786(09)	1.0850(14)	1.2528(15)	1.3424(16)
$\Delta = 5.0$	1.3461(15)	1.2744(17)	1.1205(14)	1.0276(10)	1.1211(12)	1.2745(14)	1.3460(17)

Table A.43: **homoscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.2343(16)	1.0398(11)	0.9647(09)	0.9442(08)	0.9658(10)	1.0405(11)	1.2339(16)
$\Delta = 1.0$	1.2725(15)	1.1134(11)	0.9799(10)	0.9128(08)	0.9794(10)	1.1127(13)	1.2737(15)
$\Delta = 1.5$	1.3106(16)	1.1757(12)	1.0065(11)	0.9027(09)	1.0067(11)	1.1767(14)	1.3113(16)
$\Delta = 3.0$	1.3474(18)	1.2592(15)	1.0893(12)	0.9832(10)	1.0900(14)	1.2564(14)	1.3481(16)
$\Delta = 5.0$	1.3512(17)	1.2780(15)	1.1256(15)	1.0308(10)	1.1263(14)	1.2776(14)	1.3499(16)

Table A.44: **homoscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.2401(18)	1.0278(12)	0.9648(09)	0.9440(09)	0.9664(10)	1.0264(10)	1.2388(16)
$\Delta = 1.0$	1.2776(13)	1.1155(14)	0.9814(10)	0.9133(10)	0.9812(11)	1.1143(13)	1.2771(15)
$\Delta = 1.5$	1.3165(15)	1.1784(12)	1.0075(11)	0.9035(07)	1.0083(11)	1.1788(15)	1.3164(15)
$\Delta = 3.0$	1.3499(16)	1.2602(16)	1.0912(13)	0.9855(10)	1.0912(13)	1.2610(15)	1.3519(15)
$\Delta = 5.0$	1.3524(14)	1.2805(16)	1.1283(14)	1.0340(10)	1.1278(12)	1.2824(14)	1.3529(17)

Table A.45: **homoscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3250(15)	1.2406(14)	1.0722(13)	0.9661(09)	1.0722(12)	1.2401(14)	1.3259(15)
$\sigma = 3$	1.2405(15)	1.0929(12)	1.0278(11)	1.0696(12)	1.1702(16)	1.2736(13)	1.3291(16)
$\sigma = 5$	1.2736(15)	1.1807(13)	1.1577(15)	1.1924(15)	1.2493(14)	1.3042(15)	1.3317(15)
$\sigma = 7$	1.2949(16)	1.2167(13)	1.2040(14)	1.2327(15)	1.2751(15)	1.3126(17)	1.3313(17)
$\sigma = 9$	1.3042(17)	1.2342(15)	1.2219(14)	1.2486(16)	1.2841(15)	1.3163(16)	1.3309(15)

Table A.46: **heteroscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3348(16)	1.2488(15)	1.0810(14)	0.9732(09)	1.0805(13)	1.2484(13)	1.3340(14)
$\sigma = 3$	1.2517(16)	1.0936(13)	1.0337(11)	1.0765(13)	1.1777(14)	1.2848(16)	1.3389(15)
$\sigma = 5$	1.2808(15)	1.1837(13)	1.1648(11)	1.2002(14)	1.2557(15)	1.3121(16)	1.3398(13)
$\sigma = 7$	1.3013(16)	1.2240(14)	1.2109(15)	1.2400(17)	1.2829(15)	1.3213(18)	1.3404(16)
$\sigma = 9$	1.3110(16)	1.2402(16)	1.2272(16)	1.2569(14)	1.2938(16)	1.3253(16)	1.3411(15)

Table A.47: **heteroscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3426(16)	1.2526(16)	1.0848(12)	0.9786(09)	1.0850(14)	1.2528(15)	1.3424(16)
$\sigma = 3$	1.2581(18)	1.0963(13)	1.0357(10)	1.0802(14)	1.1823(15)	1.2879(15)	1.3448(17)
$\sigma = 5$	1.2873(15)	1.1866(14)	1.1706(12)	1.2053(15)	1.2623(15)	1.3170(16)	1.3444(15)
$\sigma = 7$	1.3055(15)	1.2295(15)	1.2155(16)	1.2453(15)	1.2888(16)	1.3267(16)	1.3450(14)
$\sigma = 9$	1.3143(14)	1.2463(14)	1.2349(16)	1.2624(15)	1.2983(15)	1.3297(17)	1.3440(18)

Table A.48: **heteroscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3474(18)	1.2592(15)	1.0893(12)	0.9832(10)	1.0900(14)	1.2564(14)	1.3481(16)
$\sigma = 3$	1.2666(15)	1.0993(13)	1.0402(11)	1.0840(12)	1.1865(15)	1.2922(16)	1.3502(15)
$\sigma = 5$	1.2914(14)	1.1921(13)	1.1741(15)	1.2107(17)	1.2656(13)	1.3212(15)	1.3512(16)
$\sigma = 7$	1.3105(16)	1.2324(14)	1.2187(14)	1.2510(15)	1.2930(14)	1.3314(17)	1.3498(14)
$\sigma = 9$	1.3187(18)	1.2497(16)	1.2392(14)	1.2651(16)	1.3025(15)	1.3338(17)	1.3509(15)

Table A.49: **heteroscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.3499(16)	1.2602(16)	1.0912(13)	0.9855(10)	1.0912(13)	1.2610(15)	1.3519(15)
$\sigma = 3$	1.2705(14)	1.1015(12)	1.0422(10)	1.0866(12)	1.1893(13)	1.2941(15)	1.3513(15)
$\sigma = 5$	1.2932(15)	1.1934(13)	1.1766(16)	1.2144(15)	1.2691(14)	1.3225(16)	1.3518(15)
$\sigma = 7$	1.3115(16)	1.2347(14)	1.2228(16)	1.2523(14)	1.2930(16)	1.3336(16)	1.3519(18)
$\sigma = 9$	1.3203(15)	1.2527(15)	1.2400(16)	1.2687(17)	1.3046(18)	1.3374(15)	1.3519(17)

Table A.50: **heteroscedastic** Kolomogorov-Smirnov critical values, Case II: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.6572(16)	1.6344(14)	1.6124(14)	1.6090(14)	1.6126(12)	1.6332(16)	1.6548(15)
$\Delta = 1.0$	1.6600(16)	1.6212(15)	1.5979(16)	1.5905(14)	1.5992(13)	1.6210(16)	1.6584(14)
$\Delta = 1.5$	1.6668(15)	1.6177(16)	1.5811(15)	1.5721(14)	1.5820(14)	1.6201(14)	1.6667(12)
$\Delta = 3.0$	1.6880(14)	1.6174(12)	1.5485(15)	1.5368(15)	1.5486(14)	1.6183(13)	1.6885(15)
$\Delta = 5.0$	1.6927(14)	1.6249(15)	1.5574(15)	1.5438(15)	1.5566(13)	1.6270(13)	1.6916(15)

Table A.51: **homoscedastic** Kuiper critical values, Case II: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.6733(15)	1.6446(15)	1.6260(14)	1.6228(17)	1.6244(14)	1.6446(16)	1.6736(14)
$\Delta = 1.0$	1.6777(16)	1.6345(15)	1.6128(15)	1.6081(14)	1.6128(14)	1.6358(15)	1.6788(12)
$\Delta = 1.5$	1.6865(16)	1.6346(15)	1.5958(12)	1.5885(14)	1.5973(15)	1.6344(14)	1.6868(16)
$\Delta = 3.0$	1.7057(15)	1.6337(15)	1.5658(15)	1.5515(15)	1.5655(16)	1.6342(14)	1.7071(15)
$\Delta = 5.0$	1.7108(15)	1.6422(14)	1.5736(17)	1.5599(14)	1.5734(13)	1.6427(14)	1.7115(13)

Table A.52: **homoscedastic** Kuiper critical values, Case II: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.6853(15)	1.6498(14)	1.6332(14)	1.6327(14)	1.6338(13)	1.6494(15)	1.6843(15)
$\Delta = 1.0$	1.6911(15)	1.6420(14)	1.6222(14)	1.6185(15)	1.6214(14)	1.6423(14)	1.6910(15)
$\Delta = 1.5$	1.6995(12)	1.6443(14)	1.6055(14)	1.5978(12)	1.6069(14)	1.6430(15)	1.7003(15)
$\Delta = 3.0$	1.7200(13)	1.6442(15)	1.5750(15)	1.5633(14)	1.5767(13)	1.6445(15)	1.7193(15)
$\Delta = 5.0$	1.7221(15)	1.6544(15)	1.5840(14)	1.5698(16)	1.5857(13)	1.6546(14)	1.7215(14)

Table A.53: **homoscedastic** Kuiper critical values, Case II: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.6946(15)	1.6522(14)	1.6431(13)	1.6412(13)	1.6419(15)	1.6527(15)	1.6948(16)
$\Delta = 1.0$	1.7014(15)	1.6518(15)	1.6310(14)	1.6258(13)	1.6294(15)	1.6511(14)	1.7015(15)
$\Delta = 1.5$	1.7116(15)	1.6516(14)	1.6169(15)	1.6086(14)	1.6158(14)	1.6520(16)	1.7125(15)
$\Delta = 3.0$	1.7300(16)	1.6545(13)	1.5846(15)	1.5711(16)	1.5852(15)	1.6518(15)	1.7295(15)
$\Delta = 5.0$	1.7324(16)	1.6632(14)	1.5941(14)	1.5786(16)	1.5945(15)	1.6636(14)	1.7316(16)

Table A.54: **homoscedastic** Kuiper critical values, Case II: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.7009(17)	1.6544(15)	1.6465(14)	1.6452(15)	1.6471(16)	1.6526(13)	1.7000(16)
$\Delta = 1.0$	1.7068(15)	1.6570(15)	1.6357(13)	1.6301(17)	1.6359(17)	1.6541(15)	1.7063(15)
$\Delta = 1.5$	1.7173(15)	1.6567(15)	1.6196(14)	1.6117(14)	1.6209(13)	1.6572(15)	1.7160(15)
$\Delta = 3.0$	1.7353(16)	1.6585(15)	1.5892(15)	1.5763(17)	1.5899(14)	1.6582(14)	1.7345(15)
$\Delta = 5.0$	1.7357(15)	1.6684(16)	1.5980(15)	1.5832(14)	1.5979(14)	1.6694(13)	1.7355(16)

Table A.55: **homoscedastic** Kuiper critical values, Case II: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.6880(14)	1.6174(12)	1.5485(15)	1.5368(15)	1.5486(14)	1.6183(13)	1.6885(15)
$\sigma = 3$	1.6389(16)	1.5478(13)	1.5285(15)	1.5312(14)	1.5584(14)	1.6296(14)	1.6900(16)
$\sigma = 5$	1.6412(14)	1.5453(13)	1.5235(16)	1.5244(16)	1.5509(14)	1.6285(15)	1.6913(13)
$\sigma = 7$	1.6453(14)	1.5520(13)	1.5255(15)	1.5261(16)	1.5485(13)	1.6276(15)	1.6917(16)
$\sigma = 9$	1.6500(16)	1.5605(14)	1.5278(14)	1.5263(16)	1.5479(13)	1.6269(15)	1.6917(15)

Table A.56: **heteroscedastic** Kuiper critical values, Case II: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3$, $\sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.7057(15)	1.6337(15)	1.5658(15)	1.5515(15)	1.5655(16)	1.6342(14)	1.7071(15)
$\sigma = 3$	1.6553(17)	1.5583(14)	1.5438(15)	1.5465(12)	1.5721(14)	1.6480(15)	1.7093(13)
$\sigma = 5$	1.6556(15)	1.5582(13)	1.5402(15)	1.5410(14)	1.5661(14)	1.6433(14)	1.7106(16)
$\sigma = 7$	1.6606(16)	1.5677(14)	1.5444(15)	1.5432(16)	1.5668(14)	1.6433(14)	1.7098(16)
$\sigma = 9$	1.6671(15)	1.5774(15)	1.5458(13)	1.5453(13)	1.5674(16)	1.6427(14)	1.7113(15)

Table A.57: **heteroscedastic** Kuiper critical values, Case II: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.7200(13)	1.6442(15)	1.5750(15)	1.5633(14)	1.5767(13)	1.6445(15)	1.7193(15)
$\sigma = 3$	1.6660(15)	1.5680(14)	1.5543(14)	1.5577(14)	1.5824(14)	1.6552(15)	1.7210(14)
$\sigma = 5$	1.6653(14)	1.5661(15)	1.5512(15)	1.5536(13)	1.5777(14)	1.6551(14)	1.7227(15)
$\sigma = 7$	1.6738(13)	1.5791(15)	1.5547(15)	1.5554(15)	1.5779(14)	1.6539(14)	1.7224(16)
$\sigma = 9$	1.6777(14)	1.5887(13)	1.5590(15)	1.5564(14)	1.5784(15)	1.6535(15)	1.7232(16)

Table A.58: **heteroscedastic** Kuiper critical values, Case II: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.7300(16)	1.6545(13)	1.5846(15)	1.5711(16)	1.5852(15)	1.6518(15)	1.7295(15)
$\sigma = 3$	1.6745(16)	1.5760(16)	1.5638(15)	1.5662(15)	1.5909(14)	1.6641(15)	1.7316(17)
$\sigma = 5$	1.6758(14)	1.5766(13)	1.5611(16)	1.5622(16)	1.5863(14)	1.6629(14)	1.7317(16)
$\sigma = 7$	1.6825(16)	1.5881(16)	1.5624(11)	1.5651(15)	1.5876(14)	1.6635(14)	1.7319(15)
$\sigma = 9$	1.6880(16)	1.5968(14)	1.5677(13)	1.5659(16)	1.5873(14)	1.6631(15)	1.7324(13)

Table A.59: **heteroscedastic** Kuiper critical values, Case II: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	1.7353(16)	1.6585(15)	1.5892(15)	1.5763(17)	1.5899(14)	1.6582(14)	1.7345(15)
$\sigma = 3$	1.6797(15)	1.5799(13)	1.5677(14)	1.5709(14)	1.5963(14)	1.6698(14)	1.7352(16)
$\sigma = 5$	1.6797(13)	1.5799(14)	1.5665(15)	1.5682(15)	1.5915(14)	1.6669(15)	1.7363(15)
$\sigma = 7$	1.6871(15)	1.5918(14)	1.5694(15)	1.5689(13)	1.5909(15)	1.6677(15)	1.7368(15)
$\sigma = 9$	1.6936(14)	1.6008(14)	1.5718(16)	1.5714(15)	1.5924(15)	1.6685(14)	1.7359(15)

Table A.60: **heteroscedastic** Kuiper critical values, Case II: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.9587(63)	1.5592(45)	1.2092(30)	1.1409(31)	1.2115(31)	1.5665(51)	1.9550(59)
$\Delta = 1.0$	2.0114(60)	1.4827(44)	1.1707(31)	1.0910(28)	1.1709(25)	1.4813(43)	2.0104(68)
$\Delta = 1.5$	2.1186(61)	1.5993(44)	1.1953(29)	1.0772(26)	1.1969(31)	1.6048(41)	2.1208(61)
$\Delta = 3.0$	2.4051(68)	1.8597(52)	1.3391(34)	1.1806(28)	1.3375(32)	1.8599(59)	2.4048(66)
$\Delta = 5.0$	2.4912(68)	1.9494(57)	1.4473(37)	1.2913(27)	1.4454(38)	1.9530(57)	2.4893(74)

Table A.61: **homoscedastic** Anderson-Darling critical values, Case II: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.9621(70)	1.4597(42)	1.1457(26)	1.0998(26)	1.1432(29)	1.4597(46)	1.9687(66)
$\Delta = 1.0$	2.0287(58)	1.4353(42)	1.1457(27)	1.0688(27)	1.1470(27)	1.4380(36)	2.0365(68)
$\Delta = 1.5$	2.1526(63)	1.5922(41)	1.1767(26)	1.0633(24)	1.1802(33)	1.5934(43)	2.1578(64)
$\Delta = 3.0$	2.4270(70)	1.8536(49)	1.3350(34)	1.1749(23)	1.3345(34)	1.8505(46)	2.4202(67)
$\Delta = 5.0$	2.4858(76)	1.9475(54)	1.4416(40)	1.2901(29)	1.4441(41)	1.9510(60)	2.4902(72)

Table A.62: **homoscedastic** Anderson-Darling critical values, Case II: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.9646(65)	1.3573(36)	1.1119(25)	1.0836(25)	1.1155(27)	1.3585(34)	1.9728(55)
$\Delta = 1.0$	2.0497(61)	1.4026(32)	1.1321(26)	1.0599(23)	1.1305(28)	1.4022(37)	2.0554(68)
$\Delta = 1.5$	2.1911(58)	1.5776(39)	1.1689(26)	1.0547(27)	1.1711(32)	1.5779(45)	2.1963(69)
$\Delta = 3.0$	2.4464(69)	1.8446(55)	1.3299(33)	1.1767(25)	1.3315(32)	1.8458(53)	2.4478(80)
$\Delta = 5.0$	2.4925(60)	1.9558(54)	1.4443(35)	1.2893(27)	1.4413(32)	1.9509(56)	2.4838(76)

Table A.63: **homoscedastic** Anderson-Darling critical values, Case II: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.9725(64)	1.2466(31)	1.1026(25)	1.0765(27)	1.1006(25)	1.2476(32)	1.9693(59)
$\Delta = 1.0$	2.0774(60)	1.3871(34)	1.1237(24)	1.0535(24)	1.1262(27)	1.3854(35)	2.0797(65)
$\Delta = 1.5$	2.2311(66)	1.5635(40)	1.1663(29)	1.0523(25)	1.1657(28)	1.5659(42)	2.2324(65)
$\Delta = 3.0$	2.4618(71)	1.8504(58)	1.3262(33)	1.1734(23)	1.3280(37)	1.8386(50)	2.4619(64)
$\Delta = 5.0$	2.4919(80)	1.9521(65)	1.4410(39)	1.2871(30)	1.4421(35)	1.9466(44)	2.4916(71)

Table A.64: **homoscedastic** Anderson-Darling critical values, Case II: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	1.9857(72)	1.2005(29)	1.0978(24)	1.0743(27)	1.1007(26)	1.1950(26)	1.9828(65)
$\Delta = 1.0$	2.0881(56)	1.3840(36)	1.1239(26)	1.0506(24)	1.1242(28)	1.3803(35)	2.0871(62)
$\Delta = 1.5$	2.2481(64)	1.5631(42)	1.1639(26)	1.0507(24)	1.1645(27)	1.5621(47)	2.2485(65)
$\Delta = 3.0$	2.4640(65)	1.8444(51)	1.3300(36)	1.1744(29)	1.3296(38)	1.8473(55)	2.4729(70)
$\Delta = 5.0$	2.4868(63)	1.9545(62)	1.4443(36)	1.2878(29)	1.4425(34)	1.9581(46)	2.4960(72)

Table A.65: **homoscedastic** Anderson-Darling critical values, Case II: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	2.4051(68)	1.8597(52)	1.3391(34)	1.1806(28)	1.3375(32)	1.8599(59)	2.4048(66)
$\sigma = 3$	2.1056(61)	1.5860(39)	1.4230(36)	1.4523(31)	1.6505(54)	2.0668(61)	2.4558(78)
$\sigma = 5$	2.3521(60)	2.0652(56)	1.9513(66)	1.9342(59)	2.0169(58)	2.2460(61)	2.4789(73)
$\sigma = 7$	2.4444(75)	2.2408(66)	2.1474(64)	2.1197(53)	2.1569(62)	2.2972(77)	2.4841(78)
$\sigma = 9$	2.4830(72)	2.3145(79)	2.2314(58)	2.1981(69)	2.2151(57)	2.3230(69)	2.4870(67)

Table A.66: **heteroscedastic** Anderson-Darling critical values, Case II: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	2.4270(70)	1.8536(49)	1.3350(34)	1.1749(23)	1.3345(34)	1.8505(46)	2.4202(67)
$\sigma = 3$	2.1069(65)	1.5567(43)	1.4155(38)	1.4468(41)	1.6416(49)	2.0533(63)	2.4696(67)
$\sigma = 5$	2.3410(71)	2.0510(50)	1.9469(49)	1.9363(58)	2.0186(71)	2.2316(65)	2.4831(67)
$\sigma = 7$	2.4300(74)	2.2410(60)	2.1503(67)	2.1225(69)	2.1607(72)	2.3002(70)	2.4921(72)
$\sigma = 9$	2.4714(60)	2.3073(74)	2.2221(57)	2.1944(63)	2.2222(71)	2.3303(68)	2.4930(71)

Table A.67: **heteroscedastic** Anderson-Darling critical values, Case II: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	2.4464(69)	1.8446(55)	1.3299(33)	1.1767(25)	1.3315(32)	1.8458(53)	2.4478(80)
$\sigma = 3$	2.0981(64)	1.5430(37)	1.4158(33)	1.4448(38)	1.6386(45)	2.0452(60)	2.4750(76)
$\sigma = 5$	2.3382(62)	2.0476(57)	1.9518(47)	1.9376(54)	2.0228(58)	2.2346(73)	2.4837(66)
$\sigma = 7$	2.4219(70)	2.2359(73)	2.1495(67)	2.1195(62)	2.1682(68)	2.2980(73)	2.4854(67)
$\sigma = 9$	2.4557(75)	2.3069(66)	2.2307(66)	2.1992(65)	2.2239(68)	2.3259(75)	2.4838(79)

Table A.68: **heteroscedastic** Anderson-Darling critical values, Case II: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	0.3510(13)	0.2649(09)	0.1899(05)	0.1750(05)	0.1900(05)	0.2660(10)	0.3486(12)
$\Delta = 1.0$	0.3648(12)	0.2557(09)	0.1820(05)	0.1608(04)	0.1821(05)	0.2556(08)	0.3645(12)
$\Delta = 1.5$	0.3940(12)	0.2885(10)	0.1864(05)	0.1520(04)	0.1863(05)	0.2891(08)	0.3948(12)
$\Delta = 3.0$	0.4506(14)	0.3515(11)	0.2148(06)	0.1606(04)	0.2143(06)	0.3514(11)	0.4507(14)
$\Delta = 5.0$	0.4602(12)	0.3713(12)	0.2385(06)	0.1823(04)	0.2383(07)	0.3720(11)	0.4600(14)

Table A.71: **homoscedastic** Cramér-von Mises critical values, Case II: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	0.3512(14)	0.2441(07)	0.1768(05)	0.1667(04)	0.1765(04)	0.2441(08)	0.3518(14)
$\Delta = 1.0$	0.3695(12)	0.2475(08)	0.1765(05)	0.1559(04)	0.1766(05)	0.2476(07)	0.3708(13)
$\Delta = 1.5$	0.4022(14)	0.2873(08)	0.1815(04)	0.1489(04)	0.1822(06)	0.2888(09)	0.4023(13)
$\Delta = 3.0$	0.4540(16)	0.3516(10)	0.2147(06)	0.1600(04)	0.2147(06)	0.3516(10)	0.4532(14)
$\Delta = 5.0$	0.4598(15)	0.3714(11)	0.2381(08)	0.1830(04)	0.2386(07)	0.3724(12)	0.4603(16)

Table A.72: **homoscedastic** Cramér-von Mises critical values, Case II: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	2.4618(71)	1.8504(58)	1.3262(33)	1.1734(23)	1.3280(37)	1.8386(50)	2.4619(64)
$\sigma = 3$	2.1081(57)	1.5407(38)	1.4148(34)	1.4421(35)	1.6371(44)	2.0425(57)	2.4891(71)
$\sigma = 5$	2.3314(59)	2.0500(57)	1.9493(60)	1.9453(62)	2.0201(63)	2.2317(64)	2.4944(76)
$\sigma = 7$	2.4105(69)	2.2312(69)	2.1477(62)	2.1244(57)	2.1683(53)	2.3030(69)	2.4894(68)
$\sigma = 9$	2.4468(73)	2.3064(75)	2.2362(73)	2.1971(67)	2.2230(59)	2.3232(76)	2.4964(72)

Table A.69: **heteroscedastic** Anderson-Darling critical values, Case II: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	2.4640(65)	1.8444(51)	1.3300(36)	1.1744(29)	1.3296(38)	1.8473(55)	2.4729(70)
$\sigma = 3$	2.1145(57)	1.5408(41)	1.4111(35)	1.4414(38)	1.6398(51)	2.0413(56)	2.4849(72)
$\sigma = 5$	2.3210(57)	2.0496(55)	1.9509(61)	1.9426(57)	2.0269(59)	2.2311(64)	2.4890(65)
$\sigma = 7$	2.4133(72)	2.2334(65)	2.1492(60)	2.1235(55)	2.1574(75)	2.2977(68)	2.4936(73)
$\sigma = 9$	2.4473(67)	2.3065(73)	2.2245(68)	2.1940(73)	2.2218(73)	2.3328(70)	2.4914(79)

Table A.70: **heteroscedastic** Anderson-Darling critical values, Case II: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	0.3522(13)	0.2231(06)	0.1704(04)	0.1633(04)	0.1704(04)	0.2232(07)	0.3533(12)
$\Delta = 1.0$	0.3739(12)	0.2421(07)	0.1732(04)	0.1535(04)	0.1730(04)	0.2418(07)	0.3745(14)
$\Delta = 1.5$	0.4088(13)	0.2855(08)	0.1797(05)	0.1471(03)	0.1802(05)	0.2852(09)	0.4100(14)
$\Delta = 3.0$	0.4565(15)	0.3509(11)	0.2137(05)	0.1599(03)	0.2145(06)	0.3509(11)	0.4575(16)
$\Delta = 5.0$	0.4620(13)	0.3733(11)	0.2388(07)	0.1830(04)	0.2390(06)	0.3724(11)	0.4598(17)

Table A.73: **homoscedastic** Cramér-von Mises critical values, Case II: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	0.3529(12)	0.2013(05)	0.1681(04)	0.1611(03)	0.1677(04)	0.2013(05)	0.3525(12)
$\Delta = 1.0$	0.3795(12)	0.2391(07)	0.1716(04)	0.1521(03)	0.1716(04)	0.2388(07)	0.3800(13)
$\Delta = 1.5$	0.4161(14)	0.2828(08)	0.1790(04)	0.1466(04)	0.1788(05)	0.2833(08)	0.4164(14)
$\Delta = 3.0$	0.4582(15)	0.3516(12)	0.2135(06)	0.1597(03)	0.2140(07)	0.3500(11)	0.4585(14)
$\Delta = 5.0$	0.4610(16)	0.3727(12)	0.2390(08)	0.1832(04)	0.2394(07)	0.3720(10)	0.4608(15)

Table A.74: **homoscedastic** Cramér-von Mises critical values, Case II: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\Delta = 0.5$	0.3557(15)	0.1920(05)	0.1671(04)	0.1608(05)	0.1674(04)	0.1914(05)	0.3547(13)
$\Delta = 1.0$	0.3817(11)	0.2385(07)	0.1713(04)	0.1517(04)	0.1714(05)	0.2375(07)	0.3819(11)
$\Delta = 1.5$	0.4194(13)	0.2827(08)	0.1783(05)	0.1461(03)	0.1787(04)	0.2823(09)	0.4190(14)
$\Delta = 3.0$	0.4585(13)	0.3511(11)	0.2139(07)	0.1597(04)	0.2142(07)	0.3516(11)	0.4593(14)
$\Delta = 5.0$	0.4606(14)	0.3737(12)	0.2393(08)	0.1832(05)	0.2388(06)	0.3741(11)	0.4615(15)

Table A.75: **homoscedastic** Cramér-von Mises critical values, Case II: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = [0.5, 1, 1.5, 3, 5], \sigma_2 = 1$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	0.4506(14)	0.3515(11)	0.2148(06)	0.1606(04)	0.2143(06)	0.3514(11)	0.4507(14)
$\sigma = 3$	0.3697(12)	0.2445(06)	0.2024(05)	0.2233(05)	0.2911(10)	0.3908(12)	0.4561(15)
$\sigma = 5$	0.4164(11)	0.3494(11)	0.3277(12)	0.3386(11)	0.3741(11)	0.4271(11)	0.4590(16)
$\sigma = 7$	0.4408(15)	0.3971(13)	0.3816(11)	0.3852(11)	0.4058(14)	0.4375(16)	0.4588(16)
$\sigma = 9$	0.4517(15)	0.4179(15)	0.4040(12)	0.4047(13)	0.4186(12)	0.4430(14)	0.4590(14)

Table A.76: **heteroscedastic** Cramér-von Mises critical values, Case II: $n = 50$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	0.4540(16)	0.3516(10)	0.2147(06)	0.1600(04)	0.2147(06)	0.3516(10)	0.4532(14)
$\sigma = 3$	0.3721(13)	0.2395(08)	0.2009(06)	0.2224(07)	0.2897(09)	0.3918(13)	0.4585(13)
$\sigma = 5$	0.4165(15)	0.3466(10)	0.3271(09)	0.3391(11)	0.3736(13)	0.4268(14)	0.4603(12)
$\sigma = 7$	0.4407(14)	0.3977(13)	0.3818(13)	0.3849(14)	0.4058(14)	0.4389(14)	0.4613(14)
$\sigma = 9$	0.4510(12)	0.4172(15)	0.4028(11)	0.4055(12)	0.4204(14)	0.4448(14)	0.4607(14)

Table A.77: **heteroscedastic** Cramér-von Mises critical values, Case II: $n = 100$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	0.4565(15)	0.3509(11)	0.2137(05)	0.1599(03)	0.2145(06)	0.3509(11)	0.4575(16)
$\sigma = 3$	0.3720(13)	0.2366(06)	0.2000(05)	0.2220(07)	0.2893(09)	0.3917(13)	0.4592(17)
$\sigma = 5$	0.4171(13)	0.3462(10)	0.3287(09)	0.3389(11)	0.3747(12)	0.4264(14)	0.4600(13)
$\sigma = 7$	0.4403(14)	0.3976(15)	0.3822(12)	0.3855(12)	0.4069(12)	0.4389(14)	0.4598(13)
$\sigma = 9$	0.4499(13)	0.4177(12)	0.4047(14)	0.4064(13)	0.4196(14)	0.4438(15)	0.4599(15)

Table A.78: **heteroscedastic** Cramér-von Mises critical values, Case II: $n = 200$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	0.4582(15)	0.3516(12)	0.2135(06)	0.1597(03)	0.2140(07)	0.3500(11)	0.4585(14)
$\sigma = 3$	0.3754(12)	0.2360(07)	0.1999(05)	0.2214(06)	0.2890(08)	0.3907(12)	0.4613(14)
$\sigma = 5$	0.4171(12)	0.3469(11)	0.3282(11)	0.3405(12)	0.3739(12)	0.4263(13)	0.4621(15)
$\sigma = 7$	0.4389(14)	0.3973(14)	0.3814(12)	0.3865(12)	0.4073(11)	0.4396(15)	0.4610(13)
$\sigma = 9$	0.4488(15)	0.4173(14)	0.4059(13)	0.4053(14)	0.4191(12)	0.4436(16)	0.4620(14)

Table A.79: **heteroscedastic** Cramér-von Mises critical values, Case II: $n = 500$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\sigma = 1$	0.4585(13)	0.3511(11)	0.2139(07)	0.1597(04)	0.2142(07)	0.3516(11)	0.4593(14)
$\sigma = 3$	0.3766(12)	0.2355(07)	0.1993(05)	0.2215(07)	0.2895(09)	0.3904(12)	0.4603(15)
$\sigma = 5$	0.4160(12)	0.3469(10)	0.3285(11)	0.3403(10)	0.3750(12)	0.4258(14)	0.4608(13)
$\sigma = 7$	0.4393(15)	0.3970(14)	0.3817(11)	0.3866(12)	0.4053(15)	0.4387(15)	0.4615(16)
$\sigma = 9$	0.4491(13)	0.4182(14)	0.4044(13)	0.4056(14)	0.4194(15)	0.4447(15)	0.4607(15)

Table A.80: **heteroscedastic** Cramér-von Mises critical values, Case II: $n = 1000$, $\mu_1 = 0$, $\sigma_1 = 1$, $\mu_2 = 3, \sigma_2 = [1, 3, 5, 7, 9]$.

Appendix B

Critical Values: Two-component Weibull Mixture

Section 6.2 considered hypothesis testing for Weibull mixture models. Critical values were calculated for various two-component Weibull mixture models defined by,

$$\boldsymbol{\alpha} = \{1,\, 1\}, \quad \boldsymbol{\beta} = \{\, [0.5,\, 0.5,\, 0.5,\, 1,\, 1,\, 1.5],\, [1,\, 1.5,\, 5,\, 1.5,\, 5,\, 5]\,\} \tag{B.1}$$

with the following mixture proportions and sample sizes,

$$\pi_1 = \Big\{0,\, 0.1,\, 0.3,\, 0.5,\, 0.7,\, 0.9,\, 1\Big\}, \tag{B.2}$$

$$n = \Big\{50,\, 100,\, 200,\, 500,\, 1000\Big\}, \tag{B.3}$$

for Kolmogorov-Smirnov, Kuiper, Anderson-Darling, and Cramér-von Mises goodness-of-fit tests. The full set of critical values are presented in this appendix.

B.1 Case I: Parameters/mixture proportions known *a priori*

- Kolmogorov- Smirnov: Tables [B.1 - B.5]
- Kuiper: Tables [B.6 - B.10]
- Anderson-Darling: Tables [B.11 - B.15]
- Cramér-von Mises: Tables [B.16 - B.20]

B.2 Case II: Component parameters known *a priori*, mixture proportions requiring estimation

- Kolmogorov- Smirnov: Tables [B.21 - B.25]
- Kuiper: Tables [B.26 - B.30]
- Anderson-Darling: Tables [B.31 - B.35]
- Cramér-von Mises: Tables [B.36 - B.40]

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3324(17)	1.3288(16)	1.3249(16)	1.3216(17)	1.3230(18)	1.3284(15)	1.3314(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3326(16)	1.3271(14)	1.3180(14)	1.3114(17)	1.3123(15)	1.3214(18)	1.3326(16)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3314(16)	1.3216(15)	1.2969(14)	1.2770(15)	1.2711(14)	1.3002(13)	1.3315(15)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3303(16)	1.3308(15)	1.3304(15)	1.3299(17)	1.3287(17)	1.3320(17)	1.3315(16)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3313(15)	1.3235(15)	1.3091(16)	1.2949(15)	1.2934(15)	1.3123(15)	1.3326(17)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3317(14)	1.3274(17)	1.3151(17)	1.3080(17)	1.3079(14)	1.3206(17)	1.3329(16)

Table B.1: Kolomogorov-Smirnov critical values, Case I: $n = 50$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3400(14)	1.3381(15)	1.3327(16)	1.3319(16)	1.3320(16)	1.3365(17)	1.3413(14)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3401(17)	1.3344(16)	1.3266(18)	1.3186(15)	1.3185(17)	1.3319(16)	1.3411(16)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3399(16)	1.3291(15)	1.3062(18)	1.2858(16)	1.2798(14)	1.3082(15)	1.3380(16)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3391(15)	1.3386(14)	1.3379(16)	1.3371(16)	1.3359(18)	1.3386(16)	1.3401(16)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3388(16)	1.3328(15)	1.3169(14)	1.3038(15)	1.3021(17)	1.3204(17)	1.3400(16)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3391(16)	1.3353(17)	1.3238(14)	1.3162(18)	1.3153(15)	1.3294(16)	1.3413(15)

Table B.2: Kolomogorov-Smirnov critical values, Case I: $n = 100$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3462(15)	1.3420(13)	1.3400(15)	1.3358(15)	1.3360(17)	1.3410(15)	1.3450(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3449(18)	1.3408(16)	1.3305(17)	1.3244(16)	1.3246(15)	1.3368(15)	1.3454(15)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3459(16)	1.3357(18)	1.3141(16)	1.2895(16)	1.2853(15)	1.3150(15)	1.3466(16)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3454(16)	1.3447(18)	1.3426(17)	1.3429(16)	1.3433(15)	1.3420(18)	1.3437(16)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3443(14)	1.3363(15)	1.3225(14)	1.3076(15)	1.3060(17)	1.3266(16)	1.3454(17)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3457(18)	1.3381(16)	1.3290(16)	1.3223(15)	1.3210(16)	1.3355(16)	1.3447(16)

Table B.3: Kolomogorov-Smirnov critical values, Case I: $n = 200$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3510(17)	1.3492(16)	1.3428(15)	1.3419(13)	1.3420(15)	1.3460(16)	1.3521(16)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3483(18)	1.3457(17)	1.3370(15)	1.3292(16)	1.3298(16)	1.3410(16)	1.3506(16)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3507(15)	1.3407(16)	1.3177(15)	1.2949(16)	1.2899(16)	1.3184(16)	1.3507(17)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3499(16)	1.3480(14)	1.3480(14)	1.3479(15)	1.3473(15)	1.3483(14)	1.3500(17)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3517(16)	1.3422(16)	1.3263(15)	1.3136(16)	1.3114(15)	1.3316(17)	1.3499(16)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3506(14)	1.3437(15)	1.3340(16)	1.3267(17)	1.3265(13)	1.3386(16)	1.3510(15)

Table B.4: Kolomogorov-Smirnov critical values, Case I: $n = 500$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3521(15)	1.3517(15)	1.3463(17)	1.3428(16)	1.3436(16)	1.3485(15)	1.3530(16)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3530(12)	1.3462(17)	1.3385(15)	1.3315(18)	1.3337(17)	1.3419(15)	1.3540(16)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3534(14)	1.3424(17)	1.3195(15)	1.2973(16)	1.2941(15)	1.3207(16)	1.3521(15)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3524(17)	1.3524(16)	1.3504(15)	1.3514(15)	1.3490(15)	1.3517(16)	1.3526(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3519(14)	1.3453(17)	1.3277(17)	1.3145(16)	1.3111(16)	1.3322(18)	1.3537(16)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3527(16)	1.3469(15)	1.3352(13)	1.3283(16)	1.3281(16)	1.3404(16)	1.3526(13)

Table B.5: Kolomogorov-Smirnov critical values, Case I: $n = 1000$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.6928(15)	1.6837(14)	1.6713(14)	1.6672(15)	1.6704(13)	1.6854(14)	1.6925(14)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.6934(15)	1.6733(16)	1.6466(14)	1.6374(15)	1.6471(13)	1.6723(16)	1.6930(16)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.6928(15)	1.6438(15)	1.5856(13)	1.5715(13)	1.5878(13)	1.6452(13)	1.6933(15)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.6931(13)	1.6895(14)	1.6854(16)	1.6833(14)	1.6858(17)	1.6896(15)	1.6939(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.6926(17)	1.6585(15)	1.6170(13)	1.6031(13)	1.6155(14)	1.6581(13)	1.6929(14)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.6933(16)	1.6713(16)	1.6402(15)	1.6301(15)	1.6400(13)	1.6695(16)	1.6924(15)

Table B.6: Kuiper critical values, Case I: $n = 50$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1$, $\beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.7115(13)	1.7020(14)	1.6893(14)	1.6854(13)	1.6899(14)	1.7026(18)	1.7118(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.7114(15)	1.6906(15)	1.6643(15)	1.6553(13)	1.6642(14)	1.6915(14)	1.7101(15)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.7110(15)	1.6623(15)	1.6063(15)	1.5899(15)	1.6060(14)	1.6644(13)	1.7096(17)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.7102(14)	1.7079(15)	1.7045(14)	1.7012(15)	1.7014(16)	1.7072(15)	1.7106(14)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.7101(17)	1.6776(14)	1.6349(13)	1.6211(14)	1.6337(14)	1.6771(15)	1.7110(15)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.7106(13)	1.6879(14)	1.6596(13)	1.6487(15)	1.6576(14)	1.6863(14)	1.7120(15)

Table B.7: Kuiper critical values, Case I: $n = 100$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1$, $\beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.7211(15)	1.7109(15)	1.7003(15)	1.6970(15)	1.7002(15)	1.7118(14)	1.7223(14)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.7215(16)	1.7020(15)	1.6769(14)	1.6676(15)	1.6763(14)	1.7024(16)	1.7221(16)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.7222(16)	1.6754(17)	1.6184(15)	1.6006(12)	1.6179(13)	1.6760(13)	1.7226(14)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.7228(15)	1.7185(15)	1.7134(15)	1.7135(15)	1.7153(15)	1.7174(16)	1.7218(14)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.7201(16)	1.6869(14)	1.6461(16)	1.6319(13)	1.6456(15)	1.6896(13)	1.7212(14)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.7223(16)	1.6989(14)	1.6700(15)	1.6608(15)	1.6686(14)	1.7004(14)	1.7223(16)

Table B.8: Kuiper critical values, Case I: $n = 200$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1$, $\beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.7320(15)	1.7231(15)	1.7103(14)	1.7060(15)	1.7109(14)	1.7226(14)	1.7335(16)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.7308(15)	1.7130(16)	1.6873(15)	1.6769(14)	1.6859(12)	1.7114(14)	1.7327(14)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.7307(14)	1.6833(15)	1.6272(15)	1.6102(15)	1.6275(14)	1.6850(15)	1.7310(16)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.7305(14)	1.7271(13)	1.7247(16)	1.7222(14)	1.7230(14)	1.7280(15)	1.7320(16)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.7316(15)	1.6986(15)	1.6552(12)	1.6412(14)	1.6565(13)	1.6992(16)	1.7315(14)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.7312(18)	1.7083(13)	1.6794(13)	1.6705(15)	1.6785(13)	1.7088(14)	1.7323(15)

Table B.9: Kuiper critical values, Case I: $n = 500$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1$, $\beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.7365(16)	1.7275(15)	1.7148(13)	1.7098(14)	1.7139(16)	1.7264(15)	1.7367(16)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.7362(14)	1.7151(15)	1.6909(13)	1.6829(13)	1.6905(15)	1.7162(14)	1.7370(15)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.7366(16)	1.6879(13)	1.6325(14)	1.6158(13)	1.6314(14)	1.6899(12)	1.7369(15)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.7372(15)	1.7336(16)	1.7279(13)	1.7279(16)	1.7283(17)	1.7327(16)	1.7372(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.7360(15)	1.7025(14)	1.6608(14)	1.6465(14)	1.6581(14)	1.7013(14)	1.7361(17)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.7361(15)	1.7138(15)	1.6823(14)	1.6738(15)	1.6833(15)	1.7137(16)	1.7360(13)

Table B.10: Kuiper critical values, Case I: $n = 1000$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1$, $\beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	2.4959(73)	2.4704(74)	2.4472(67)	2.4421(75)	2.4512(69)	2.4810(75)	2.4942(73)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	2.4981(81)	2.4356(61)	2.3951(64)	2.3978(69)	2.4203(70)	2.4602(79)	2.4965(68)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	2.4966(70)	2.3716(74)	2.2955(77)	2.2981(71)	2.3451(64)	2.4333(75)	2.4914(66)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	2.4941(73)	2.4835(59)	2.4821(73)	2.4751(72)	2.4803(62)	2.4946(73)	2.4925(73)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	2.4958(71)	2.3987(66)	2.3455(70)	2.3464(77)	2.3838(72)	2.4476(69)	2.4968(83)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	2.4960(60)	2.4346(77)	2.3834(78)	2.3819(69)	2.4075(71)	2.4604(71)	2.4972(75)

Table B.11: Anderson-Darling critical values, Case I: $n = 50$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	2.4936(68)	2.4680(66)	2.4407(77)	2.4485(74)	2.4534(73)	2.4770(78)	2.5003(73)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	2.4948(74)	2.4365(69)	2.4032(66)	2.3952(72)	2.4135(62)	2.4695(72)	2.4971(69)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	2.4883(70)	2.3726(72)	2.2992(71)	2.3014(77)	2.3467(70)	2.4330(72)	2.4880(63)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	2.4916(65)	2.4823(69)	2.4784(75)	2.4707(69)	2.4745(77)	2.4904(78)	2.4973(68)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	2.4879(69)	2.4117(70)	2.3425(72)	2.3503(72)	2.3818(74)	2.4408(69)	2.4915(69)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	2.4897(67)	2.4317(72)	2.3895(64)	2.3791(72)	2.4026(73)	2.4626(73)	2.4986(74)

Table B.12: Anderson-Darling critical values, Case I: $n = 100$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	2.4944(72)	2.4703(67)	2.4478(73)	2.4403(74)	2.4556(80)	2.4719(74)	2.4932(64)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	2.4923(78)	2.4383(65)	2.4019(72)	2.3975(59)	2.4142(69)	2.4679(68)	2.4915(70)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	2.4973(69)	2.3743(74)	2.3064(71)	2.2967(66)	2.3481(67)	2.4348(70)	2.4946(67)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	2.4934(66)	2.4822(75)	2.4734(70)	2.4788(77)	2.4752(71)	2.4736(77)	2.4863(77)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	2.4929(61)	2.4038(66)	2.3472(59)	2.3386(66)	2.3751(68)	2.4492(74)	2.4900(78)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	2.4888(78)	2.4296(77)	2.3871(76)	2.3888(85)	2.4078(71)	2.4692(65)	2.4983(75)

Table B.13: Anderson-Darling critical values, Case I: $n = 200$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	2.4954(73)	2.4699(71)	2.4465(72)	2.4440(68)	2.4531(69)	2.4764(72)	2.4957(64)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	2.4842(77)	2.4456(69)	2.4014(78)	2.3942(76)	2.4186(78)	2.4654(70)	2.4916(73)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	2.4969(67)	2.3814(67)	2.3036(64)	2.3040(74)	2.3461(67)	2.4284(75)	2.4922(73)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	2.4902(67)	2.4760(64)	2.4760(67)	2.4756(74)	2.4753(69)	2.4836(71)	2.4940(75)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	2.4936(70)	2.4069(72)	2.3496(70)	2.3483(73)	2.3745(64)	2.4486(81)	2.4923(83)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	2.4957(70)	2.4314(74)	2.3883(72)	2.3881(84)	2.4076(70)	2.4591(67)	2.4951(64)

Table B.14: Anderson-Darling critical values, Case I: $n = 500$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	2.4917(66)	2.4760(67)	2.4444(67)	2.4384(74)	2.4512(66)	2.4757(68)	2.4921(66)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	2.4938(65)	2.4390(76)	2.3958(67)	2.3939(79)	2.4227(74)	2.4554(70)	2.4924(71)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	2.4959(60)	2.3785(78)	2.3080(67)	2.2994(80)	2.3468(62)	2.4271(69)	2.4905(75)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	2.4989(68)	2.4836(67)	2.4759(67)	2.4788(71)	2.4764(68)	2.4894(69)	2.4917(64)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	2.4866(68)	2.4080(76)	2.3501(82)	2.3414(73)	2.3725(72)	2.4510(66)	2.4958(67)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	2.4957(73)	2.4390(68)	2.3878(72)	2.3844(76)	2.4039(77)	2.4594(70)	2.4902(70)

Table B.15: Anderson-Darling critical values, Case I: $n = 1000$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	0.4599(15)	0.4574(15)	0.4540(13)	0.4527(14)	0.4533(13)	0.4578(14)	0.4594(14)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	0.4600(15)	0.4543(14)	0.4480(13)	0.4459(15)	0.4481(15)	0.4534(17)	0.4602(13)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	0.4598(14)	0.4494(13)	0.4319(15)	0.4259(14)	0.4298(13)	0.4459(15)	0.4592(14)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	0.4588(14)	0.4582(12)	0.4590(14)	0.4577(14)	0.4580(14)	0.4597(14)	0.4591(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	0.4592(15)	0.4515(14)	0.4404(15)	0.4359(15)	0.4394(14)	0.4496(14)	0.4602(17)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	0.4593(13)	0.4554(15)	0.4463(16)	0.4434(15)	0.4452(13)	0.4531(13)	0.4599(14)

Table B.16: Cramér-von Mises critical values, Case I: $n = 50$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	0.4605(14)	0.4578(13)	0.4536(16)	0.4549(15)	0.4545(15)	0.4578(16)	0.4617(14)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	0.4613(15)	0.4548(14)	0.4497(14)	0.4458(15)	0.4466(13)	0.4562(15)	0.4609(13)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	0.4597(15)	0.4494(15)	0.4331(15)	0.4270(15)	0.4306(13)	0.4460(15)	0.4592(13)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	0.4597(13)	0.4589(14)	0.4589(15)	0.4579(15)	0.4575(15)	0.4600(15)	0.4614(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	0.4592(15)	0.4533(14)	0.4409(14)	0.4371(14)	0.4399(14)	0.4504(14)	0.4599(14)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	0.4595(13)	0.4544(15)	0.4474(13)	0.4435(15)	0.4449(15)	0.4547(14)	0.4609(14)

Table B.17: Cramér-von Mises critical values, Case I: $n = 100$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	0.4617(15)	0.4586(13)	0.4552(15)	0.4532(15)	0.4546(17)	0.4575(14)	0.4615(14)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	0.4601(16)	0.4557(15)	0.4499(14)	0.4470(13)	0.4471(14)	0.4565(15)	0.4605(15)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	0.4623(15)	0.4499(15)	0.4344(14)	0.4264(14)	0.4318(13)	0.4481(15)	0.4614(14)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	0.4614(13)	0.4597(14)	0.4585(15)	0.4595(15)	0.4584(14)	0.4573(16)	0.4598(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	0.4606(13)	0.4523(14)	0.4414(13)	0.4360(13)	0.4391(14)	0.4522(15)	0.4600(15)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	0.4608(16)	0.4535(15)	0.4473(15)	0.4453(15)	0.4461(15)	0.4562(13)	0.4616(15)

Table B.18: Cramér-von Mises critical values, Case I: $n = 200$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	0.4620(16)	0.4594(14)	0.4550(15)	0.4547(13)	0.4551(14)	0.4593(14)	0.4617(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	0.4588(15)	0.4569(15)	0.4494(16)	0.4465(15)	0.4485(16)	0.4561(15)	0.4608(14)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	0.4619(15)	0.4516(13)	0.4338(13)	0.4273(14)	0.4324(13)	0.4476(16)	0.4605(14)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	0.4602(14)	0.4588(13)	0.4593(14)	0.4599(14)	0.4577(13)	0.4593(14)	0.4612(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	0.4613(15)	0.4532(14)	0.4419(13)	0.4377(14)	0.4392(14)	0.4522(16)	0.4607(17)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	0.4608(14)	0.4546(15)	0.4478(15)	0.4454(17)	0.4465(13)	0.4546(15)	0.4623(13)

Table B.19: Cramér-von Mises critical values, Case I: $n = 500$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	0.4613(14)	0.4600(14)	0.4551(14)	0.4538(16)	0.4550(13)	0.4592(13)	0.4617(14)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	0.4623(12)	0.4556(15)	0.4483(14)	0.4467(16)	0.4496(14)	0.4540(13)	0.4614(13)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	0.4623(14)	0.4502(16)	0.4353(13)	0.4270(16)	0.4320(12)	0.4464(15)	0.4607(15)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	0.4625(14)	0.4598(14)	0.4593(14)	0.4601(14)	0.4589(14)	0.4614(14)	0.4616(14)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	0.4607(13)	0.4535(15)	0.4420(16)	0.4360(15)	0.4379(15)	0.4516(14)	0.4624(13)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	0.4623(14)	0.4562(13)	0.4472(13)	0.4440(15)	0.4462(16)	0.4550(14)	0.4604(14)

Table B.20: Cramér-von Mises critical values, Case I: $n = 1000$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3203(15)	1.3126(17)	1.2936(15)	1.2734(16)	1.2600(16)	1.2735(14)	1.3082(16)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3268(16)	1.3157(19)	1.2915(14)	1.2675(17)	1.2430(16)	1.2472(15)	1.3031(17)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3328(18)	1.3173(16)	1.2870(16)	1.2522(15)	1.2277(15)	1.2398(17)	1.3069(16)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3083(15)	1.3035(15)	1.2941(15)	1.2843(15)	1.2818(17)	1.2967(16)	1.3161(18)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3291(17)	1.3164(19)	1.2872(17)	1.2586(15)	1.2310(17)	1.2318(17)	1.2992(17)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3259(15)	1.3156(15)	1.2906(18)	1.2637(15)	1.2408(16)	1.2428(14)	1.3002(17)

Table B.21: Kolomogorov-Smirnov critical values, Case II: $n = 50$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3312(16)	1.3215(15)	1.3017(16)	1.2841(16)	1.2671(15)	1.2699(15)	1.3121(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3369(15)	1.3236(15)	1.2986(15)	1.2752(13)	1.2507(15)	1.2430(16)	1.3043(17)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3418(15)	1.3262(15)	1.2954(16)	1.2600(18)	1.2344(15)	1.2469(16)	1.3127(16)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3218(15)	1.3154(16)	1.3041(16)	1.2915(16)	1.2840(15)	1.2960(16)	1.3177(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3381(15)	1.3244(15)	1.2954(16)	1.2658(16)	1.2381(16)	1.2335(15)	1.3033(17)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3375(17)	1.3232(16)	1.2980(16)	1.2723(14)	1.2471(14)	1.2387(14)	1.3031(17)

Table B.22: Kolomogorov-Smirnov critical values, Case II: $n = 100$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3372(16)	1.3290(15)	1.3068(17)	1.2879(15)	1.2701(18)	1.2649(16)	1.3125(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3425(15)	1.3294(17)	1.3035(17)	1.2801(17)	1.2564(15)	1.2417(18)	1.3060(17)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3439(15)	1.3305(15)	1.2994(17)	1.2649(17)	1.2396(14)	1.2512(14)	1.3171(14)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3313(15)	1.3223(15)	1.3096(16)	1.2964(15)	1.2868(15)	1.2905(17)	1.3190(16)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3448(17)	1.3308(15)	1.3012(16)	1.2719(15)	1.2450(13)	1.2363(15)	1.3072(16)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3439(15)	1.3287(16)	1.3033(15)	1.2782(17)	1.2520(13)	1.2382(18)	1.3055(15)

Table B.23: Kolomogorov-Smirnov critical values, Case II: $n = 200$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3455(16)	1.3319(15)	1.3117(15)	1.2932(14)	1.2754(15)	1.2612(14)	1.3142(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3499(17)	1.3326(15)	1.3087(20)	1.2843(16)	1.2612(17)	1.2437(15)	1.3065(15)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3507(17)	1.3374(17)	1.3054(16)	1.2706(16)	1.2449(15)	1.2553(16)	1.3203(17)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3416(15)	1.3298(17)	1.3137(16)	1.3001(15)	1.2894(15)	1.2852(17)	1.3222(16)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3502(16)	1.3342(16)	1.3066(16)	1.2758(18)	1.2487(16)	1.2410(17)	1.3092(16)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3486(14)	1.3335(18)	1.3073(16)	1.2840(17)	1.2570(15)	1.2423(16)	1.3080(15)

Table B.24: Kolomogorov-Smirnov critical values, Case II: $n = 500$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.3491(15)	1.3346(14)	1.3131(16)	1.2945(16)	1.2771(16)	1.2630(16)	1.3136(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.3508(13)	1.3364(15)	1.3112(16)	1.2875(14)	1.2627(17)	1.2456(16)	1.3091(13)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.3525(15)	1.3388(17)	1.3065(15)	1.2713(14)	1.2476(17)	1.2578(16)	1.3222(15)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.3437(17)	1.3325(18)	1.3158(17)	1.3047(15)	1.2914(16)	1.2834(16)	1.3226(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.3514(16)	1.3373(18)	1.3080(16)	1.2794(17)	1.2506(16)	1.2426(13)	1.3121(16)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.3514(16)	1.3364(19)	1.3111(15)	1.2835(16)	1.2609(15)	1.2432(15)	1.3082(14)

Table B.25: Kolomogorov-Smirnov critical values, Case II: $n = 1000$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.6689(16)	1.6280(13)	1.5721(13)	1.5394(14)	1.5272(16)	1.5592(14)	1.6224(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.6819(14)	1.6303(16)	1.5603(14)	1.5281(16)	1.5183(14)	1.5429(14)	1.6278(15)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.6920(15)	1.6226(15)	1.5471(12)	1.5261(14)	1.5287(15)	1.5640(14)	1.6493(13)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.6526(14)	1.6221(12)	1.5827(14)	1.5591(13)	1.5527(12)	1.5835(14)	1.6242(16)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.6886(15)	1.6284(15)	1.5491(13)	1.5238(13)	1.5201(14)	1.5411(14)	1.6346(17)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.6835(15)	1.6290(13)	1.5569(16)	1.5255(14)	1.5181(15)	1.5411(15)	1.6281(16)

Table B.26: Kuiper critical values, Case II: $n = 50$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1$, $\beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.6897(16)	1.6455(13)	1.5877(13)	1.5562(14)	1.5427(15)	1.5643(15)	1.6417(14)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.7021(15)	1.6465(14)	1.5758(13)	1.5461(13)	1.5379(14)	1.5488(14)	1.6442(14)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.7114(13)	1.6404(13)	1.5668(13)	1.5444(13)	1.5464(15)	1.5807(15)	1.6680(16)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.6737(14)	1.6393(15)	1.5993(14)	1.5716(14)	1.5607(14)	1.5905(15)	1.6415(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.7071(14)	1.6422(13)	1.5675(14)	1.5428(14)	1.5393(15)	1.5538(16)	1.6519(16)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.7046(13)	1.6445(12)	1.5733(15)	1.5442(13)	1.5376(14)	1.5481(14)	1.6457(15)

Table B.27: Kuiper critical values, Case II: $n = 100$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1$, $\beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.7051(17)	1.6575(13)	1.5955(13)	1.5674(15)	1.5533(15)	1.5630(12)	1.6521(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.7158(13)	1.6551(14)	1.5864(14)	1.5567(16)	1.5496(14)	1.5527(16)	1.6559(17)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.7205(16)	1.6523(14)	1.5775(15)	1.5559(16)	1.5580(13)	1.5917(14)	1.6787(14)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.6908(13)	1.6498(14)	1.6071(14)	1.5821(15)	1.5665(13)	1.5874(14)	1.6535(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.7210(16)	1.6537(13)	1.5782(13)	1.5532(13)	1.5510(15)	1.5621(14)	1.6623(13)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.7187(14)	1.6546(14)	1.5855(13)	1.5551(15)	1.5490(14)	1.5530(15)	1.6564(15)

Table B.28: Kuiper critical values, Case II: $n = 200$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1$, $\beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.7195(14)	1.6651(14)	1.6059(15)	1.5760(12)	1.5629(13)	1.5607(16)	1.6645(16)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.7292(13)	1.6632(14)	1.5946(14)	1.5669(15)	1.5599(15)	1.5602(15)	1.6649(15)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.7311(15)	1.6634(12)	1.5880(14)	1.5665(14)	1.5688(15)	1.5999(14)	1.6890(15)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.7061(12)	1.6593(13)	1.6146(15)	1.5889(12)	1.5760(14)	1.5792(15)	1.6650(16)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.7313(15)	1.6626(14)	1.5887(12)	1.5633(16)	1.5593(15)	1.5702(12)	1.6715(15)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.7289(12)	1.6631(13)	1.5935(15)	1.5664(15)	1.5577(14)	1.5615(16)	1.6658(15)

Table B.29: Kuiper critical values, Case II: $n = 500$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1$, $\beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	1.7285(13)	1.6681(16)	1.6100(15)	1.5804(14)	1.5675(15)	1.5641(15)	1.6683(17)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	1.7340(15)	1.6687(14)	1.5994(14)	1.5708(13)	1.5628(16)	1.5641(15)	1.6713(14)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	1.7359(17)	1.6677(14)	1.5918(14)	1.5699(12)	1.5726(18)	1.6043(13)	1.6938(14)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	1.7132(15)	1.6649(15)	1.6176(14)	1.5932(14)	1.5774(13)	1.5770(14)	1.6696(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	1.7359(15)	1.6675(15)	1.5916(14)	1.5679(14)	1.5646(15)	1.5768(14)	1.6777(16)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	1.7346(16)	1.6683(15)	1.5986(15)	1.5696(15)	1.5639(14)	1.5671(14)	1.6704(14)

Table B.30: Kuiper critical values, Case II: $n = 1000$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1$, $\beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	2.3486(66)	2.3205(85)	2.2679(68)	2.2175(67)	2.2126(72)	2.3200(64)	2.4825(75)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	2.4078(69)	2.3441(68)	2.2424(57)	2.2105(63)	2.2109(69)	2.2902(71)	2.4893(86)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	2.4768(74)	2.3229(72)	2.2285(67)	2.2156(61)	2.2572(61)	2.3321(73)	2.4927(74)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	2.2907(59)	2.2763(64)	2.2630(68)	2.2464(67)	2.2732(71)	2.3755(68)	2.4962(84)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	2.4489(70)	2.3331(71)	2.2239(69)	2.2120(67)	2.2269(69)	2.2939(66)	2.4793(66)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	2.4115(71)	2.3386(67)	2.2420(77)	2.2049(65)	2.2132(62)	2.2949(62)	2.4777(78)

Table B.31: Anderson-Darling critical values, Case II: $n = 50$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	2.3869(72)	2.3409(75)	2.2550(67)	2.2246(68)	2.2190(78)	2.2863(66)	2.4569(73)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	2.4380(73)	2.3456(63)	2.2422(68)	2.2151(68)	2.2135(75)	2.2567(72)	2.4511(72)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	2.4937(59)	2.3329(70)	2.2367(66)	2.2186(73)	2.2514(70)	2.3325(76)	2.4734(75)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	2.3349(70)	2.3144(72)	2.2796(73)	2.2442(65)	2.2503(71)	2.3361(77)	2.4576(60)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	2.4623(72)	2.3344(62)	2.2395(66)	2.2144(72)	2.2243(67)	2.2853(66)	2.4643(84)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	2.4455(80)	2.3428(71)	2.2400(74)	2.2115(59)	2.2163(66)	2.2589(62)	2.4547(74)

Table B.32: Anderson-Darling critical values, Case II: $n = 100$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	2.4107(74)	2.3506(74)	2.2616(79)	2.2246(73)	2.2085(71)	2.2462(73)	2.4285(68)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	2.4541(62)	2.3427(69)	2.2449(74)	2.2160(78)	2.2161(64)	2.2422(76)	2.4321(77)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	2.4840(74)	2.3314(66)	2.2336(71)	2.2217(61)	2.2533(66)	2.3311(67)	2.4638(68)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	2.3673(61)	2.3317(73)	2.2817(72)	2.2454(68)	2.2352(63)	2.2893(72)	2.4288(77)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	2.4772(72)	2.3397(72)	2.2346(63)	2.2201(66)	2.2256(65)	2.2731(73)	2.4436(78)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	2.4707(69)	2.3392(69)	2.2422(70)	2.2142(72)	2.2149(57)	2.2433(81)	2.4350(72)

Table B.33: Anderson-Darling critical values, Case II: $n = 200$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	2.4452(72)	2.3490(73)	2.2678(71)	2.2305(61)	2.2150(67)	2.2185(65)	2.4083(71)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	2.4713(72)	2.3349(67)	2.2495(74)	2.2193(71)	2.2184(72)	2.2362(67)	2.4151(66)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	2.4906(81)	2.3424(79)	2.2416(75)	2.2261(65)	2.2524(71)	2.3222(70)	2.4550(77)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	2.4111(69)	2.3412(73)	2.2814(63)	2.2477(70)	2.2298(67)	2.2481(79)	2.4159(73)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	2.4867(73)	2.3326(77)	2.2427(77)	2.2163(67)	2.2304(69)	2.2771(79)	2.4296(71)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	2.4774(67)	2.3371(68)	2.2447(78)	2.2239(68)	2.2160(73)	2.2421(67)	2.4253(66)

Table B.34: Anderson-Darling critical values, Case II: $n = 500$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	2.4543(63)	2.3443(63)	2.2582(66)	2.2249(73)	2.2116(65)	2.2206(72)	2.3978(67)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	2.4784(70)	2.3450(67)	2.2499(69)	2.2228(70)	2.2172(70)	2.2430(69)	2.4084(65)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	2.4923(73)	2.3420(72)	2.2411(66)	2.2193(69)	2.2588(74)	2.3276(67)	2.4470(71)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	2.4225(76)	2.3405(81)	2.2780(77)	2.2541(75)	2.2295(61)	2.2333(71)	2.3996(70)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	2.4837(66)	2.3350(73)	2.2413(61)	2.2201(73)	2.2251(67)	2.2802(66)	2.4297(74)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	2.4804(67)	2.3396(69)	2.2472(69)	2.2146(69)	2.2250(69)	2.2403(69)	2.4079(65)

Table B.35: Anderson-Darling critical values, Case II: $n = 1000$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	0.4468(14)	0.4417(16)	0.4309(14)	0.4196(14)	0.4151(14)	0.4284(13)	0.4529(13)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	0.4534(14)	0.4444(15)	0.4266(12)	0.4159(13)	0.4098(14)	0.4198(15)	0.4531(18)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	0.4589(15)	0.4432(14)	0.4219(14)	0.4103(12)	0.4118(12)	0.4237(15)	0.4546(14)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	0.4379(13)	0.4346(13)	0.4309(14)	0.4256(14)	0.4275(15)	0.4396(14)	0.4562(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	0.4558(13)	0.4442(14)	0.4228(14)	0.4128(14)	0.4091(14)	0.4172(14)	0.4511(14)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	0.4531(14)	0.4439(13)	0.4265(16)	0.4140(12)	0.4100(12)	0.4202(11)	0.4510(15)

Table B.36: Cramér-von Mises critical values, Case II: $n = 50$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	0.4507(13)	0.4433(16)	0.4293(14)	0.4210(14)	0.4151(14)	0.4231(14)	0.4503(14)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	0.4568(15)	0.4454(13)	0.4266(13)	0.4170(13)	0.4110(14)	0.4148(15)	0.4481(15)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	0.4623(14)	0.4455(15)	0.4231(12)	0.4108(14)	0.4107(13)	0.4244(15)	0.4521(15)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	0.4435(14)	0.4394(15)	0.4334(14)	0.4249(12)	0.4246(14)	0.4342(15)	0.4505(12)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	0.4576(13)	0.4451(13)	0.4248(14)	0.4135(14)	0.4086(12)	0.4165(13)	0.4502(15)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	0.4573(17)	0.4451(15)	0.4268(14)	0.4154(13)	0.4105(13)	0.4138(11)	0.4493(14)

Table B.37: Cramér-von Mises critical values, Case II: $n = 100$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	0.4530(16)	0.4462(15)	0.4298(15)	0.4205(15)	0.4142(13)	0.4174(14)	0.4461(14)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	0.4579(13)	0.4459(14)	0.4274(14)	0.4173(16)	0.4114(14)	0.4120(14)	0.4456(16)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	0.4603(15)	0.4452(13)	0.4226(14)	0.4118(13)	0.4122(14)	0.4248(13)	0.4526(13)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	0.4473(13)	0.4417(15)	0.4331(14)	0.4258(14)	0.4219(13)	0.4272(15)	0.4479(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	0.4593(14)	0.4464(15)	0.4240(14)	0.4142(14)	0.4095(13)	0.4152(13)	0.4475(16)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	0.4599(15)	0.4454(14)	0.4261(13)	0.4158(14)	0.4103(11)	0.4115(17)	0.4459(15)

Table B.38: Cramér-von Mises critical values, Case II: $n = 200$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	0.4568(14)	0.4459(14)	0.4311(14)	0.4216(12)	0.4153(14)	0.4125(13)	0.4438(15)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	0.4596(14)	0.4449(13)	0.4272(15)	0.4172(14)	0.4119(14)	0.4115(13)	0.4437(13)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	0.4608(17)	0.4471(16)	0.4246(14)	0.4131(14)	0.4122(14)	0.4235(14)	0.4507(16)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	0.4532(14)	0.4433(15)	0.4333(13)	0.4260(15)	0.4207(13)	0.4214(15)	0.4463(15)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	0.4607(15)	0.4453(16)	0.4254(15)	0.4138(14)	0.4105(14)	0.4159(16)	0.4457(16)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	0.4596(13)	0.4453(14)	0.4271(15)	0.4176(14)	0.4104(14)	0.4116(14)	0.4457(13)

Table B.39: Cramér-von Mises critical values, Case II: $n = 500$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

	$\pi_1 = 0$	$\pi_1 = 0.1$	$\pi_1 = 0.3$	$\pi_1 = 0.5$	$\pi_1 = 0.7$	$\pi_1 = 0.9$	$\pi_1 = 1$
$\beta_1 = 0.5,\ \beta_2 = 1.0$	0.4575(13)	0.4442(14)	0.4294(14)	0.4207(15)	0.4144(13)	0.4131(15)	0.4420(14)
$\beta_1 = 0.5,\ \beta_2 = 1.5$	0.4597(14)	0.4463(14)	0.4273(13)	0.4183(13)	0.4114(14)	0.4121(13)	0.4424(13)
$\beta_1 = 0.5,\ \beta_2 = 5.0$	0.4618(15)	0.4469(14)	0.4241(13)	0.4113(13)	0.4131(15)	0.4239(14)	0.4498(14)
$\beta_1 = 1.0,\ \beta_2 = 1.5$	0.4540(15)	0.4436(16)	0.4326(16)	0.4270(14)	0.4204(13)	0.4191(14)	0.4434(13)
$\beta_1 = 1.0,\ \beta_2 = 5.0$	0.4601(14)	0.4449(15)	0.4248(13)	0.4142(14)	0.4098(13)	0.4163(12)	0.4453(15)
$\beta_1 = 1.5,\ \beta_2 = 5.0$	0.4601(15)	0.4453(14)	0.4273(14)	0.4157(14)	0.4123(14)	0.4116(13)	0.4424(13)

Table B.40: Cramér-von Mises critical values, Case II: $n = 1000$, $\alpha_1 = 1$, $\beta_1 = [0.5, 0.5, 0.5, 1, 1, 1.5]$, $\alpha_2 = 1, \beta_2 = [1, 1.5, 5, 1.5, 5, 5]$.

Appendix C

Limit-Order Arrival Time Parameter Estimates

Chapter 13 presented a representative sample of mixed distribution parameter estimates and model selection results for limit order inter-arrival time data. The full set are provided in this appendix.

C.1 Removal of "zero inflated" Data

- Exponential/Weibull mixtures: Tables [C.1 - C.6]
- Gamma/Weibull mixtures: Tables [C.7 - C.12]
- Loglogistic/Weibull mixtures: Tables [C.13 - C.18]
- Weibull/Weibull mixtures: Tables [C.19 - C.24]
- Model Selection via AIC and BIC Results: Tables [C.25 - C.54]

C.2 Distributing "zero inflated" Data Uniformly

- Exponential/Uniform/Weibull mixtures: Tables [C.55 - C.60]
- Gamma/Uniform/Weibull mixtures: Tables [C.61 - C.66]
- Loglogistic/Uniform/Weibull mixtures: Tables [C.67 - C.72]
- Weibull/Uniform/Weibull mixtures: Tables [C.73 - C.78]
- Model Selection via AIC and BIC Results: Tables [C.79 - C.108]

C.3 Distributing "zero inflated" Data Exponentially

- Exponential/Exponential/Weibull mixtures: Tables [C.109 - C.114]
- Gamma/Exponential/Weibull mixtures: Tables [C.115 - C.120]
- Loglogistic/Exponential/Weibull mixtures: Tables [C.121 - C.126]
- Weibull/Exponential/Weibull mixtures: Tables [C.127 - C.132]
- Model Selection via AIC and BIC Results: Tables [C.133 - C.162]

C.4 g-component Exponential Mixture

- 2-component Exponential mixtures: Tables [C.163 - C.168]
- 4-component Exponential mixtures: Tables [C.169 - C.174]
- 6-component Exponential mixtures: Tables [C.175 - C.180]
- 10-component Exponential mixtures: Tables [C.181 - C.186]
- Model Selection via AIC and BIC Results: Tables [C.187 - C.216]

C.5 Censoring "zero inflated" Data

- 3-component Exponential mixtures with censoring [0, 0.5]: Tables [C.217 - C.222]
- 3-component Exponential mixtures with censoring [0, 0.5, 1.5, 2.5, 10]: Tables [C.223 - C.228]
- 4-component Exponential mixtures with censoring [0, 0.5]: Tables [C.229 - C.234]
- 4-component Exponential mixtures with censoring [0, 0.5, 1.5, 2.5, 10]: Tables [C.235 - C.240]
- Exponential/Exponential/Weibull mixtures with censoring [0, 0.5]: Tables [C.241 - C.246]
- Model Selection via AIC and BIC Results: Tables [C.247 - C.264]

n	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	- - - - -	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.177	0.187	0.104	0.030	0.344	- - - - -	2995.250	2408.577	2336.835	982.472	5030.795	0.837	0.443	0.754	0.568	1.068	0.277	0.146	0.263	0.723	0.146	0.737	-386.037	-430.303	-341.455	0.001
100	0.169	0.160	0.115	0.034	0.322	- - - - -	2818.944	2044.407	2263.647	1078.428	4603.310	0.754	0.276	0.699	0.555	0.939	0.264	0.126	0.252	0.736	0.126	0.748	-777.693	-850.950	-706.487	0.000
200	0.161	0.136	0.123	0.036	0.296	- - - - -	2686.100	1769.016	2235.697	1151.199	4233.459	0.704	0.179	0.671	0.548	0.855	0.254	0.112	0.241	0.746	0.112	0.759	-1562.652	-1684.059	-1449.254	0.000
500	0.150	0.109	0.129	0.041	0.265	- - - - -	2571.316	1474.622	2204.063	1209.829	4000.539	0.668	0.129	0.649	0.550	0.785	0.248	0.100	0.235	0.752	0.100	0.765	-3920.479	-4168.391	-3687.222	0.001
1000	0.144	0.097	0.134	0.044	0.248	- - - - -	2508.572	1329.117	2176.096	1269.657	3810.964	0.650	0.106	0.637	0.552	0.740	0.244	0.091	0.234	0.756	0.091	0.766	-7851.259	-8311.136	-7431.180	0.001

Table C.1: exponential-weibull mixture on ABFLN limit order arrival data: removal of zeros.

n	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	- - - - -	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.259	0.164	0.223	0.108	0.413	- - - - -	336.670	293.082	254.628	121.761	538.605	0.832	0.858	0.728	0.603	0.966	0.410	0.184	0.394	0.590	0.184	0.606	-270.857	-317.996	-223.867	0.001
100	0.250	0.131	0.227	0.129	0.369	- - - - -	300.646	213.779	243.715	132.092	462.866	0.703	0.245	0.673	0.587	0.799	0.385	0.154	0.373	0.615	0.154	0.627	-546.249	-625.776	-467.494	0.000
200	0.244	0.104	0.228	0.148	0.338	- - - - -	280.715	167.575	238.790	140.189	418.536	0.655	0.098	0.645	0.579	0.727	0.370	0.132	0.359	0.630	0.132	0.641	-1098.073	-1235.960	-960.475	0.000
500	0.239	0.080	0.229	0.165	0.312	- - - - -	268.016	135.916	235.944	149.190	386.206	0.629	0.057	0.627	0.575	0.683	0.360	0.113	0.351	0.640	0.113	0.649	-2755.391	-3050.904	-2460.992	0.000
1000	0.236	0.068	0.229	0.173	0.299	- - - - -	263.249	122.237	235.324	153.957	371.598	0.619	0.047	0.617	0.574	0.664	0.356	0.102	0.347	0.644	0.102	0.653	-5519.866	-6060.022	-4983.277	0.000

Table C.2: exponential-weibull mixture on BARC limit order arrival data: removal of zeros.

n	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	- - - - -	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.181	0.172	0.127	0.035	0.334	- - - - -	1986.091	1648.697	1511.851	692.104	3296.782	0.784	0.948	0.697	0.547	0.944	0.290	0.162	0.264	0.710	0.162	0.736	-365.748	-410.954	-323.878	0.002
100	0.170	0.143	0.132	0.040	0.302	- - - - -	1809.212	1283.898	1463.490	768.323	2866.403	0.684	0.303	0.646	0.533	0.814	0.274	0.140	0.252	0.726	0.140	0.748	-736.708	-809.889	-671.430	0.001
200	0.162	0.119	0.139	0.046	0.275	- - - - -	1685.451	1031.766	1410.974	832.081	2574.555	0.642	0.278	0.616	0.526	0.743	0.261	0.125	0.242	0.739	0.125	0.758	-1479.961	-1599.433	-1376.533	0.001
500	0.152	0.093	0.140	0.053	0.246	- - - - -	1579.313	807.923	1375.169	899.611	2288.123	0.604	0.087	0.596	0.522	0.684	0.251	0.113	0.237	0.749	0.113	0.763	-3710.397	-3956.044	-3499.520	0.001
1000	0.148	0.081	0.143	0.059	0.235	- - - - -	1522.586	689.578	1356.898	932.828	2146.806	0.588	0.070	0.583	0.521	0.653	0.246	0.107	0.231	0.754	0.107	0.769	-7432.457	-7852.754	-7072.367	0.001

Table C.3: exponential-weibull mixture on RRLN limit order arrival data: removal of zeros.

n	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	- - - - -	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.260	0.192	0.213	0.083	0.445	- - - - -	735.042	645.163	549.451	261.152	1173.018	0.736	0.387	0.683	0.562	0.877	0.303	0.150	0.287	0.697	0.150	0.713	-320.159	-362.792	-278.481	0.001
100	0.252	0.157	0.220	0.107	0.396	- - - - -	659.446	486.642	527.053	279.066	1019.641	0.662	0.278	0.638	0.551	0.755	0.280	0.122	0.267	0.720	0.122	0.733	-644.649	-715.669	-576.096	0.000
200	0.246	0.127	0.224	0.131	0.360	- - - - -	613.121	384.548	512.899	291.202	927.376	0.628	0.476	0.614	0.545	0.697	0.264	0.102	0.255	0.736	0.102	0.745	-1294.498	-1415.001	-1179.332	0.000
500	0.240	0.095	0.226	0.155	0.324	- - - - -	580.498	311.794	508.616	307.190	852.874	0.599	0.060	0.597	0.541	0.658	0.253	0.085	0.245	0.747	0.085	0.755	-3246.626	-3491.630	-3010.746	0.000
1000	0.238	0.078	0.227	0.168	0.306	- - - - -	565.639	274.052	505.592	310.037	823.558	0.590	0.052	0.590	0.538	0.640	0.248	0.074	0.240	0.752	0.074	0.760	-6503.238	-6934.267	-6088.929	0.000

Table C.4: exponential-weibull mixture on VOD limit order arrival data: removal of zeros.

n	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.246	0.172	0.202	0.092	0.409	-	-	-	-	-	308.567	302.677	212.049	95.933	512.668	0.850	0.928	0.723	0.603	0.958	0.390	0.183	0.371	0.610	0.183	0.629	-271.566	-316.272	-228.435	0.002
100	0.236	0.138	0.206	0.112	0.360	-	-	-	-	-	272.040	229.311	200.293	98.544	442.682	0.707	0.340	0.672	0.589	0.790	0.359	0.148	0.345	0.641	0.148	0.655	-547.402	-625.001	-473.441	0.000
200	0.229	0.112	0.208	0.130	0.323	-	-	-	-	-	252.855	188.923	196.171	100.042	409.182	0.658	0.104	0.647	0.582	0.726	0.340	0.122	0.330	0.660	0.122	0.670	-1099.869	-1238.568	-968.876	0.000
500	0.222	0.085	0.209	0.147	0.291	-	-	-	-	-	240.651	159.848	195.308	101.024	384.761	0.634	0.060	0.630	0.578	0.687	0.327	0.100	0.320	0.673	0.100	0.680	-2759.156	-3069.621	-2466.625	0.000
1000	0.219	0.071	0.209	0.157	0.276	-	-	-	-	-	235.971	147.456	196.689	102.248	376.528	0.625	0.051	0.623	0.576	0.672	0.322	0.088	0.315	0.678	0.088	0.685	-5526.681	-6111.485	-4978.064	0.000

Table C.5: exponential-weibull mixture on RIOTINTO limit order arrival data: removal of zeros.

n	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.179	0.186	0.112	0.027	0.349	-	-	-	-	-	2520.890	1907.696	2034.634	874.153	4180.733	0.840	0.656	0.732	0.560	1.066	0.257	0.146	0.239	0.743	0.146	0.761	-385.839	-427.446	-343.768	0.001
100	0.169	0.156	0.122	0.031	0.320	-	-	-	-	-	2346.282	1554.859	1965.611	974.846	3755.318	0.739	0.300	0.679	0.546	0.922	0.241	0.126	0.224	0.759	0.126	0.776	-777.377	-845.771	-709.624	0.000
200	0.160	0.130	0.129	0.036	0.292	-	-	-	-	-	2202.217	1272.244	1905.552	1063.069	3392.634	0.684	0.168	0.646	0.541	0.829	0.229	0.111	0.212	0.771	0.111	0.788	-1562.348	-1673.254	-1452.136	0.000
500	0.150	0.105	0.135	0.041	0.256	-	-	-	-	-	2086.366	1030.383	1858.960	1139.809	3051.445	0.646	0.116	0.623	0.542	0.753	0.218	0.096	0.203	0.782	0.096	0.797	-3920.927	-4140.310	-3704.877	0.000
1000	0.143	0.093	0.132	0.045	0.239	-	-	-	-	-	2032.187	892.406	1835.339	1187.570	2920.400	0.629	0.092	0.617	0.540	0.716	0.213	0.088	0.200	0.787	0.088	0.800	-7857.638	-8247.671	-7504.331	0.000

Table C.6: exponential-weibull mixture on SSELN limit order arrival data: removal of zeros.

n	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.253	3.717	2.796	2.117	9.098	0.938	0.168	0.957	0.776	1.139	1856.272	1602.879	1456.156	711.543	3645.585	0.594	0.147	0.573	0.474	0.770	0.149	0.105	0.123	0.851	0.105	0.877	-384.651	-426.981	-319.556	0.074
100	3.723	3.059	2.641	2.076	5.437	0.965	0.142	0.987	0.828	1.103	1798.784	1340.284	1469.072	807.887	2768.737	0.567	0.106	0.555	0.468	0.670	0.134	0.089	0.113	0.866	0.089	0.887	-779.933	-854.241	-702.344	0.013
200	3.389	2.865	2.554	2.084	3.894	0.989	0.116	1.003	0.879	1.098	1731.693	1105.889	1466.802	894.749	2527.986	0.550	0.085	0.542	0.466	0.632	0.128	0.075	0.113	0.872	0.075	0.887	-1567.927	-1694.289	-1449.218	0.002
500	2.950	2.204	2.502	2.129	3.132	1.005	0.087	1.014	0.917	1.091	1671.832	886.431	1478.917	957.025	2315.197	0.538	0.069	0.533	0.465	0.607	0.126	0.060	0.116	0.874	0.060	0.884	-3930.492	-4181.021	-3691.385	0.001
1000	2.779	1.842	2.497	2.179	2.968	1.013	0.072	1.020	0.945	1.084	1644.226	758.827	1498.270	1006.197	2220.159	0.532	0.062	0.527	0.469	0.596	0.126	0.052	0.116	0.874	0.052	0.884	-7867.266	-8332.875	-7434.983	0.001

Table C.7: gamma-weibull mixture on ABFLN limit order arrival data: removal of zeros.

n	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.037	1.894	2.404	1.984	3.669	0.931	0.148	0.952	0.784	1.080	235.748	208.967	175.358	81.343	395.595	0.659	0.257	0.634	0.543	0.765	0.281	0.160	0.251	0.719	0.160	0.749	-269.425	-316.921	-219.263	0.014
100	2.749	1.421	2.357	1.979	3.062	0.949	0.127	0.968	0.825	1.071	222.063	163.933	176.643	92.657	346.442	0.613	0.096	0.605	0.536	0.687	0.270	0.139	0.247	0.730	0.139	0.753	-544.480	-624.958	-464.293	0.001
200	2.558	1.016	2.337	1.991	2.843	0.961	0.107	0.976	0.856	1.063	214.267	135.153	179.638	102.190	323.001	0.593	0.062	0.590	0.534	0.650	0.265	0.123	0.247	0.735	0.123	0.753	-1094.691	-1234.630	-955.714	0.000
500	2.411	0.607	2.327	2.011	2.695	0.969	0.087	0.981	0.882	1.052	208.922	113.245	182.196	110.964	304.165	0.579	0.046	0.578	0.534	0.623	0.262	0.108	0.249	0.738	0.108	0.751	-2746.742	-3045.769	-2449.117	0.000
1000	2.365	0.438	2.331	2.016	2.646	0.971	0.078	0.983	0.891	1.045	206.493	102.478	183.359	116.343	294.513	0.573	0.040	0.573	0.534	0.612	0.261	0.101	0.249	0.739	0.101	0.751	-5502.352	-6050.760	-4962.160	0.000

Table C.8: gamma-weibull mixture on BARC limit order arrival data: removal of zeros.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.247	3.574	2.839	2.125	7.960	0.942	0.158	0.962	0.790	1.114	1225.479	987.191	975.069	480.886	2188.196	0.593	0.270	0.561	0.463	0.732	0.160	0.120	0.131	0.840	0.120	0.869	-364.305	-408.796	-312.008	0.046
100	3.709	2.888	2.699	2.108	5.186	0.971	0.131	0.991	0.846	1.095	1159.558	764.589	987.676	561.086	1727.365	0.556	0.151	0.537	0.457	0.645	0.146	0.102	0.121	0.854	0.102	0.879	-737.536	-811.965	-669.858	0.006
200	3.318	2.322	2.622	2.137	3.895	0.994	0.105	1.009	0.892	1.094	1110.310	597.855	992.377	628.912	1567.923	0.537	0.112	0.522	0.457	0.610	0.140	0.088	0.120	0.860	0.088	0.880	-1481.908	-1604.676	-1376.255	0.001
500	2.939	1.734	2.585	2.199	3.204	1.012	0.080	1.020	0.936	1.089	1077.574	473.051	992.409	699.848	1466.872	0.523	0.074	0.514	0.458	0.588	0.137	0.073	0.120	0.863	0.073	0.880	-3715.314	-3964.120	-3498.645	0.001
1000	2.731	0.847	2.565	2.252	3.020	1.019	0.065	1.022	0.956	1.085	1064.746	408.115	998.361	744.674	1402.598	0.517	0.064	0.511	0.458	0.575	0.136	0.063	0.123	0.864	0.063	0.877	-7440.074	-7861.989	-7062.765	0.001

Table C.9: gamma-weibull mixture on RRLN limit order arrival data: removal of zeros.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.198	2.237	2.393	1.977	4.847	0.916	0.162	0.937	0.751	1.092	531.373	432.830	409.905	197.307	945.373	0.616	0.167	0.599	0.508	0.737	0.200	0.118	0.178	0.800	0.118	0.822	-318.884	-360.954	-270.512	0.036
100	2.816	1.668	2.308	1.953	3.146	0.932	0.143	0.952	0.789	1.072	501.449	344.680	413.379	216.617	791.415	0.584	0.096	0.576	0.505	0.660	0.188	0.098	0.172	0.812	0.098	0.828	-643.621	-715.205	-572.873	0.006
200	2.561	1.203	2.273	1.946	2.823	0.946	0.120	0.962	0.824	1.064	483.388	285.686	415.743	235.085	732.259	0.568	0.083	0.564	0.505	0.626	0.183	0.083	0.172	0.817	0.083	0.828	-1292.629	-1414.311	-1176.254	0.000
500	2.375	0.883	2.257	1.960	2.640	0.954	0.095	0.966	0.856	1.049	471.084	241.546	422.106	250.406	695.900	0.555	0.050	0.555	0.505	0.603	0.180	0.069	0.172	0.820	0.069	0.828	-3241.483	-3486.326	-3003.474	0.000
1000	2.302	0.463	2.252	1.969	2.567	0.956	0.083	0.966	0.868	1.038	465.062	218.562	424.679	257.794	670.640	0.550	0.044	0.551	0.504	0.593	0.179	0.061	0.171	0.821	0.061	0.829	-6492.323	-6927.016	-6072.989	0.000

Table C.10: gamma-weibull mixture on VOD limit order arrival data: removal of zeros.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.150	2.053	2.404	1.974	4.231	0.922	0.152	0.946	0.767	1.075	212.394	216.185	142.168	64.209	369.146	0.663	0.284	0.639	0.546	0.768	0.241	0.133	0.217	0.759	0.133	0.783	-270.579	-315.210	-225.226	0.015
100	2.844	1.582	2.351	1.964	3.199	0.942	0.131	0.964	0.807	1.070	199.794	175.852	143.471	70.074	327.158	0.621	0.108	0.612	0.540	0.697	0.227	0.112	0.210	0.773	0.112	0.790	-546.320	-624.467	-471.814	0.001
200	2.628	1.180	2.332	1.972	2.908	0.957	0.112	0.974	0.839	1.067	192.955	150.359	145.838	74.447	314.830	0.602	0.073	0.599	0.537	0.665	0.222	0.095	0.209	0.778	0.095	0.791	-1097.842	-1236.863	-966.214	0.000
500	2.457	0.750	2.328	1.990	2.741	0.966	0.093	0.980	0.864	1.058	188.286	130.978	148.221	78.180	303.288	0.590	0.055	0.588	0.537	0.643	0.220	0.079	0.210	0.780	0.079	0.790	-2753.887	-3064.895	-2460.531	0.000
1000	2.391	0.526	2.328	2.006	2.668	0.969	0.084	0.984	0.874	1.051	186.146	121.718	150.859	80.157	298.202	0.585	0.049	0.582	0.536	0.634	0.218	0.070	0.210	0.782	0.070	0.790	-5515.901	-6102.478	-4968.428	0.000

Table C.11: gamma-weibull mixture on RIOTINTO limit order arrival data: removal of zeros.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.221	3.678	2.809	2.120	9.216	0.929	0.167	0.949	0.768	1.137	1604.399	1266.901	1296.740	641.226	3242.693	0.600	0.164	0.577	0.485	0.782	0.133	0.092	0.112	0.867	0.092	0.888	-383.855	-423.477	-321.183	0.083
100	3.662	2.940	2.638	2.083	5.377	0.957	0.138	0.973	0.821	1.094	1548.177	1011.753	1315.684	737.018	2370.645	0.571	0.109	0.556	0.478	0.669	0.117	0.076	0.101	0.883	0.076	0.899	-778.736	-847.106	-705.461	0.015
200	3.230	2.334	2.572	2.107	3.733	0.982	0.110	0.994	0.872	1.089	1495.949	834.802	1307.050	830.386	2116.104	0.554	0.083	0.544	0.478	0.627	0.111	0.062	0.100	0.889	0.062	0.900	-1565.675	-1677.823	-1451.863	0.002
500	2.887	1.528	2.529	2.152	3.182	1.000	0.083	1.005	0.917	1.083	1454.318	662.495	1308.523	921.570	1953.647	0.541	0.065	0.535	0.480	0.604	0.108	0.047	0.101	0.892	0.047	0.899	-3926.990	-4149.210	-3704.419	0.000
1000	2.747	1.362	2.547	2.201	3.016	1.008	0.070	1.012	0.937	1.078	1437.363	570.942	1313.409	976.669	1889.762	0.536	0.056	0.529	0.481	0.589	0.107	0.038	0.102	0.893	0.038	0.898	-7866.433	-8259.509	-7509.747	0.000

Table C.12: gamma-weibull mixture on SSELN limit order arrival data: removal of zeros.

n	$\hat\alpha_{\log\ ave}$	$\pm\hat\alpha_{\log\ ave}$	$\hat\alpha_{\log\ median}$	$\hat\alpha_{\log\ lower}$	$\hat\alpha_{\log\ upper}$	$\hat\beta_{\log\ ave}$	$\pm\hat\beta_{\log\ ave}$	$\hat\beta_{\log\ median}$	$\hat\beta_{\log\ lower}$	$\hat\beta_{\log\ upper}$	$\hat\alpha_{ave}$	$\pm\hat\alpha_{ave}$	$\hat\alpha_{median}$	$\hat\alpha_{lower}$	$\hat\alpha_{upper}$	$\hat\beta_{ave}$	$\pm\hat\beta_{ave}$	$\hat\beta_{median}$	$\hat\beta_{lower}$	$\hat\beta_{upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	14.543	16.846	7.863	1.810	28.860	9.308	22.439	1.425	0.955	3.992	3349.586	2784.789	2610.683	1013.739	5769.188	0.964	1.082	0.807	0.581	1.237	0.317	0.175	0.301	0.683	0.175	0.699	-381.398	-427.550	-333.945	0.007
100	13.188	14.803	7.693	1.823	25.650	8.483	21.231	1.274	0.934	3.124	3207.716	2379.565	2617.631	1094.263	5401.202	0.854	0.705	0.762	0.566	1.088	0.309	0.158	0.300	0.691	0.158	0.700	-770.707	-847.263	-693.717	0.007
200	11.984	12.690	7.413	1.912	23.021	7.772	20.134	1.185	0.911	2.612	3111.722	2083.617	2609.951	1186.092	5143.200	0.795	0.385	0.738	0.562	1.004	0.304	0.147	0.296	0.696	0.147	0.704	-1551.522	-1677.702	-1427.139	0.006
500	10.628	10.506	7.462	2.093	19.766	6.972	18.852	1.111	0.914	2.167	3029.408	1787.198	2681.664	1243.859	4928.381	0.767	0.894	0.711	0.566	0.924	0.300	0.137	0.301	0.700	0.137	0.699	-3899.407	-4153.441	-3657.127	0.004
1000	10.076	9.641	7.315	2.254	18.166	5.547	16.405	1.087	0.909	1.997	2998.029	1642.339	2674.579	1331.325	4710.112	0.733	0.229	0.706	0.571	0.875	0.301	0.129	0.301	0.699	0.129	0.699	-7816.308	-8278.674	-7391.552	0.000

Table C.13: loglogistic-weibull mixture on ABFLN limit order arrival data: removal of zeros.

n	$\hat\alpha_{\log\ ave}$	$\pm\hat\alpha_{\log\ ave}$	$\hat\alpha_{\log\ median}$	$\hat\alpha_{\log\ lower}$	$\hat\alpha_{\log\ upper}$	$\hat\beta_{\log\ ave}$	$\pm\hat\beta_{\log\ ave}$	$\hat\beta_{\log\ median}$	$\hat\beta_{\log\ lower}$	$\hat\beta_{\log\ upper}$	$\hat\alpha_{ave}$	$\pm\hat\alpha_{ave}$	$\hat\alpha_{median}$	$\hat\alpha_{lower}$	$\hat\alpha_{upper}$	$\hat\beta_{ave}$	$\pm\hat\beta_{ave}$	$\hat\beta_{median}$	$\hat\beta_{lower}$	$\hat\beta_{upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.354	5.468	2.686	1.001	6.846	17.129	33.285	2.106	1.312	82.163	383.288	404.188	273.724	130.131	622.169	0.908	1.294	0.715	0.586	1.020	0.402	0.214	0.378	0.598	0.214	0.622	-268.743	-316.903	-219.395	0.008
100	3.622	4.173	2.483	1.000	5.365	21.119	36.434	2.070	1.342	86.576	324.877	284.561	250.490	127.770	509.938	0.728	0.591	0.661	0.567	0.814	0.374	0.188	0.356	0.626	0.188	0.644	-536.213	-620.973	-451.850	0.001
200	3.048	2.968	2.345	1.000	4.444	24.248	38.112	2.071	1.406	87.298	284.134	196.962	233.899	124.205	440.088	0.652	0.211	0.631	0.558	0.732	0.349	0.163	0.337	0.651	0.163	0.663	-1068.603	-1222.809	-916.096	0.000
500	2.658	2.080	2.274	1.000	3.778	25.442	38.123	2.055	1.496	85.706	258.532	151.111	224.311	122.874	392.417	0.617	0.070	0.609	0.554	0.681	0.329	0.141	0.328	0.671	0.141	0.672	-2669.069	-3012.470	-2329.633	0.000
1000	2.508	1.745	2.263	1.000	3.505	24.950	37.356	2.034	1.562	84.060	250.635	136.631	224.276	123.172	373.418	0.606	0.056	0.600	0.554	0.660	0.322	0.130	0.325	0.678	0.130	0.675	-5347.303	-5985.339	-4710.460	0.000

Table C.14: loglogistic-weibull mixture on BARC limit order arrival data: removal of zeros.

n	$\hat\alpha_{\log\ ave}$	$\pm\hat\alpha_{\log\ ave}$	$\hat\alpha_{\log\ median}$	$\hat\alpha_{\log\ lower}$	$\hat\alpha_{\log\ upper}$	$\hat\beta_{\log\ ave}$	$\pm\hat\beta_{\log\ ave}$	$\hat\beta_{\log\ median}$	$\hat\beta_{\log\ lower}$	$\hat\beta_{\log\ upper}$	$\hat\alpha_{ave}$	$\pm\hat\alpha_{ave}$	$\hat\alpha_{median}$	$\hat\alpha_{lower}$	$\hat\alpha_{upper}$	$\hat\beta_{ave}$	$\pm\hat\beta_{ave}$	$\hat\beta_{median}$	$\hat\beta_{lower}$	$\hat\beta_{upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	12.099	14.563	5.877	1.799	24.296	9.623	22.964	1.549	1.001	4.057	2308.229	2197.227	1680.718	717.993	3887.837	0.858	0.843	0.735	0.560	1.069	0.327	0.191	0.297	0.673	0.191	0.703	-362.010	-408.655	-319.326	0.003
100	10.911	12.507	5.733	1.870	21.308	8.401	21.256	1.387	0.972	2.985	2163.350	1772.467	1670.211	788.572	3571.566	0.751	0.383	0.691	0.549	0.920	0.319	0.172	0.297	0.681	0.172	0.703	-730.815	-807.329	-662.828	0.001
200	9.621	10.312	5.675	1.991	18.303	7.758	20.202	1.299	0.962	2.478	2032.705	1420.004	1635.888	852.141	3282.834	0.708	0.384	0.661	0.545	0.839	0.309	0.157	0.295	0.691	0.157	0.705	-1469.999	-1595.136	-1361.010	0.000
500	8.305	7.449	5.765	2.256	15.185	6.545	18.185	1.238	0.966	2.116	1937.776	1158.529	1626.972	927.349	3009.785	0.663	0.155	0.641	0.547	0.763	0.303	0.143	0.293	0.697	0.143	0.707	-3691.878	-3944.236	-3470.095	0.000
1000	7.687	6.268	5.750	2.360	13.641	6.091	17.347	1.215	0.971	1.971	1879.050	1020.339	1647.091	957.958	2820.011	0.645	0.141	0.627	0.548	0.729	0.299	0.136	0.294	0.701	0.136	0.706	-7399.614	-7829.518	-7022.247	0.000

Table C.15: loglogistic-weibull mixture on RRLN limit order arrival data: removal of zeros.

n	$\hat\alpha_{\log\ ave}$	$\pm\hat\alpha_{\log\ ave}$	$\hat\alpha_{\log\ median}$	$\hat\alpha_{\log\ lower}$	$\hat\alpha_{\log\ upper}$	$\hat\beta_{\log\ ave}$	$\pm\hat\beta_{\log\ ave}$	$\hat\beta_{\log\ median}$	$\hat\beta_{\log\ lower}$	$\hat\beta_{\log\ upper}$	$\hat\alpha_{ave}$	$\pm\hat\alpha_{ave}$	$\hat\alpha_{median}$	$\hat\alpha_{lower}$	$\hat\alpha_{upper}$	$\hat\beta_{ave}$	$\pm\hat\beta_{ave}$	$\hat\beta_{median}$	$\hat\beta_{lower}$	$\hat\beta_{upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	5.265	8.132	2.501	1.000	8.348	24.498	37.010	2.335	1.245	83.699	784.933	823.738	546.242	251.116	1247.481	0.767	0.672	0.668	0.548	0.903	0.295	0.182	0.257	0.705	0.182	0.743	-313.623	-359.716	-268.850	0.002
100	4.372	6.415	2.328	1.000	6.630	26.284	37.455	2.176	1.239	82.666	695.843	620.137	515.855	256.153	1087.256	0.678	0.422	0.629	0.536	0.778	0.275	0.156	0.245	0.725	0.156	0.755	-629.586	-709.044	-552.530	0.001
200	3.552	4.654	2.194	1.000	5.345	28.480	37.621	2.152	1.283	81.535	632.270	513.614	494.189	261.231	971.669	0.627	0.177	0.603	0.529	0.711	0.254	0.134	0.232	0.746	0.134	0.768	-1262.464	-1401.648	-1127.970	0.000
500	2.844	3.005	2.072	1.000	4.225	30.491	37.246	2.167	1.365	79.481	577.572	388.605	467.209	266.864	883.499	0.595	0.184	0.581	0.522	0.660	0.233	0.117	0.219	0.767	0.117	0.781	-3166.074	-3457.772	-2884.517	0.000
1000	2.509	2.332	1.986	1.000	3.741	30.785	36.704	2.206	1.445	78.016	554.275	341.678	457.313	267.774	842.768	0.578	0.070	0.571	0.521	0.636	0.221	0.106	0.213	0.779	0.106	0.787	-6344.986	-6875.510	-5824.655	0.000

Table C.16: loglogistic-weibull mixture on VOD limit order arrival data: removal of zeros.

n	$\hat\alpha_{\log\ ave}$	$\pm\hat\alpha_{\log\ ave}$	$\hat\alpha_{\log\ median}$	$\hat\alpha_{\log\ lower}$	$\hat\alpha_{\log\ upper}$	$\hat\beta_{\log\ ave}$	$\pm\hat\beta_{\log\ ave}$	$\hat\beta_{\log\ median}$	$\hat\beta_{\log\ lower}$	$\hat\beta_{\log\ upper}$	$\hat\alpha_{ave}$	$\pm\hat\alpha_{ave}$	$\hat\alpha_{median}$	$\hat\alpha_{lower}$	$\hat\alpha_{upper}$	$\hat\beta_{ave}$	$\pm\hat\beta_{ave}$	$\hat\beta_{median}$	$\hat\beta_{lower}$	$\hat\beta_{upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	5.086	6.314	3.036	1.227	8.523	16.558	32.948	1.972	1.248	80.382	377.750	431.419	247.362	111.618	632.006	1.006	1.800	0.719	0.592	1.051	0.397	0.225	0.361	0.603	0.225	0.639	-271.287	-316.916	-225.527	0.010
100	4.275	4.798	2.820	1.000	6.862	21.032	36.765	1.905	1.244	87.862	320.403	314.624	228.233	109.471	520.034	0.788	1.010	0.670	0.577	0.826	0.370	0.203	0.339	0.630	0.203	0.661	-542.076	-624.512	-462.918	0.001
200	3.591	3.615	2.639	1.000	5.675	25.340	39.367	1.911	1.284	89.784	278.033	228.055	212.296	103.044	447.692	0.682	0.437	0.643	0.570	0.745	0.342	0.179	0.316	0.658	0.179	0.684	-1079.495	-1233.610	-931.282	0.000
500	3.012	2.484	2.485	1.000	4.700	29.301	40.932	1.936	1.366	90.135	246.473	175.790	200.604	93.763	399.648	0.635	0.096	0.624	0.568	0.699	0.314	0.154	0.294	0.686	0.154	0.706	-2681.991	-3049.966	-2323.703	0.000
1000	2.816	2.117	2.472	1.000	4.291	29.797	40.672	1.927	1.413	88.795	236.217	158.099	197.610	91.223	383.974	0.623	0.063	0.616	0.567	0.681	0.304	0.142	0.289	0.696	0.142	0.711	-5357.216	-6068.840	-4654.168	0.000

Table C.17: loglogistic-weibull mixture on RIOTINTO limit order arrival data: removal of zeros.

n	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	15.515	18.466	7.246	1.753	32.944	10.030	23.329	1.502	0.956	4.657	2828.325	2309.255	2255.782	906.766	4819.488	0.956	1.124	0.778	0.570	1.230	0.297	0.180	0.272	0.703	0.180	0.728	-381.971	-424.789	-336.795	0.009
100	14.640	17.048	7.217	1.781	30.653	8.602	21.416	1.294	0.914	3.358	2716.452	1900.857	2286.070	1017.602	4522.382	0.859	0.914	0.733	0.561	1.097	0.293	0.169	0.272	0.707	0.169	0.728	-771.644	-841.888	-698.380	0.009
200	13.627	15.356	7.507	1.804	27.159	8.219	20.766	1.181	0.890	2.851	2617.738	1588.847	2274.096	1094.809	4253.257	0.787	0.365	0.712	0.551	1.010	0.290	0.160	0.274	0.710	0.160	0.726	-1552.629	-1666.812	-1435.466	0.006
500	12.174	12.945	8.044	1.988	22.908	7.964	20.251	1.095	0.883	2.501	2527.081	1344.391	2274.972	1202.429	3944.350	0.738	0.218	0.693	0.552	0.913	0.285	0.150	0.279	0.715	0.150	0.721	-3899.969	-4128.300	-3669.859	0.008
1000	11.576	11.643	8.484	2.063	21.129	7.197	19.042	1.075	0.885	2.264	2502.119	1217.528	2297.112	1247.244	3822.950	0.717	0.178	0.680	0.553	0.873	0.284	0.144	0.286	0.716	0.144	0.714	-7822.834	-8215.230	-7455.299	0.002

Table C.18: loglogistic-weibull mixture on SSELN limit order arrival data: removal of zeros.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	20.341	45.934	9.763	3.186	37.627	1.649	2.535	1.142	0.805	2.359	3042.790	2494.755	2357.591	1016.327	5676.640	0.876	0.833	0.755	0.571	1.189	0.282	0.157	0.263	0.718	0.157	0.737	-385.004	-427.641	-331.645	0.042
100	18.844	33.035	10.246	3.126	34.166	1.278	1.012	1.014	0.768	1.848	2955.459	2161.430	2398.967	1100.654	5132.834	0.792	0.468	0.722	0.558	1.026	0.277	0.140	0.263	0.723	0.140	0.737	-776.888	-849.072	-696.323	0.028
200	17.389	29.530	11.442	3.190	31.392	1.142	0.847	0.930	0.741	1.592	2877.713	1892.878	2430.425	1169.869	4757.527	0.741	0.249	0.705	0.553	0.939	0.273	0.125	0.263	0.727	0.125	0.737	-1561.100	-1681.412	-1438.073	0.019
500	15.406	13.468	13.308	3.343	28.131	1.034	0.444	0.854	0.728	1.448	2816.131	1599.736	2507.399	1244.955	4511.877	0.706	0.149	0.690	0.559	0.864	0.273	0.113	0.272	0.727	0.113	0.728	-3916.245	-4160.858	-3669.581	0.017
1000	14.450	10.509	13.506	3.350	25.363	1.005	0.411	0.815	0.723	1.406	2769.957	1468.680	2565.970	1255.009	4293.436	0.689	0.125	0.686	0.558	0.817	0.271	0.108	0.276	0.729	0.108	0.724	-7839.843	-8297.809	-7404.775	0.009

Table C.19: weibull-weibull mixture on ABFLN limit order arrival data: removal of zeros.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	5.792	7.128	3.963	2.142	11.457	1.740	1.388	1.434	1.031	2.993	322.536	295.388	238.297	114.025	683.363	0.807	0.755	0.699	0.588	1.081	0.382	0.184	0.359	0.618	0.184	0.641	-269.206	-314.595	-201.647	0.078
100	5.258	5.989	3.836	2.128	9.535	1.572	0.808	1.363	1.020	2.671	292.267	230.365	231.293	122.796	565.474	0.692	0.308	0.650	0.569	0.846	0.361	0.158	0.343	0.639	0.158	0.657	-544.308	-621.208	-433.882	0.072
200	4.723	4.310	3.740	2.132	8.244	1.517	0.600	1.335	1.029	2.486	266.748	174.454	223.641	129.141	491.592	0.637	0.118	0.620	0.556	0.749	0.342	0.134	0.329	0.658	0.134	0.671	-1095.092	-1229.027	-901.183	0.074
500	4.252	3.055	3.631	2.154	7.300	1.499	0.568	1.332	1.046	2.446	246.227	131.576	215.455	135.651	435.880	0.604	0.065	0.597	0.546	0.692	0.325	0.109	0.318	0.675	0.109	0.682	-2748.939	-3035.417	-2314.729	0.081
1000	4.041	2.386	3.616	2.231	6.982	1.476	0.535	1.330	1.062	2.332	238.476	116.440	213.175	138.820	414.087	0.592	0.053	0.586	0.543	0.668	0.317	0.096	0.312	0.683	0.096	0.688	-5508.060	-6029.852	-4678.674	0.086

Table C.20: weibull-weibull mixture on BARC limit order arrival data: removal of zeros.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	15.340	27.474	7.596	3.034	31.007	1.706	2.237	1.226	0.854	2.475	1986.502	1716.673	1489.425	693.057	3660.772	0.776	0.538	0.692	0.542	1.006	0.287	0.168	0.253	0.713	0.168	0.747	-364.469	-408.745	-315.095	0.038
100	14.254	20.837	7.420	3.064	28.158	1.352	1.049	1.096	0.811	1.917	1894.885	1485.563	1486.224	763.493	3225.055	0.698	0.268	0.655	0.532	0.865	0.278	0.151	0.252	0.722	0.151	0.748	-735.734	-808.113	-662.683	0.024
200	12.976	14.605	7.739	3.126	25.341	1.201	0.560	1.018	0.783	1.693	1800.141	1178.920	1492.246	826.248	2943.076	0.658	0.195	0.633	0.525	0.792	0.272	0.136	0.253	0.728	0.136	0.747	-1478.182	-1597.052	-1365.379	0.018
500	12.043	10.223	8.367	3.272	22.578	1.115	0.455	0.948	0.763	1.522	1724.816	960.564	1476.964	875.082	2701.962	0.625	0.100	0.618	0.524	0.725	0.269	0.126	0.256	0.731	0.126	0.744	-3705.489	-3950.813	-3483.582	0.012
1000	11.671	9.116	8.838	3.305	21.072	1.082	0.431	0.905	0.757	1.472	1684.739	854.458	1497.432	900.569	2542.658	0.611	0.085	0.611	0.523	0.696	0.267	0.123	0.254	0.733	0.123	0.746	-7423.648	-7840.804	-7047.167	0.008

Table C.21: weibull-weibull mixture on RRLN limit order arrival data: removal of zeros.

n	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{2\text{ave}}$	$\pm\hat{\alpha}_{2\text{ave}}$	$\hat{\alpha}_{2\text{median}}$	$\hat{\alpha}_{2\text{lower}}$	$\hat{\alpha}_{2\text{upper}}$	$\hat{\beta}_{2\text{ave}}$	$\pm\hat{\beta}_{2\text{ave}}$	$\hat{\beta}_{2\text{median}}$	$\hat{\beta}_{2\text{lower}}$	$\hat{\beta}_{2\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	7.570	12.924	4.429	2.262	18.135	1.809	1.929	1.379	0.975	3.074	721.461	645.977	542.169	265.801	1597.719	0.739	0.558	0.668	0.556	1.020	0.291	0.156	0.266	0.709	0.156	0.734	-319.327	-359.629	-256.371	0.091
100	6.695	10.043	4.218	2.209	15.348	1.540	0.918	1.293	0.952	2.962	665.481	512.380	529.243	286.194	1370.689	0.663	0.254	0.629	0.541	0.852	0.273	0.132	0.253	0.727	0.132	0.747	-644.527	-711.773	-541.133	0.089
200	5.793	6.775	4.077	2.175	14.372	1.469	0.722	1.255	0.959	2.970	617.547	408.371	515.517	299.885	1274.666	0.623	0.495	0.603	0.531	0.774	0.257	0.111	0.240	0.743	0.111	0.760	-1295.182	-1409.283	-1108.235	0.101
500	4.846	4.205	3.893	2.136	12.500	1.467	0.628	1.251	0.989	3.005	574.220	324.838	501.285	315.972	1172.832	0.587	0.076	0.579	0.519	0.712	0.239	0.092	0.228	0.761	0.092	0.772	-3250.733	-3480.724	-2824.930	0.117
1000	4.387	3.190	3.805	2.123	12.325	1.472	0.616	1.251	1.019	3.029	556.976	281.190	494.959	326.105	1199.549	0.572	0.062	0.565	0.512	0.689	0.230	0.081	0.223	0.770	0.081	0.777	-6518.153	-6922.002	-5637.460	0.132

Table C.22: weibull-weibull mixture on VOD limit order arrival data: removal of zeros.

n	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{2\text{ave}}$	$\pm\hat{\alpha}_{2\text{ave}}$	$\hat{\alpha}_{2\text{median}}$	$\hat{\alpha}_{2\text{lower}}$	$\hat{\alpha}_{2\text{upper}}$	$\hat{\beta}_{2\text{ave}}$	$\pm\hat{\beta}_{2\text{ave}}$	$\hat{\beta}_{2\text{median}}$	$\hat{\beta}_{2\text{lower}}$	$\hat{\beta}_{2\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	6.802	8.733	4.462	2.277	15.585	1.724	1.510	1.378	1.006	3.020	310.128	325.510	209.167	96.313	731.915	0.881	1.214	0.707	0.596	1.224	0.371	0.192	0.338	0.629	0.192	0.662	-270.595	-312.954	-209.105	0.088
100	6.238	7.096	4.344	2.273	13.537	1.526	0.775	1.300	0.988	2.951	282.233	251.673	207.324	101.925	622.730	0.731	0.555	0.661	0.579	0.913	0.351	0.171	0.321	0.649	0.171	0.679	-547.449	-621.190	-443.092	0.088
200	5.668	5.563	4.263	2.322	12.531	1.463	0.621	1.262	0.994	2.960	258.766	193.628	205.234	105.757	561.530	0.662	0.202	0.634	0.567	0.798	0.332	0.149	0.308	0.668	0.149	0.692	-1102.575	-1233.255	-910.719	0.097
500	5.075	3.910	4.192	2.496	13.219	1.429	0.578	1.238	1.017	3.005	240.541	151.431	205.849	111.401	546.613	0.621	0.082	0.611	0.558	0.746	0.313	0.122	0.298	0.687	0.122	0.702	-2774.968	-3060.724	-2291.981	0.122
1000	4.838	3.298	4.166	2.670	16.222	1.405	0.553	1.235	1.032	3.057	234.891	135.289	208.955	114.813	605.430	0.607	0.062	0.600	0.554	0.748	0.305	0.107	0.293	0.695	0.107	0.707	-5570.652	-6098.386	-4468.056	0.145

Table C.23: weibull-weibull mixture on RIOTINTO limit order arrival data: removal of zeros.

n	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{2\text{ave}}$	$\pm\hat{\alpha}_{2\text{ave}}$	$\hat{\alpha}_{2\text{median}}$	$\hat{\alpha}_{2\text{lower}}$	$\hat{\alpha}_{2\text{upper}}$	$\hat{\beta}_{2\text{ave}}$	$\pm\hat{\beta}_{2\text{ave}}$	$\hat{\beta}_{2\text{median}}$	$\hat{\beta}_{2\text{lower}}$	$\hat{\beta}_{2\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	18.307	42.006	7.572	2.918	38.729	1.923	2.897	1.283	0.855	3.029	2432.467	1928.652	1932.807	881.275	4647.296	0.835	0.839	0.707	0.552	1.160	0.248	0.156	0.214	0.752	0.156	0.786	-384.058	-424.215	-330.190	0.056
100	18.401	39.109	7.276	2.800	35.693	1.455	1.455	1.126	0.796	2.156	2379.661	1630.765	1969.448	978.089	4091.873	0.770	0.658	0.674	0.540	0.998	0.243	0.145	0.212	0.757	0.145	0.788	-776.181	-843.546	-698.256	0.030
200	16.474	23.958	7.451	2.789	32.811	1.263	0.741	1.032	0.762	1.871	2291.922	1362.530	1982.226	1054.678	3736.425	0.707	0.237	0.655	0.533	0.906	0.237	0.131	0.210	0.763	0.131	0.790	-1560.898	-1670.795	-1440.125	0.019
500	14.970	18.094	8.400	2.872	29.541	1.163	0.550	0.928	0.735	1.695	2229.943	1145.346	1995.249	1141.441	3463.261	0.673	0.148	0.646	0.531	0.833	0.235	0.120	0.212	0.765	0.120	0.788	-3917.505	-4136.074	-3688.044	0.017
1000	14.664	12.805	10.244	2.968	28.594	1.115	0.528	0.865	0.723	1.604	2218.033	1020.984	2058.268	1197.138	3350.023	0.663	0.126	0.650	0.532	0.803	0.236	0.115	0.225	0.764	0.115	0.775	-7852.498	-8237.078	-7488.737	0.013

Table C.24: weibull-weibull mixture on SSELN limit order arrival data: removal of zeros.

	AIC			BIC		
exponential/weibull	0.4364	-	-	0.8291	-	-
gamma/weibull	0.0224	0.0351	0.2269	0.0015	0.0351	0.2269
loglogistic/weibull	0.4266	0.6243	-	0.1517	0.6243	-
weibull/weibull	0.1144	0.3404	0.7544	0.0175	0.3404	0.7544

Table C.25: ABFLN, N = 50: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.2250	0.4520	-	0.5995	-	-
gamma/weibull	0.0169	0.2839	0.3899	0.0055	0.0206	0.3899
loglogistic/weibull	0.6807	-	-	0.3753	0.8324	-
weibull/weibull	0.0773	0.2639	0.6053	0.0196	0.1469	0.6053

Table C.26: BARC, N = 50: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.4391	-	-	0.8279	-	-
gamma/weibull	0.0206	0.0333	0.2478	0.0027	0.0333	0.2478
loglogistic/weibull	0.4219	0.6490	-	0.1520	0.6490	-
weibull/weibull	0.1184	0.3176	0.7401	0.0173	0.3176	0.7401

Table C.27: RRLN, N = 50: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.2903	0.5371	-	0.6434	-	-
gamma/weibull	0.0130	0.2203	0.3506	0.0017	0.0176	0.3506
loglogistic/weibull	0.6108	-	-	0.3391	0.7978	-
weibull/weibull	0.0854	0.2421	0.6358	0.0153	0.1842	0.6358

Table C.28: VOD, N = 50: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.2580	0.4887	-	0.6261	-	-
gamma/weibull	0.0158	0.2540	0.3687	0.0037	0.0200	0.3687
loglogistic/weibull	0.6411	-	-	0.3526	0.8105	-
weibull/weibull	0.0850	0.2573	0.6261	0.0176	0.1694	0.6261

Table C.29: RIOTINTO, N = 50: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.4435	-	-	0.8303	-	-
gamma/weibull	0.0175	0.0322	0.2327	0.0013	0.0322	0.2327
loglogistic/weibull	0.4138	0.6445	-	0.1498	0.6445	-
weibull/weibull	0.1251	0.3228	0.7410	0.0184	0.3228	0.7410

Table C.30: SSELN, N = 50: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.3007	0.5145	-	0.7845	-	-
gamma/weibull	0.0193	0.1956	0.2519	0.0019	0.0243	0.2519
loglogistic/weibull	0.5838	-	-	0.2020	0.7119	-
weibull/weibull	0.0961	0.2897	0.7452	0.0115	0.2637	0.7452

Table C.31: ABFLN, N = 100: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0932	0.3440	0.5022	0.4899	0.7617	-
gamma/weibull	0.0086	0.4078	-	0.0041	0.1253	0.4688
loglogistic/weibull	0.8647	-	-	0.4979	-	-
weibull/weibull	0.0335	0.2482	0.4978	0.0081	0.1129	0.5309

Table C.32: BARC, N = 100: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.2979	0.5330	-	0.7954	-	-
gamma/weibull	0.0187	0.2084	0.2742	0.0028	0.0228	0.2742
loglogistic/weibull	0.5968	-	-	0.1905	0.7519	-
weibull/weibull	0.0867	0.2585	0.7245	0.0113	0.2252	0.7245

Table C.33: RRLN, N = 100: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.1512	0.4190	-	0.5673	-	-
gamma/weibull	0.0084	0.3456	0.4279	0.0022	0.0100	0.4279
loglogistic/weibull	0.7980	-	-	0.4223	0.9071	-
weibull/weibull	0.0423	0.2352	0.5700	0.0081	0.0827	0.5700

Table C.34: VOD, N = 100: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.1074	0.3810	-	0.5508	-	-
gamma/weibull	0.0077	0.3661	0.4335	0.0028	0.0088	0.4335
loglogistic/weibull	0.8445	-	-	0.4388	0.9241	-
weibull/weibull	0.0404	0.2529	0.5662	0.0076	0.0671	0.5662

Table C.35: RIOTINTO, N = 100: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.3090	0.5309	-	0.7996	-	-
gamma/weibull	0.0192	0.1840	0.2618	0.0010	0.0251	0.2618
loglogistic/weibull	0.5696	-	-	0.1843	0.7244	-
weibull/weibull	0.1023	0.2851	0.7346	0.0151	0.2505	0.7346

Table C.36: SSELN, N = 100: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.1495	0.3512	0.6500	0.6339	-	-
gamma/weibull	0.0141	0.2511	0.3498	0.0044	0.0151	0.2774
loglogistic/weibull	0.7524	-	-	0.3506	0.8016	-
weibull/weibull	0.0839	0.3975	-	0.0111	0.1833	0.7224

Table C.37: ABFLN, N = 200: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0183	0.2467	0.4235	0.3080	0.6258	-
gamma/weibull	0.0027	0.5168	-	0.0016	0.2615	0.5475
loglogistic/weibull	0.9696	-	-	0.6880	-	-
weibull/weibull	0.0093	0.2365	0.5765	0.0024	0.1126	0.4524

Table C.38: BARC, N = 200: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.1358	0.3933	-	0.6843	-	-
gamma/weibull	0.0140	0.2638	0.2955	0.0041	0.0147	0.2955
loglogistic/weibull	0.7862	-	-	0.3046	0.8483	-
weibull/weibull	0.0640	0.3426	0.7041	0.0070	0.1370	0.7041

Table C.39: RRLN, N = 200: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0404	0.3102	0.5269	0.4116	0.7419	-
gamma/weibull	0.0035	0.4582	-	0.0015	0.1527	0.5027
loglogistic/weibull	0.9422	-	-	0.5845	-	-
weibull/weibull	0.0139	0.2316	0.4731	0.0023	0.1053	0.4971

Table C.40: VOD, N = 200: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0158	0.2827	0.4870	0.3919	0.7106	-
gamma/weibull	0.0025	0.4646	-	0.0016	0.1851	0.4988
loglogistic/weibull	0.9676	-	-	0.6040	-	-
weibull/weibull	0.0141	0.2527	0.5130	0.0025	0.1043	0.5012

Table C.41: RIOTINTO, N = 200: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.1318	0.3735	0.6406	0.6772	-	-
gamma/weibull	0.0160	0.2466	0.3594	0.0019	0.0188	0.2941
loglogistic/weibull	0.7597	-	-	0.3066	0.8147	-
weibull/weibull	0.0925	0.3799	-	0.0143	0.1665	0.7056

Table C.42: SSELN, N = 200: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0368	0.1813	0.6249	0.2956	0.5916	-
gamma/weibull	0.0050	0.2840	0.3746	0.0035	0.1531	0.2956
loglogistic/weibull	0.8706	-	-	0.6717	-	-
weibull/weibull	0.0876	0.5342	-	0.0292	0.2548	0.7039

Table C.43: ABFLN, N = 500: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0005	0.1431	0.3212	0.0768	0.4111	0.6326
gamma/weibull	0.0003	0.6324	-	0.0003	0.4902	-
loglogistic/weibull	0.9982	-	-	0.9225	-	-
weibull/weibull	0.0010	0.2244	0.6787	0.0004	0.0985	0.3673

Table C.44: BARC, N = 500: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0265	0.2254	0.6086	0.3294	0.6953	-
gamma/weibull	0.0053	0.3018	0.3907	0.0039	0.1546	0.3135
loglogistic/weibull	0.9239	-	-	0.6536	-	-
weibull/weibull	0.0442	0.4720	-	0.0131	0.1493	0.6858

Table C.45: RRLN, N = 500: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0019	0.1880	0.4352	0.1278	0.5214	-
gamma/weibull	0.0004	0.5844	-	0.0003	0.3644	0.6040
loglogistic/weibull	0.9954	-	-	0.8714	-	-
weibull/weibull	0.0024	0.2276	0.5648	0.0005	0.1143	0.3960

Table C.46: VOD, N = 500: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0001	0.1676	0.3982	0.0747	0.5008	-
gamma/weibull	0.0003	0.5782	-	0.0002	0.4039	0.5903
loglogistic/weibull	0.9973	-	-	0.9241	-	-
weibull/weibull	0.0022	0.2542	0.6018	0.0009	0.0953	0.4097

Table C.47: RIOTINTO, N = 500: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0225	0.1884	0.5858	0.3057	0.6537	-
gamma/weibull	0.0053	0.3092	0.4142	0.0031	0.1240	0.3348
loglogistic/weibull	0.9047	-	-	0.6617	-	-
weibull/weibull	0.0675	0.5024	-	0.0296	0.2223	0.6652

Table C.48: SSELN, N = 500: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0134	0.1146	0.6058	0.0857	0.3550	0.7317
gamma/weibull	0.0031	0.2941	0.3932	0.0031	0.2198	0.2673
loglogistic/weibull	0.9082	-	-	0.8669	-	-
weibull/weibull	0.0753	0.5903	-	0.0444	0.4241	-

Table C.49: ABFLN, N = 1000: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0000	0.0879	0.2597	0.0084	0.2750	0.5231
gamma/weibull	0.0002	0.7102	-	0.0001	0.6344	-
loglogistic/weibull	0.9995	-	-	0.9913	-	-
weibull/weibull	0.0003	0.2018	0.7402	0.0002	0.0905	0.4768

Table C.50: BARC, N = 1000: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0050	0.1507	0.5955	0.0894	0.4895	-
gamma/weibull	0.0014	0.3158	0.4030	0.0014	0.2228	0.3208
loglogistic/weibull	0.9712	-	-	0.8969	-	-
weibull/weibull	0.0224	0.5321	-	0.0123	0.2862	0.6777

Table C.51: RRLN, N = 1000: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0000	0.1161	0.3821	0.0208	0.3578	0.6628
gamma/weibull	0.0000	0.6724	-	0.0000	0.5342	-
loglogistic/weibull	0.9995	-	-	0.9789	-	-
weibull/weibull	0.0005	0.2116	0.6179	0.0003	0.1081	0.3372

Table C.52: VOD, N = 1000: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0000	0.1099	0.3471	0.0033	0.3490	0.6676
gamma/weibull	0.0000	0.6431	-	0.0000	0.5536	-
loglogistic/weibull	0.9997	-	-	0.9964	-	-
weibull/weibull	0.0003	0.2470	0.6529	0.0003	0.0975	0.3324

Table C.53: RIOTINTO, N = 1000: limit order arrival times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0081	0.1123	0.5517	0.0800	0.3953	-
gamma/weibull	0.0027	0.3405	0.4483	0.0027	0.2165	0.3522
loglogistic/weibull	0.9425	-	-	0.8868	-	-
weibull/weibull	0.0467	0.5472	-	0.0305	0.3881	0.6478

Table C.54: SSELN, N = 1000: limit order arrival times, removal of zeros.

n	θ	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	-	-	-	-	-	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	9.243	200.917	0.444	0.072	2.827	-	-	-	-	-	2953.907	3675.802	1878.821	621.452	5210.225	0.895	2.467	0.715	0.491	1.109	0.132	0.112	0.105	0.493	0.162	0.497	0.375	0.140	0.368	-193.235	-262.305	-123.644	0.012
100	0.5	3.831	98.193	0.371	0.058	1.287	-	-	-	-	-	2663.124	2737.105	1818.468	726.959	4670.064	0.719	0.769	0.650	0.471	0.944	0.121	0.094	0.101	0.502	0.136	0.505	0.376	0.117	0.373	-391.184	-505.590	-277.076	0.009
200	0.5	8.072	718.964	0.340	0.054	0.964	-	-	-	-	-	2445.343	2169.727	1772.340	801.479	4119.164	0.655	0.217	0.611	0.465	0.847	0.115	0.081	0.098	0.508	0.115	0.509	0.377	0.101	0.376	-784.649	-977.416	-595.374	0.002
500	0.5	0.437	1.493	0.315	0.055	0.740	-	-	-	-	-	2265.817	1781.072	1700.560	897.153	3712.886	0.615	0.165	0.585	0.461	0.767	0.109	0.064	0.095	0.512	0.089	0.513	0.379	0.080	0.380	-1975.488	-2347.574	-1607.062	0.000
1000	0.5	0.378	0.342	0.319	0.058	0.653	-	-	-	-	-	2137.601	1458.678	1651.301	950.825	3491.495	0.593	0.141	0.569	0.460	0.725	0.105	0.056	0.092	0.513	0.071	0.514	0.382	0.067	0.384	-3967.236	-4605.798	-3342.752	0.000

Table C.55: exponential-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) ABFLN data.

n	θ	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	-	-	-	-	-	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	3.893	584.917	0.474	0.159	1.531	-	-	-	-	-	313.616	395.771	192.760	60.734	554.951	1.040	2.348	0.723	0.514	1.197	0.221	0.157	0.191	0.449	0.196	0.485	0.329	0.152	0.320	-141.324	-191.692	-88.004	0.012
100	0.5	1.253	40.016	0.454	0.181	1.246	-	-	-	-	-	265.181	286.569	179.655	68.262	448.994	0.717	1.209	0.618	0.485	0.853	0.201	0.136	0.172	0.457	0.182	0.494	0.342	0.135	0.334	-286.368	-373.391	-197.106	0.003
200	0.5	0.699	6.774	0.452	0.211	1.125	-	-	-	-	-	228.114	207.399	168.172	74.268	374.763	0.598	0.395	0.565	0.474	0.704	0.185	0.121	0.157	0.459	0.172	0.499	0.355	0.120	0.346	-578.572	-730.191	-421.476	0.002
500	0.5	0.594	0.404	0.457	0.254	1.056	-	-	-	-	-	198.283	148.591	156.681	78.977	314.284	0.541	0.081	0.530	0.469	0.611	0.172	0.108	0.145	0.461	0.163	0.503	0.367	0.107	0.356	-1454.446	-1786.281	-1109.940	0.000
1000	0.5	0.593	0.362	0.457	0.282	1.043	-	-	-	-	-	185.520	122.926	152.130	82.229	287.853	0.523	0.060	0.517	0.468	0.576	0.167	0.102	0.141	0.461	0.159	0.504	0.373	0.101	0.360	-2917.584	-3501.274	-2273.318	0.000

Table C.56: exponential-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	-	-	-	-	-	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	13.816	963.236	0.402	0.084	1.967	-	-	-	-	-	1933.213	2255.420	1249.541	416.139	3433.865	0.842	1.877	0.690	0.473	1.043	0.142	0.128	0.110	0.486	0.164	0.495	0.372	0.137	0.365	-185.505	-245.702	-125.577	0.010
100	0.5	1.777	23.550	0.340	0.078	1.181	-	-	-	-	-	1690.513	1674.505	1178.130	491.203	2914.829	0.678	0.864	0.613	0.452	0.858	0.128	0.109	0.103	0.494	0.137	0.502	0.378	0.114	0.373	-376.365	-472.756	-281.154	0.007
200	0.5	0.890	23.516	0.314	0.078	0.889	-	-	-	-	-	1518.743	1423.129	1124.358	552.369	2486.624	0.601	0.190	0.571	0.443	0.756	0.119	0.093	0.098	0.499	0.113	0.505	0.382	0.096	0.381	-756.841	-913.422	-606.789	0.003
500	0.5	1.725	101.711	0.298	0.087	0.699	-	-	-	-	-	1346.253	949.744	1068.000	627.257	2057.900	0.558	0.190	0.539	0.439	0.673	0.111	0.083	0.094	0.502	0.088	0.508	0.386	0.076	0.386	-1902.605	-2193.902	-1616.395	0.001
1000	0.5	0.361	0.271	0.298	0.100	0.599	-	-	-	-	-	1237.380	736.718	1027.051	663.608	1809.543	0.534	0.099	0.522	0.435	0.635	0.105	0.074	0.091	0.503	0.073	0.509	0.391	0.063	0.392	-3818.382	-4301.821	-3333.267	0.001

Table C.57: exponential-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) RRLN data.

n	θ	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	-	-	-	-	-	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	5.126	132.436	0.610	0.166	1.954	-	-	-	-	-	682.530	844.352	430.673	139.308	1216.269	0.950	3.719	0.684	0.488	1.052	0.155	0.112	0.135	0.506	0.144	0.512	0.339	0.126	0.336	-150.253	-200.728	-98.496	0.017
100	0.5	2.441	143.683	0.549	0.180	1.551	-	-	-	-	-	590.870	623.135	408.748	164.782	1022.251	0.727	2.105	0.604	0.468	0.826	0.136	0.087	0.124	0.515	0.115	0.518	0.348	0.105	0.347	-305.478	-387.099	-222.608	0.016
200	0.5	0.880	10.744	0.523	0.207	1.339	-	-	-	-	-	523.409	462.639	391.448	181.502	879.148	0.592	0.547	0.560	0.461	0.705	0.124	0.069	0.114	0.520	0.094	0.520	0.357	0.087	0.357	-617.254	-754.127	-478.420	0.016
500	0.5	0.671	0.457	0.534	0.252	1.201	-	-	-	-	-	464.595	339.484	373.640	195.363	733.641	0.539	0.109	0.527	0.458	0.617	0.114	0.055	0.106	0.526	0.081	0.523	0.361	0.076	0.365	-1532.585	-1831.877	-1264.174	0.003
1000	0.5	0.662	0.390	0.541	0.286	1.136	-	-	-	-	-	435.021	276.716	365.787	199.678	667.441	0.520	0.067	0.514	0.457	0.581	0.108	0.047	0.102	0.527	0.068	0.524	0.364	0.064	0.369	-3067.009	-3595.862	-2576.061	0.000

Table C.58: exponential-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) VOD data.

n	θ	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	4.513	213.849	0.632	0.177	1.804	-	-	-	-	-	271.123	384.527	144.482	46.708	481.234	1.041	2.290	0.713	0.507	1.171	0.180	0.128	0.160	0.503	0.143	0.513	0.316	0.129	0.317	-127.093	-171.447	-81.371	0.010
100	0.5	1.923	134.062	0.571	0.190	1.475	-	-	-	-	-	229.001	285.649	133.541	49.932	392.460	0.711	1.077	0.612	0.482	0.834	0.158	0.104	0.142	0.513	0.118	0.519	0.330	0.109	0.332	-257.834	-333.684	-182.809	0.002
200	0.5	0.902	7.424	0.548	0.214	1.310	-	-	-	-	-	196.634	216.805	123.341	51.664	335.247	0.592	0.301	0.562	0.474	0.693	0.141	0.086	0.128	0.516	0.098	0.520	0.343	0.089	0.345	-520.821	-654.054	-391.195	0.000
500	0.5	0.679	0.595	0.533	0.251	1.202	-	-	-	-	-	171.090	159.616	115.928	53.203	291.939	0.541	0.080	0.530	0.473	0.606	0.128	0.068	0.117	0.518	0.079	0.520	0.354	0.070	0.357	-1311.588	-1604.337	-1032.852	0.000
1000	0.5	0.667	0.397	0.525	0.278	1.162	-	-	-	-	-	159.805	135.378	111.932	53.578	274.054	0.524	0.057	0.518	0.472	0.574	0.122	0.059	0.112	0.518	0.067	0.520	0.360	0.059	0.363	-2631.790	-3181.299	-2120.564	0.000

Table C.59: exponential-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) RIOTINTO data.

n	θ	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	18.465	1499.067	0.503	0.087	3.503	-	-	-	-	-	2526.896	3100.010	1667.896	541.158	4235.471	0.816	0.887	0.701	0.493	1.060	0.122	0.106	0.096	0.490	0.162	0.496	0.388	0.144	0.380	-194.290	-263.380	-127.134	0.002
100	0.5	4.449	129.382	0.402	0.072	1.366	-	-	-	-	-	2222.482	2207.920	1586.843	628.567	3723.776	0.694	0.321	0.631	0.471	0.901	0.108	0.087	0.089	0.500	0.136	0.504	0.392	0.121	0.384	-393.838	-506.683	-282.179	0.000
200	0.5	0.855	9.211	0.362	0.072	0.991	-	-	-	-	-	1990.306	1652.210	1506.488	713.516	3268.690	0.635	0.215	0.588	0.462	0.804	0.100	0.073	0.083	0.504	0.112	0.507	0.396	0.101	0.391	-795.253	-979.994	-610.046	0.000
500	0.5	0.426	0.447	0.334	0.075	0.745	-	-	-	-	-	1803.555	1195.465	1451.427	805.689	2839.477	0.592	0.148	0.558	0.460	0.728	0.093	0.057	0.079	0.507	0.085	0.510	0.401	0.079	0.397	-2004.527	-2362.654	-1645.035	0.000
1000	0.5	0.380	0.281	0.328	0.085	0.647	-	-	-	-	-	1709.977	1015.716	1405.093	869.742	2606.672	0.571	0.121	0.544	0.458	0.687	0.088	0.048	0.077	0.508	0.067	0.508	0.404	0.066	0.403	-4022.419	-4605.778	-3441.622	0.000

Table C.60: exponential-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) SSELN data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	3.197	3.719	2.104	0.498	8.789	0.897	0.197	0.939	0.644	1.128	2333.366	3387.712	1416.268	488.036	5521.470	0.750	2.934	0.605	0.439	0.990	0.088	0.083	0.068	0.498	0.164	0.500	0.415	0.152	0.405	-193.970	-259.749	-102.310	0.082
100	0.5	3.290	3.380	2.310	0.822	6.354	0.964	0.142	0.977	0.879	1.090	1987.771	2293.264	1370.182	573.265	3431.708	0.583	0.307	0.551	0.422	0.754	0.076	0.065	0.060	0.504	0.137	0.507	0.421	0.131	0.415	-393.854	-506.651	-269.310	0.032
200	0.5	3.055	2.961	2.264	1.062	4.718	1.003	0.090	0.997	0.943	1.083	1780.628	1642.180	1330.703	652.571	2778.567	0.540	0.128	0.524	0.416	0.661	0.069	0.054	0.056	0.510	0.116	0.512	0.421	0.115	0.418	-788.164	-981.580	-594.937	0.005
500	0.5	2.684	2.323	2.173	1.322	3.335	1.025	0.055	1.015	0.977	1.080	1604.531	1158.140	1301.988	760.209	2370.965	0.513	0.094	0.505	0.418	0.606	0.063	0.040	0.054	0.514	0.090	0.515	0.423	0.091	0.421	-1981.298	-2358.389	-1611.150	0.000
1000	0.5	2.462	1.949	2.143	1.447	2.927	1.029	0.044	1.021	0.987	1.072	1511.993	900.823	1290.178	811.750	2166.957	0.501	0.081	0.499	0.419	0.583	0.060	0.031	0.054	0.515	0.072	0.518	0.424	0.074	0.423	-3976.509	-4614.007	-3354.297	0.000

Table C.61: gamma-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) ABFLN data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	2.595	2.496	1.871	0.732	4.598	0.952	0.121	0.963	0.868	1.054	234.132	311.374	138.486	42.281	423.049	0.757	1.614	0.623	0.466	0.911	0.166	0.136	0.130	0.458	0.194	0.492	0.376	0.150	0.363	-141.291	-191.533	-86.794	0.018
100	0.5	2.381	2.089	1.835	0.818	3.542	0.981	0.077	0.980	0.921	1.047	199.578	221.088	132.560	50.804	335.728	0.599	1.140	0.552	0.449	0.705	0.149	0.117	0.119	0.463	0.179	0.499	0.388	0.131	0.375	-286.812	-373.704	-197.423	0.004
200	0.5	2.147	1.665	1.784	0.889	2.938	0.994	0.054	0.989	0.948	1.045	175.652	162.359	128.642	57.973	285.017	0.534	0.351	0.518	0.443	0.615	0.139	0.103	0.113	0.465	0.168	0.503	0.396	0.118	0.382	-579.174	-730.832	-421.978	0.003
500	0.5	1.902	1.147	1.762	0.941	2.546	1.000	0.040	0.994	0.964	1.041	158.360	120.213	125.434	65.960	245.973	0.501	0.061	0.497	0.442	0.559	0.132	0.091	0.108	0.467	0.159	0.508	0.401	0.106	0.386	-1455.377	-1787.429	-1111.431	0.000
1000	0.5	1.786	0.848	1.763	0.963	2.410	1.001	0.035	0.997	0.968	1.036	150.863	98.095	124.764	70.865	227.878	0.490	0.048	0.488	0.443	0.537	0.129	0.085	0.107	0.468	0.154	0.509	0.403	0.100	0.387	-2919.108	-3505.272	-2275.038	0.000

Table C.62: gamma-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	3.355	3.671	2.282	0.603	8.328	0.909	0.183	0.945	0.739	1.110	1463.381	1919.783	905.464	316.144	3319.859	0.719	2.722	0.585	0.424	0.950	0.094	0.094	0.071	0.489	0.165	0.497	0.417	0.145	0.408	-186.393	-244.021	-109.978	0.071
100	0.5	3.389	3.298	2.422	0.907	6.410	0.970	0.124	0.978	0.893	1.079	1235.316	1291.340	878.354	383.975	2111.924	0.571	0.635	0.528	0.407	0.716	0.080	0.076	0.062	0.496	0.136	0.504	0.424	0.121	0.418	-378.212	-473.306	-277.108	0.024
200	0.5	3.116	2.839	2.326	1.137	4.897	1.002	0.077	0.995	0.945	1.070	1084.294	885.623	859.406	446.487	1664.986	0.516	0.144	0.497	0.402	0.627	0.072	0.063	0.057	0.501	0.112	0.507	0.427	0.101	0.424	-759.008	-915.946	-607.390	0.004
500	0.5	2.742	2.217	2.226	1.373	3.586	1.019	0.051	1.010	0.975	1.069	962.677	574.264	828.992	515.446	1378.676	0.487	0.089	0.474	0.401	0.572	0.066	0.049	0.054	0.504	0.087	0.510	0.430	0.079	0.429	-1906.038	-2199.713	-1618.870	0.001
1000	0.5	2.477	1.529	2.175	1.503	3.035	1.024	0.042	1.016	0.984	1.067	916.892	460.478	817.173	555.776	1265.551	0.477	0.076	0.466	0.403	0.553	0.063	0.040	0.054	0.505	0.073	0.511	0.432	0.065	0.430	-3823.592	-4305.257	-3335.132	0.001

Table C.63: gamma-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) RRLN data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	2.529	2.703	1.683	0.606	5.389	0.930	0.150	0.950	0.834	1.070	546.626	685.887	345.328	110.986	1083.588	0.782	2.799	0.613	0.450	0.918	0.117	0.094	0.096	0.515	0.141	0.518	0.369	0.123	0.364	-150.516	-199.921	-94.001	0.042
100	0.5	2.365	2.286	1.718	0.717	3.928	0.974	0.095	0.975	0.907	1.056	473.966	480.158	335.398	135.643	834.152	0.636	1.749	0.551	0.437	0.721	0.103	0.073	0.088	0.520	0.113	0.522	0.377	0.101	0.375	-306.486	-387.314	-222.369	0.020
200	0.5	2.098	1.786	1.677	0.794	3.030	0.993	0.062	0.987	0.944	1.052	425.462	352.342	329.218	156.455	714.506	0.542	0.749	0.519	0.434	0.634	0.095	0.058	0.085	0.523	0.092	0.523	0.383	0.084	0.382	-618.633	-755.016	-479.056	0.017
500	0.5	1.798	1.233	1.607	0.850	2.452	0.999	0.043	0.991	0.964	1.041	391.435	263.128	326.271	176.141	600.893	0.505	0.080	0.499	0.436	0.574	0.089	0.045	0.083	0.529	0.078	0.526	0.382	0.077	0.385	-1533.358	-1833.142	-1265.061	0.003
1000	0.5	1.668	0.941	1.571	0.898	2.285	1.000	0.035	0.993	0.971	1.034	374.744	220.396	324.590	185.553	560.511	0.494	0.059	0.491	0.437	0.548	0.087	0.037	0.082	0.531	0.066	0.527	0.382	0.064	0.386	-3068.710	-3598.054	-2576.628	0.000

Table C.64: gamma-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) VOD data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	2.558	2.563	1.720	0.664	4.877	0.943	0.121	0.956	0.861	1.043	210.186	317.655	110.364	35.535	372.957	0.758	1.547	0.635	0.472	0.935	0.133	0.105	0.110	0.514	0.140	0.520	0.353	0.119	0.351	-127.056	-171.248	-80.291	0.016
100	0.5	2.357	2.196	1.691	0.741	3.761	0.974	0.075	0.974	0.916	1.037	178.183	229.620	104.361	40.039	297.669	0.601	0.611	0.561	0.454	0.719	0.116	0.083	0.098	0.518	0.116	0.523	0.367	0.098	0.366	-258.198	-334.109	-183.097	0.002
200	0.5	2.119	1.794	1.633	0.802	3.017	0.988	0.052	0.983	0.944	1.037	155.749	172.840	99.831	43.721	257.438	0.540	0.126	0.526	0.449	0.626	0.105	0.068	0.092	0.520	0.096	0.523	0.376	0.081	0.376	-521.375	-654.740	-391.789	0.000
500	0.5	1.863	1.316	1.595	0.861	2.553	0.995	0.040	0.988	0.960	1.036	139.169	127.195	97.708	47.449	229.512	0.510	0.064	0.504	0.448	0.571	0.097	0.052	0.089	0.521	0.076	0.523	0.382	0.064	0.382	-1312.618	-1605.759	-1033.493	0.000
1000	0.5	1.723	1.002	1.566	0.887	2.386	0.996	0.035	0.989	0.965	1.033	132.452	108.724	95.917	49.575	218.013	0.499	0.051	0.496	0.449	0.549	0.094	0.043	0.088	0.521	0.065	0.522	0.385	0.054	0.385	-2633.471	-3183.508	-2121.989	0.000

Table C.65: gamma-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) RIOTINTO data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	3.121	3.596	2.056	0.478	8.336	0.888	0.196	0.931	0.636	1.112	2048.007	2849.103	1291.428	442.040	4601.912	0.673	0.659	0.613	0.451	0.978	0.081	0.075	0.062	0.495	0.163	0.499	0.424	0.153	0.413	-195.032	-260.431	-106.854	0.078
100	0.5	3.212	3.237	2.293	0.789	6.077	0.958	0.138	0.972	0.874	1.078	1729.291	1904.994	1222.502	516.280	2910.139	0.582	0.191	0.556	0.432	0.740	0.068	0.056	0.054	0.502	0.136	0.506	0.431	0.130	0.424	-395.767	-507.132	-274.873	0.024
200	0.5	3.017	2.884	2.248	1.061	4.634	0.997	0.087	0.991	0.939	1.070	1524.963	1290.406	1194.547	603.442	2331.288	0.541	0.136	0.524	0.427	0.655	0.059	0.043	0.050	0.506	0.112	0.509	0.435	0.108	0.430	-797.451	-983.403	-609.302	0.003
500	0.5	2.605	1.972	2.175	1.348	3.363	1.019	0.053	1.007	0.974	1.069	1367.069	840.988	1163.896	694.220	1969.075	0.514	0.091	0.502	0.426	0.601	0.053	0.031	0.047	0.509	0.085	0.513	0.438	0.083	0.433	-2008.231	-2368.368	-1647.344	0.000
1000	0.5	2.415	1.609	2.145	1.465	2.972	1.024	0.043	1.015	0.984	1.067	1307.882	680.846	1146.219	751.567	1849.974	0.503	0.077	0.496	0.428	0.577	0.051	0.024	0.046	0.510	0.068	0.511	0.439	0.067	0.437	-4027.949	-4611.757	-3442.805	0.000

Table C.66: gamma-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) SSELN data.

n	θ	$\hat{\alpha}_{\text{log ave}}$	$\pm\hat{\alpha}_{\text{log ave}}$	$\hat{\alpha}_{\text{log median}}$	$\hat{\alpha}_{\text{log lower}}$	$\hat{\alpha}_{\text{log upper}}$	$\hat{\beta}_{\text{log ave}}$	$\pm\hat{\beta}_{\text{log ave}}$	$\hat{\beta}_{\text{log median}}$	$\hat{\beta}_{\text{log lower}}$	$\hat{\beta}_{\text{log upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	39.707	979.946	7.390	1.358	56.721	9.378	25.906	1.285	0.767	19.404	3943.351	4385.172	2761.692	925.787	8417.308	1.430	3.937	0.968	0.634	2.008	0.198	0.135	0.170	0.490	0.157	0.498	0.312	0.127	0.307	-194.901	-258.496	-111.476	0.072
100	0.5	26.607	505.496	6.740	1.405	35.083	6.407	18.850	1.049	0.709	3.458	3728.029	3622.731	2801.718	1031.732	6870.542	1.041	1.339	0.857	0.590	1.392	0.188	0.120	0.168	0.501	0.132	0.505	0.311	0.110	0.309	-391.211	-500.518	-267.600	0.039
200	0.5	22.447	366.490	6.271	1.490	27.226	5.890	17.970	0.917	0.680	2.146	3476.803	2757.309	2833.853	1140.420	5993.467	0.913	0.986	0.805	0.576	1.168	0.182	0.107	0.168	0.509	0.114	0.510	0.309	0.095	0.306	-781.260	-968.558	-585.634	0.019
500	0.5	11.397	16.698	6.092	1.789	21.811	4.601	15.497	0.813	0.664	1.418	3348.385	2253.415	2867.517	1283.434	5535.055	0.846	0.917	0.776	0.583	1.037	0.182	0.093	0.174	0.510	0.089	0.511	0.307	0.080	0.303	-1964.695	-2329.669	-1582.581	0.010
1000	0.5	10.290	15.215	6.160	2.005	19.064	3.348	12.741	0.773	0.663	1.166	3301.036	1944.418	2941.870	1401.257	5272.828	0.801	0.342	0.761	0.596	0.978	0.184	0.084	0.181	0.510	0.072	0.509	0.306	0.072	0.300	-3948.676	-4576.713	-3327.923	0.004

Table C.67: loglogistic-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) ABFLN data.

n	θ	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.5	5.980	95.596	2.240	1.000	9.718	23.023	42.536	2.042	1.039	99.672	416.247	605.919	238.478	75.217	849.747	1.432	3.590	0.795	0.546	1.947	0.225	0.157	0.187	0.473	0.188	0.508	0.303	0.160	0.291	-139.516	-187.962	-79.834	0.048
100	0.5	4.712	55.295	2.143	1.000	6.295	23.024	41.074	1.728	0.955	92.248	368.140	495.404	227.139	78.935	639.195	0.968	1.909	0.675	0.509	1.146	0.214	0.145	0.183	0.479	0.173	0.514	0.308	0.151	0.301	-280.122	-366.509	-187.433	0.012
200	0.5	3.610	25.354	2.059	1.000	4.733	23.696	40.017	1.612	0.935	89.966	311.946	346.376	213.310	84.302	524.440	0.731	0.757	0.611	0.494	0.877	0.201	0.131	0.175	0.481	0.162	0.518	0.318	0.141	0.312	-564.433	-717.047	-404.273	0.004
500	0.5	2.537	3.947	1.960	1.000	3.518	26.352	39.993	1.626	0.967	88.572	256.734	217.725	194.844	88.703	422.942	0.607	0.214	0.566	0.485	0.710	0.182	0.114	0.162	0.484	0.153	0.522	0.334	0.130	0.326	-1416.848	-1751.074	-1067.391	0.000
1000	0.5	2.178	2.515	1.913	1.000	3.038	27.994	39.825	1.651	1.023	86.947	230.606	167.870	184.020	90.989	374.524	0.569	0.159	0.548	0.483	0.645	0.170	0.103	0.156	0.484	0.148	0.524	0.345	0.122	0.334	-2841.210	-3440.933	-2181.504	0.000

Table C.68: loglogistic-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.5	25.081	256.510	5.633	1.271	40.066	10.949	232.923	1.414	0.810	12.282	2715.459	3184.591	1779.496	594.107	5584.729	1.241	2.990	0.876	0.578	1.693	0.202	0.147	0.167	0.485	0.159	0.497	0.313	0.131	0.307	-186.787	-243.829	-117.859	0.054
100	0.5	20.420	326.989	5.239	1.387	27.353	7.179	20.384	1.123	0.745	3.515	2526.902	2608.669	1785.558	688.635	4548.669	0.918	1.375	0.774	0.543	1.183	0.192	0.132	0.165	0.495	0.132	0.504	0.313	0.113	0.310	-376.772	-470.765	-276.971	0.024
200	0.5	12.120	46.952	5.064	1.555	21.255	6.207	18.559	0.975	0.716	2.258	2325.510	1991.130	1764.010	768.832	3939.567	0.785	0.591	0.722	0.531	0.988	0.186	0.118	0.166	0.502	0.111	0.507	0.313	0.098	0.310	-754.710	-910.110	-603.361	0.007
500	0.5	8.879	10.403	4.961	1.817	16.707	5.156	16.585	0.873	0.704	1.658	2154.113	1494.999	1751.873	868.991	3494.347	0.711	0.320	0.681	0.531	0.865	0.183	0.105	0.172	0.504	0.087	0.508	0.314	0.081	0.312	-1895.999	-2189.802	-1607.869	0.001
1000	0.5	7.823	7.570	5.040	2.032	14.399	3.877	13.881	0.827	0.703	1.381	2082.534	1270.222	1750.593	927.901	3278.594	0.688	0.358	0.668	0.543	0.805	0.183	0.097	0.176	0.503	0.072	0.508	0.314	0.073	0.312	-3806.872	-4290.448	-3323.819	0.000

Table C.69: loglogistic-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) RRLN data.

n	θ	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.5	8.711	86.291	2.386	1.000	15.099	18.732	37.068	1.777	0.896	84.587	964.991	1335.386	570.118	182.565	1843.225	1.368	4.978	0.804	0.546	1.621	0.185	0.126	0.156	0.519	0.139	0.527	0.296	0.130	0.293	-148.881	-198.349	-93.092	0.037
100	0.5	7.353	111.580	2.247	1.000	10.064	17.537	35.123	1.403	0.821	79.858	864.843	1066.164	547.607	203.697	1497.729	0.992	4.263	0.696	0.514	1.115	0.172	0.112	0.150	0.526	0.112	0.530	0.301	0.115	0.303	-302.023	-384.227	-217.599	0.016
200	0.5	4.809	19.642	2.112	1.000	7.239	18.517	33.933	1.256	0.798	80.167	757.415	785.493	521.086	219.970	1297.512	0.724	0.987	0.636	0.500	0.899	0.160	0.098	0.144	0.531	0.093	0.532	0.310	0.100	0.313	-609.322	-749.256	-468.016	0.015
500	0.5	3.173	7.115	1.938	1.000	4.487	22.473	35.494	1.250	0.807	79.712	650.117	570.587	482.537	226.159	1049.717	0.620	0.177	0.589	0.486	0.746	0.143	0.085	0.134	0.538	0.083	0.534	0.319	0.088	0.324	-1508.054	-1819.476	-1229.676	0.003
1000	0.5	2.513	2.581	1.853	1.000	3.626	25.155	36.062	1.318	0.833	78.897	593.854	472.095	450.291	225.236	955.197	0.586	0.147	0.567	0.480	0.688	0.133	0.078	0.127	0.540	0.069	0.536	0.327	0.076	0.331	-3019.024	-3576.328	-2502.327	0.001

Table C.70: loglogistic-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) VOD data.

n	θ	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.5	6.815	89.107	2.335	1.000	11.636	25.240	44.875	2.132	1.057	105.450	379.676	614.198	183.983	60.148	822.731	1.506	3.729	0.798	0.550	2.151	0.192	0.132	0.159	0.527	0.138	0.537	0.280	0.134	0.282	-125.532	-168.503	-72.682	0.054
100	0.5	5.392	65.192	2.179	1.000	7.492	26.715	44.316	1.774	0.958	98.404	335.721	511.469	170.535	59.561	602.377	1.017	2.107	0.676	0.514	1.154	0.179	0.122	0.149	0.534	0.114	0.541	0.287	0.123	0.295	-252.334	-329.612	-172.524	0.013
200	0.5	3.806	13.795	2.042	1.000	5.546	29.218	44.281	1.635	0.930	96.580	281.680	377.537	156.791	61.178	486.548	0.758	1.022	0.612	0.501	0.851	0.165	0.111	0.138	0.537	0.095	0.542	0.298	0.109	0.308	-507.622	-647.321	-371.780	0.002
500	0.5	2.662	3.280	1.896	1.000	3.986	34.256	44.645	1.676	0.952	94.860	226.637	241.003	141.177	62.510	391.361	0.612	0.311	0.570	0.496	0.688	0.146	0.094	0.123	0.539	0.076	0.543	0.315	0.090	0.325	-1275.064	-1589.954	-980.531	0.000
1000	0.5	2.412	18.952	1.838	1.000	3.343	37.725	44.461	1.765	1.001	93.363	201.881	188.397	134.089	63.034	349.164	0.571	0.122	0.554	0.494	0.635	0.134	0.083	0.113	0.540	0.066	0.544	0.326	0.077	0.334	-2553.646	-3156.237	-1998.979	0.000

Table C.71: loglogistic-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) RIOTINTO data.

n	θ	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.5	41.004	558.676	7.479	1.278	61.960	10.132	25.250	1.369	0.788	22.963	3392.207	3764.991	2434.716	760.795	6866.588	1.351	2.824	0.938	0.612	1.926	0.189	0.134	0.159	0.489	0.157	0.499	0.322	0.130	0.315	-195.540	-259.257	-115.703	0.064
100	0.5	30.153	441.173	6.798	1.330	41.091	7.150	20.138	1.072	0.716	3.930	3159.593	3043.071	2433.179	876.068	5616.614	1.070	1.616	0.827	0.564	1.376	0.179	0.121	0.155	0.500	0.131	0.505	0.321	0.113	0.315	-393.325	-500.918	-273.111	0.032
200	0.5	20.135	141.674	6.826	1.503	32.749	6.015	18.187	0.907	0.681	2.319	2957.621	2354.344	2465.610	1009.054	4995.091	0.948	1.356	0.773	0.550	1.180	0.175	0.111	0.156	0.506	0.109	0.509	0.320	0.099	0.314	-790.511	-970.326	-596.902	0.017
500	0.5	14.344	25.460	6.883	1.794	27.195	5.294	16.787	0.803	0.660	1.665	2785.429	1735.674	2434.606	1174.763	4539.542	0.881	1.373	0.742	0.548	1.065	0.174	0.098	0.163	0.507	0.084	0.510	0.318	0.086	0.313	-1991.293	-2345.284	-1624.035	0.013
1000	0.5	12.780	20.487	7.335	2.041	23.634	4.560	15.412	0.753	0.656	1.330	2731.246	1481.843	2445.367	1271.546	4317.138	0.800	0.638	0.734	0.559	1.007	0.177	0.090	0.169	0.506	0.068	0.508	0.317	0.079	0.312	-4003.036	-4582.474	-3407.011	0.008

Table C.72: loglogistic-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) SSELN data.

n	θ	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	30.873	99.422	10.242	2.111	83.333	1.806	4.015	0.937	0.569	4.017	3807.332	4135.148	2690.687	968.145	9496.950	1.296	3.656	0.928	0.638	2.129	0.194	0.127	0.165	0.487	0.155	0.495	0.319	0.128	0.315	-194.558	-257.851	-97.204	0.101
100	0.5	31.962	98.349	11.170	2.393	52.532	1.214	2.356	0.774	0.522	1.761	3645.383	3362.146	2757.979	1079.251	6926.581	0.998	1.441	0.834	0.594	1.354	0.181	0.114	0.159	0.500	0.134	0.503	0.319	0.109	0.319	-392.220	-501.167	-260.979	0.050
200	0.5	26.690	66.416	11.077	2.517	40.390	0.885	1.020	0.683	0.503	1.192	3429.972	2766.187	2797.883	1177.288	5980.814	0.857	0.612	0.785	0.575	1.116	0.175	0.100	0.160	0.504	0.115	0.504	0.321	0.094	0.321	-787.776	-971.893	-582.820	0.030
500	0.5	19.865	47.370	11.169	2.834	31.131	0.719	0.486	0.612	0.500	0.893	3213.105	2119.212	2784.817	1297.345	5174.398	0.783	0.640	0.752	0.578	0.961	0.172	0.080	0.164	0.505	0.090	0.506	0.323	0.076	0.323	-1977.614	-2338.257	-1588.766	0.017
1000	0.5	15.782	22.887	11.090	3.001	26.000	0.659	0.270	0.589	0.503	0.770	3088.449	1754.131	2791.934	1394.470	4811.696	0.740	0.160	0.730	0.586	0.898	0.171	0.069	0.169	0.504	0.072	0.505	0.325	0.065	0.322	-3961.434	-4590.172	-3329.483	0.008

Table C.73: weibull-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) ABFLN data.

n	θ	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	8.192	20.527	3.700	1.608	26.447	1.712	3.500	1.057	0.675	3.558	414.612	527.279	259.160	95.127	1202.681	1.545	3.510	0.851	0.588	3.636	0.257	0.160	0.225	0.459	0.191	0.494	0.284	0.154	0.270	-141.834	-187.297	-64.012	0.109
100	0.5	7.622	20.141	3.530	1.713	18.888	1.330	2.132	0.925	0.638	2.584	386.133	462.627	255.605	104.352	913.464	1.064	2.173	0.711	0.537	1.721	0.243	0.145	0.217	0.465	0.178	0.501	0.292	0.146	0.284	-286.764	-366.085	-165.065	0.088
200	0.5	6.484	18.466	3.303	1.807	16.474	1.126	1.088	0.865	0.632	2.347	340.882	367.491	240.716	111.139	795.515	0.802	1.133	0.632	0.512	1.229	0.230	0.129	0.208	0.466	0.167	0.504	0.304	0.137	0.297	-579.505	-715.503	-359.252	0.095
500	0.5	4.758	7.415	3.044	1.922	14.142	1.018	0.670	0.841	0.651	2.366	278.126	237.020	217.917	116.551	676.913	0.635	0.398	0.576	0.495	0.953	0.214	0.110	0.195	0.465	0.158	0.505	0.321	0.125	0.311	-1459.068	-1736.177	-914.848	0.118
1000	0.5	3.913	4.865	2.915	2.008	14.366	0.974	0.440	0.839	0.677	2.524	245.348	165.192	206.239	119.708	647.904	0.578	0.149	0.553	0.490	0.865	0.205	0.099	0.187	0.464	0.155	0.503	0.332	0.116	0.318	-2927.173	-3388.401	-1758.340	0.138

Table C.74: weibull-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	25.339	70.785	8.269	2.091	62.380	1.912	4.310	0.984	0.594	3.725	2563.764	2804.523	1744.458	648.026	6234.425	1.190	3.061	0.850	0.586	1.837	0.200	0.141	0.164	0.482	0.158	0.492	0.318	0.128	0.314	-186.630	-242.807	-104.403	0.091
100	0.5	25.398	85.932	8.608	2.354	42.069	1.268	2.342	0.813	0.545	1.882	2444.934	2424.940	1763.277	724.064	4600.008	0.905	1.488	0.755	0.548	1.168	0.186	0.127	0.158	0.493	0.134	0.501	0.321	0.110	0.320	-377.211	-470.243	-269.684	0.042
200	0.5	20.450	48.028	8.595	2.548	33.405	0.947	1.179	0.715	0.528	1.307	2234.520	1818.478	1745.454	798.272	3914.770	0.756	0.464	0.702	0.533	0.962	0.179	0.111	0.160	0.497	0.112	0.502	0.324	0.095	0.324	-758.928	-910.009	-596.101	0.027
500	0.5	14.876	34.306	8.422	2.769	24.942	0.762	0.384	0.646	0.525	0.979	2020.078	1374.490	1693.482	863.570	3276.069	0.678	0.221	0.660	0.527	0.825	0.174	0.098	0.162	0.497	0.088	0.502	0.329	0.078	0.328	-1903.880	-2189.477	-1603.732	0.015
1000	0.5	12.042	13.033	8.705	2.920	21.021	0.703	0.292	0.619	0.527	0.843	1917.899	1082.113	1678.461	902.737	2988.459	0.650	0.125	0.646	0.534	0.764	0.173	0.090	0.164	0.497	0.073	0.501	0.331	0.067	0.330	-3815.092	-4289.968	-3320.708	0.010

Table C.75: weibull-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) RRLN data.

n	θ	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	12.507	47.626	4.104	1.491	49.230	1.939	5.150	1.008	0.627	5.130	931.051	1140.221	595.967	224.696	2926.941	1.354	4.143	0.823	0.578	2.954	0.197	0.123	0.172	0.512	0.138	0.518	0.290	0.127	0.286	-150.707	-196.634	-70.449	0.117
100	0.5	11.442	30.575	3.885	1.628	35.252	1.394	3.202	0.853	0.587	2.694	867.200	968.415	588.192	252.350	2190.371	1.013	3.503	0.712	0.536	1.518	0.183	0.108	0.164	0.520	0.112	0.522	0.297	0.113	0.300	-306.778	-382.810	-189.570	0.098
200	0.5	9.433	21.909	3.563	1.731	30.611	1.077	1.626	0.775	0.577	2.518	782.775	766.951	567.489	272.595	1970.466	0.756	1.092	0.647	0.516	1.158	0.173	0.094	0.158	0.521	0.092	0.521	0.307	0.098	0.311	-622.229	-748.968	-417.496	0.105
500	0.5	6.487	13.789	3.187	1.828	26.770	0.933	0.552	0.738	0.582	2.559	678.969	578.822	534.088	289.676	1910.423	0.636	0.324	0.599	0.500	0.970	0.160	0.078	0.150	0.521	0.076	0.520	0.319	0.082	0.323	-1564.610	-1822.435	-1031.283	0.130
1000	0.5	4.824	6.984	2.983	1.911	108.583	0.900	0.489	0.729	0.598	2.931	621.840	425.025	509.736	298.746	3924.665	0.591	0.117	0.576	0.493	1.531	0.152	0.067	0.147	0.521	0.066	0.519	0.326	0.071	0.329	-3130.110	-3584.266	-316.566	0.160

Table C.76: weibull-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) VOD data.

n	θ	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	8.812	20.661	3.983	1.568	35.336	1.837	3.592	1.088	0.681	5.197	375.936	523.767	207.779	75.346	1350.411	1.692	3.889	0.862	0.588	5.649	0.222	0.135	0.196	0.517	0.139	0.527	0.261	0.130	0.258	-127.325	-167.808	-54.353	0.123
100	0.5	8.664	31.051	3.851	1.678	27.936	1.407	2.346	0.926	0.634	2.823	360.997	473.796	211.583	81.149	1044.629	1.201	2.590	0.713	0.538	2.579	0.209	0.125	0.185	0.523	0.116	0.529	0.268	0.123	0.274	-258.911	-329.496	-144.429	0.107
200	0.5	7.647	17.979	3.618	1.804	29.821	1.168	1.318	0.852	0.622	2.645	331.380	406.258	208.482	85.532	1035.687	0.915	1.640	0.635	0.517	2.107	0.199	0.115	0.177	0.524	0.098	0.529	0.277	0.115	0.288	-525.087	-648.185	-308.650	0.122
500	0.5	6.016	9.736	3.340	2.002	-	1.022	0.789	0.815	0.636	-	280.105	285.730	201.592	93.209	-	0.696	0.751	0.580	0.504	-	0.185	0.099	0.166	0.522	0.081	0.527	0.292	0.103	0.305	-1333.949	-1595.042	-	0.171
1000	0.5	5.073	7.290	3.196	2.200	-	0.959	0.502	0.808	0.660	-	250.677	204.048	202.091	102.232	-	0.613	0.310	0.557	0.500	-	0.178	0.087	0.163	0.519	0.071	0.523	0.303	0.092	0.313	-2701.770	-3169.237	-	0.225

Table C.77: weibull-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) RIOTINTO data.

n	θ	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	31.193	145.153	9.011	1.891	90.694	2.019	4.132	1.016	0.594	5.049	3221.389	3486.388	2319.274	815.892	8056.935	1.217	1.971	0.893	0.613	2.149	0.183	0.125	0.151	0.485	0.155	0.494	0.332	0.132	0.325	-195.252	-257.970	-97.709	0.105
100	0.5	33.208	88.548	10.361	2.245	59.970	1.345	2.490	0.824	0.537	2.079	3088.448	2880.824	2400.842	927.444	5741.895	1.003	1.406	0.801	0.567	1.342	0.170	0.116	0.144	0.499	0.133	0.504	0.331	0.113	0.326	-393.820	-501.054	-263.909	0.050
200	0.5	32.237	260.121	10.969	2.413	49.470	0.961	1.232	0.703	0.508	1.337	2893.159	2188.582	2427.235	1031.468	4963.295	0.869	0.946	0.753	0.550	1.124	0.166	0.103	0.147	0.502	0.112	0.505	0.332	0.097	0.330	-794.051	-973.783	-592.394	0.029
500	0.5	24.285	50.327	12.207	2.700	38.319	0.758	0.549	0.615	0.496	0.973	2690.395	1569.371	2409.127	1185.225	4293.308	0.800	0.943	0.722	0.545	0.970	0.164	0.086	0.154	0.502	0.086	0.505	0.334	0.081	0.332	-2000.318	-2353.674	-1628.612	0.015
1000	0.5	19.095	29.222	13.040	2.921	32.333	0.695	0.401	0.583	0.495	0.827	2596.438	1317.509	2433.328	1277.274	3947.130	0.739	0.606	0.716	0.555	0.896	0.162	0.074	0.158	0.502	0.068	0.502	0.336	0.071	0.333	-4016.419	-4589.832	-3410.358	0.013

Table C.78: weibull-uniform-weibull mixture on limit order arrival zero inflated (uniformly distributed) SSELN data.

	AIC			BIC		
exponential/uniform/weibull	0.4559	-	-	0.8127	-	-
gamma/uniform/weibull	0.0191	0.0686	0.1594	0.0017	0.0686	0.1594
loglogistic/uniform/weibull	0.2611	0.4098	0.8260	0.1014	0.4098	0.8260
weibull/uniform/weibull	0.2545	0.5084	-	0.0749	0.5084	-

Table C.79: ABFLN, N = 50: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4533	-	-	0.7670	-	-
gamma/uniform/weibull	0.0240	0.0669	0.2495	0.0054	0.0669	0.2495
loglogistic/uniform/weibull	0.3576	0.5262	-	0.2014	0.5262	-
weibull/uniform/weibull	0.1618	0.4031	0.7419	0.0229	0.4031	0.7419

Table C.80: BARC, N = 50: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4767	-	-	0.8371	-	-
gamma/uniform/weibull	0.0201	0.0709	0.1615	0.0022	0.0709	0.1615
loglogistic/uniform/weibull	0.2609	0.4196	0.8275	0.0998	0.4196	0.8275
weibull/uniform/weibull	0.2348	0.4995	-	0.0535	0.4995	-

Table C.81: RRLN, N = 50: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4495	-	-	0.7828	-	-
gamma/uniform/weibull	0.0193	0.0560	0.2247	0.0030	0.0560	0.2247
loglogistic/uniform/weibull	0.3391	0.5116	-	0.1767	0.5116	-
weibull/uniform/weibull	0.1835	0.4220	0.7523	0.0289	0.4220	0.7523

Table C.82: VOD, N = 50: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4500	-	-	0.7565	-	-
gamma/uniform/weibull	0.0222	0.0686	0.2522	0.0047	0.0686	0.2522
loglogistic/uniform/weibull	0.3467	0.5051	-	0.2094	0.5051	-
weibull/uniform/weibull	0.1787	0.4236	0.7412	0.0271	0.4236	0.7412

Table C.83: RIOTINTO, N = 50: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4520	-	-	0.8119	-	-
gamma/uniform/weibull	0.0156	0.0649	0.1647	0.0010	0.0649	0.1647
loglogistic/uniform/weibull	0.2703	0.4198	0.8297	0.1107	0.4198	0.8297
weibull/uniform/weibull	0.2611	0.5110	-	0.0754	0.5110	-

Table C.84: SSELN, N = 50: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4340	-	-	0.8238	-	-
gamma/uniform/weibull	0.0161	0.0549	0.1190	0.0004	0.0549	0.1190
loglogistic/uniform/weibull	0.2666	0.4206	0.8702	0.0938	0.4206	0.8702
weibull/uniform/weibull	0.2758	0.5150	-	0.0744	0.5150	-

Table C.85: ABFLN, N = 100: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4066	0.6795	-	0.7566	-	-
gamma/uniform/weibull	0.0234	0.0615	0.2591	0.0013	0.0525	0.2591
loglogistic/uniform/weibull	0.4236	-	-	0.2287	0.5996	-
weibull/uniform/weibull	0.1449	0.2566	0.7380	0.0119	0.3463	0.7380

Table C.86: BARC, N = 100: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4609	-	-	0.8554	-	-
gamma/uniform/weibull	0.0208	0.0581	0.1188	0.0011	0.0581	0.1188
loglogistic/uniform/weibull	0.2610	0.4309	0.8718	0.0892	0.4309	0.8718
weibull/uniform/weibull	0.2510	0.5031	-	0.0480	0.5031	-

Table C.87: RRLN, N = 100: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4268	-	-	0.7824	-	-
gamma/uniform/weibull	0.0192	0.0499	0.2280	0.0009	0.0499	0.2280
loglogistic/uniform/weibull	0.3683	0.5532	-	0.1893	0.5532	-
weibull/uniform/weibull	0.1757	0.3858	0.7546	0.0174	0.3858	0.7546

Table C.88: VOD, N = 100: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4129	-	-	0.7342	-	-
gamma/uniform/weibull	0.0219	0.0561	0.2665	0.0009	0.0561	0.2665
loglogistic/uniform/weibull	0.4019	0.5722	-	0.2494	0.5722	-
weibull/uniform/weibull	0.1628	0.3712	0.7326	0.0150	0.3712	0.7326

Table C.89: RIOTINTO, N = 100: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4372	-	-	0.8207	-	-
gamma/uniform/weibull	0.0169	0.0560	0.1291	0.0005	0.0560	0.1291
loglogistic/uniform/weibull	0.2636	0.4226	0.8680	0.0992	0.4226	0.8680
weibull/uniform/weibull	0.2821	0.5193	-	0.0793	0.5193	-

Table C.90: SSELN, N = 100: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.3333	0.4913	-	0.7888	-	-
gamma/uniform/weibull	0.0121	0.0241	0.1329	0.0003	0.0356	0.0762
loglogistic/uniform/weibull	0.3475	-	-	0.1184	0.4768	0.9225
weibull/uniform/weibull	0.3068	0.4834	0.8649	0.0922	0.4869	-

Table C.91: ABFLN, N = 200: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.3192	0.6395	-	0.7279	-	-
gamma/uniform/weibull	0.0251	0.0846	0.2887	0.0008	0.0446	0.2887
loglogistic/uniform/weibull	0.5279	-	-	0.2624	0.6819	-
weibull/uniform/weibull	0.1258	0.2736	0.7089	0.0070	0.2715	0.7089

Table C.92: BARC, N = 200: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.3758	-	-	0.8396	-	-
gamma/uniform/weibull	0.0202	0.0460	0.0834	0.0003	0.0460	0.0834
loglogistic/uniform/weibull	0.3193	0.4669	0.9145	0.0977	0.4669	0.9145
weibull/uniform/weibull	0.2830	0.4850	-	0.0607	0.4850	-

Table C.93: RRLN, N = 200: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.3438	0.6313	-	0.7577	-	-
gamma/uniform/weibull	0.0202	0.0456	0.2441	0.0008	0.0427	0.2441
loglogistic/uniform/weibull	0.4594	-	-	0.2155	0.6260	-
weibull/uniform/weibull	0.1639	0.3075	0.7391	0.0134	0.3180	0.7391

Table C.94: VOD, N = 200: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.3411	0.6656	-	0.6971	-	-
gamma/uniform/weibull	0.0270	0.0643	0.3001	0.0004	0.0516	0.3001
loglogistic/uniform/weibull	0.4886	-	-	0.2928	0.6491	-
weibull/uniform/weibull	0.1432	0.2699	0.6997	0.0095	0.2992	0.6997

Table C.95: RIOTINTO, N = 200: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.3490	-	-	0.7881	-	-
gamma/uniform/weibull	0.0140	0.0424	0.0969	0.0003	0.0424	0.0969
loglogistic/uniform/weibull	0.3137	0.4489	0.9026	0.1136	0.4489	0.9026
weibull/uniform/weibull	0.3232	0.5083	-	0.0978	0.5083	-

Table C.96: SSELN, N = 200: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.1466	0.3027	0.9383	0.6391	-	-
gamma/uniform/weibull	0.0067	0.0193	0.0614	0.0002	0.0161	0.0963
loglogistic/uniform/weibull	0.5060	-	-	0.2068	0.5675	-
weibull/uniform/weibull	0.3406	0.6780	-	0.1538	0.4165	0.9037

Table C.97: ABFLN, N = 500: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.1592	0.5574	-	0.6481	-	-
gamma/uniform/weibull	0.0212	0.1389	0.3329	0.0004	0.0285	0.3329
loglogistic/uniform/weibull	0.7301	-	-	0.3463	0.8098	-
weibull/uniform/weibull	0.0894	0.3035	0.6668	0.0050	0.1616	0.6668

Table C.98: BARC, N = 500: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.1741	0.3473	0.9071	0.7403	-	-
gamma/uniform/weibull	0.0173	0.0470	0.0922	0.0002	0.0249	0.1241
loglogistic/uniform/weibull	0.4744	-	-	0.1482	0.5525	-
weibull/uniform/weibull	0.3342	0.6057	-	0.1113	0.4226	0.8759

Table C.99: RRLN, N = 500: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.1634	0.5452	-	0.6828	-	-
gamma/uniform/weibull	0.0189	0.0761	0.2997	0.0005	0.0289	0.2997
loglogistic/uniform/weibull	0.6838	-	-	0.2976	0.7721	-
weibull/uniform/weibull	0.1322	0.3759	0.6972	0.0173	0.1971	0.6972

Table C.100: VOD, N = 500: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.1887	0.6055	-	0.6210	-	-
gamma/uniform/weibull	0.0252	0.1109	0.3604	0.0005	0.0360	0.3604
loglogistic/uniform/weibull	0.6810	-	-	0.3717	0.7739	-
weibull/uniform/weibull	0.1052	0.2836	0.6396	0.0068	0.1901	0.6396

Table C.101: RIOTINTO, N = 500: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.1560	0.3155	0.9159	0.6603	-	-
gamma/uniform/weibull	0.0138	0.0343	0.0839	0.0002	0.0267	0.1143
loglogistic/uniform/weibull	0.4562	-	-	0.1798	0.5269	-
weibull/uniform/weibull	0.3740	0.6502	-	0.1597	0.4465	0.8857

Table C.102: SSELN, N = 500: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0367	0.1589	0.9290	0.4034	-	-
gamma/uniform/weibull	0.0061	0.0249	0.0705	0.0000	0.0080	0.0762
loglogistic/uniform/weibull	0.6060	-	-	0.3568	0.6229	-
weibull/uniform/weibull	0.3512	0.8157	-	0.2398	0.3691	0.9234

Table C.103: ABFLN, N = 1000: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0686	0.4854	-	0.5424	-	-
gamma/uniform/weibull	0.0132	0.1823	0.3581	0.0006	0.0160	0.3581
loglogistic/uniform/weibull	0.8592	-	-	0.4518	0.8926	-
weibull/uniform/weibull	0.0589	0.3321	0.6417	0.0051	0.0914	0.6417

Table C.104: BARC, N = 1000: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0602	0.2155	0.8819	0.5423	-	-
gamma/uniform/weibull	0.0107	0.0485	0.1174	0.0000	0.0117	0.0930
loglogistic/uniform/weibull	0.6035	-	-	0.2596	0.6316	-
weibull/uniform/weibull	0.3255	0.7357	-	0.1981	0.3566	0.9067

Table C.105: RRLN, N = 1000: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0640	0.4643	-	0.5639	-	-
gamma/uniform/weibull	0.0092	0.1015	0.3262	0.0001	0.0144	0.3262
loglogistic/uniform/weibull	0.8289	-	-	0.4103	0.8605	-
weibull/uniform/weibull	0.0977	0.4340	0.6736	0.0255	0.1249	0.6736

Table C.106: VOD, N = 1000: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0836	0.5393	-	0.5388	-	-
gamma/uniform/weibull	0.0156	0.1598	0.4097	0.0006	0.0194	0.4097
loglogistic/uniform/weibull	0.8257	-	-	0.4529	0.8627	-
weibull/uniform/weibull	0.0751	0.3009	0.5903	0.0076	0.1179	0.5903

Table C.107: RIOTINTO, N = 1000: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0605	0.1910	0.8945	0.4587	-	-
gamma/uniform/weibull	0.0075	0.0421	0.1055	0.0000	0.0129	0.1009
loglogistic/uniform/weibull	0.5530	-	-	0.2840	0.5817	-
weibull/uniform/weibull	0.3791	0.7669	-	0.2573	0.4053	0.8991

Table C.108: SSELN, N = 1000: limit order arrival times, zero inflated (uniformly distributed, $\theta = 0.5$).

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	-	-	-	-	-	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.332	3.358	1.850	0.342	7.718	-	-	-	-	-	12.046	26.556	9.319	7.345	13.032	2386.799	3180.591	1456.539	482.837	4070.571	0.732	1.549	0.609	0.427	0.914	0.158	0.108	0.132	0.428	0.149	0.436	0.413	0.153	0.405	-182.891	-255.610	-109.811	0.011
100	2.631	2.972	1.197	0.282	6.965	-	-	-	-	-	10.506	18.370	9.096	7.647	11.379	2116.424	2403.161	1418.615	587.489	3532.974	0.617	0.446	0.565	0.416	0.788	0.140	0.095	0.118	0.448	0.133	0.455	0.413	0.132	0.407	-370.610	-490.802	-251.312	0.009
200	1.952	2.548	0.812	0.224	4.345	-	-	-	-	-	9.405	6.442	8.905	7.848	10.294	1972.341	1815.964	1421.407	685.620	3181.836	0.580	0.191	0.547	0.418	0.726	0.124	0.080	0.107	0.469	0.114	0.473	0.407	0.113	0.405	-743.160	-944.515	-545.538	0.002
500	1.179	1.867	0.533	0.175	1.676	-	-	-	-	-	8.968	10.054	8.710	7.970	9.615	1907.920	1479.069	1451.479	796.853	3029.213	0.565	0.155	0.542	0.424	0.696	0.109	0.061	0.096	0.489	0.090	0.492	0.402	0.089	0.399	-1870.458	-2258.002	-1484.204	0.001
1000	0.823	1.350	0.468	0.160	1.175	-	-	-	-	-	8.678	0.633	8.655	8.057	9.301	1882.608	1319.379	1481.099	863.086	2939.460	0.558	0.139	0.538	0.428	0.677	0.103	0.051	0.093	0.497	0.072	0.498	0.400	0.074	0.398	-3753.097	-4422.251	-3100.930	0.001

Table C.109: exponential-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) ABFLN data.

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	-	-	-	-	-	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	2.075	2.679	0.935	0.276	4.819	-	-	-	-	-	17.060	1457.169	9.939	7.718	14.439	263.137	341.494	158.640	46.841	464.038	0.876	1.962	0.657	0.473	1.021	0.242	0.143	0.216	0.399	0.166	0.423	0.359	0.149	0.353	-130.273	-183.633	-74.464	0.009
100	1.674	2.262	0.800	0.287	2.441	-	-	-	-	-	12.892	67.012	9.740	8.115	12.518	226.059	245.185	150.986	54.017	386.599	0.647	0.900	0.576	0.450	0.781	0.213	0.126	0.187	0.417	0.158	0.446	0.370	0.132	0.365	-264.853	-356.547	-170.728	0.003
200	1.311	1.810	0.694	0.301	1.867	-	-	-	-	-	11.509	31.990	9.584	8.369	11.530	201.095	187.718	145.514	60.169	336.437	0.562	0.308	0.536	0.444	0.667	0.192	0.111	0.166	0.431	0.151	0.463	0.378	0.118	0.371	-535.756	-695.079	-369.267	0.002
500	0.970	1.158	0.611	0.324	1.548	-	-	-	-	-	10.359	9.322	9.448	8.606	10.767	179.886	140.399	139.461	66.893	291.640	0.518	0.081	0.508	0.444	0.591	0.175	0.097	0.151	0.441	0.145	0.476	0.384	0.105	0.376	-1347.705	-1692.080	-983.085	0.000
1000	0.851	0.792	0.577	0.344	1.478	-	-	-	-	-	10.146	4.712	9.401	8.748	10.415	169.396	117.279	136.709	71.002	268.311	0.502	0.060	0.496	0.445	0.559	0.169	0.091	0.145	0.443	0.142	0.480	0.388	0.098	0.378	-2703.942	-3307.073	-2020.932	0.000

Table C.110: exponential-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) BARC data.

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	-	-	-	-	-	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.794	3.715	2.302	0.409	8.124	-	-	-	-	-	12.923	43.628	9.393	7.380	13.451	1445.104	1901.037	888.217	285.706	2489.908	0.691	2.155	0.569	0.400	0.857	0.174	0.119	0.148	0.402	0.145	0.410	0.424	0.151	0.418	-175.428	-238.460	-112.243	0.010
100	3.140	3.236	1.542	0.332	7.670	-	-	-	-	-	12.377	247.291	9.170	7.697	11.465	1232.521	1283.002	859.023	359.756	2052.446	0.575	0.663	0.522	0.391	0.728	0.151	0.105	0.129	0.423	0.129	0.430	0.426	0.128	0.421	-356.119	-456.228	-255.690	0.008
200	2.473	2.967	0.989	0.274	7.160	-	-	-	-	-	10.004	24.763	8.956	7.870	10.371	1125.934	979.276	860.834	430.334	1749.656	0.529	0.176	0.501	0.390	0.662	0.131	0.090	0.112	0.446	0.113	0.450	0.423	0.109	0.420	-715.530	-879.113	-557.484	0.003
500	1.660	2.491	0.609	0.229	2.318	-	-	-	-	-	9.156	16.247	8.797	8.069	9.671	1062.075	727.641	871.832	511.312	1556.024	0.510	0.189	0.489	0.396	0.620	0.113	0.078	0.098	0.470	0.090	0.476	0.418	0.087	0.416	-1797.370	-2102.820	-1498.292	0.001
1000	1.146	1.973	0.495	0.220	1.390	-	-	-	-	-	8.936	4.211	8.765	8.197	9.370	1048.369	633.378	873.886	570.620	1510.948	0.501	0.100	0.487	0.404	0.601	0.105	0.071	0.091	0.481	0.076	0.487	0.415	0.072	0.412	-3604.762	-4108.249	-3100.089	0.001

Table C.111: exponential-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) RRLN data.

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	-	-	-	-	-	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	2.250	2.670	1.232	0.319	5.318	-	-	-	-	-	11.813	68.689	9.552	7.575	13.096	591.138	746.928	368.038	112.448	1060.875	0.858	8.689	0.630	0.451	0.941	0.170	0.111	0.146	0.466	0.137	0.473	0.364	0.128	0.361	-138.295	-191.328	-84.308	0.016
100	1.701	2.055	1.030	0.311	2.706	-	-	-	-	-	10.178	11.711	9.370	7.918	11.514	516.058	540.460	357.875	137.418	903.410	0.923	64.478	0.567	0.439	0.765	0.151	0.089	0.136	0.480	0.113	0.484	0.369	0.107	0.368	-282.000	-367.716	-195.449	0.015
200	1.268	1.400	0.893	0.314	2.050	-	-	-	-	-	9.515	5.367	9.251	8.184	10.673	471.009	418.206	350.280	157.218	801.621	0.556	0.279	0.533	0.437	0.671	0.136	0.069	0.127	0.491	0.091	0.492	0.373	0.089	0.373	-570.884	-713.524	-425.799	0.016
500	1.010	0.783	0.830	0.341	1.714	-	-	-	-	-	9.235	0.862	9.182	8.442	10.021	429.778	324.470	343.561	174.241	681.075	0.518	0.107	0.509	0.439	0.596	0.125	0.052	0.119	0.502	0.076	0.498	0.373	0.078	0.378	-1415.862	-1727.578	-1133.474	0.004
1000	0.957	0.598	0.817	0.372	1.618	-	-	-	-	-	9.179	0.607	9.147	8.595	9.772	404.384	267.230	334.318	181.076	624.462	0.502	0.066	0.497	0.440	0.562	0.120	0.043	0.115	0.504	0.064	0.501	0.376	0.065	0.380	-2833.409	-3383.235	-2324.448	0.000

Table C.112: exponential-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) VOD data.

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	-	-	-	-	-	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	2.368	2.771	1.313	0.351	5.646	-	-	-	-	-	11.695	35.832	9.711	7.703	13.103	225.221	335.291	116.392	35.394	390.401	0.837	1.661	0.647	0.467	0.985	0.189	0.122	0.162	0.459	0.132	0.469	0.352	0.130	0.352	-115.045	-161.861	-67.509	0.006
100	1.866	2.258	1.124	0.334	2.677	-	-	-	-	-	10.336	16.125	9.516	8.044	11.554	194.486	253.655	110.628	39.358	334.251	0.632	0.592	0.571	0.449	0.764	0.166	0.102	0.147	0.474	0.111	0.480	0.361	0.108	0.362	-234.207	-313.915	-155.409	0.002
200	1.425	1.712	0.992	0.330	1.996	-	-	-	-	-	9.654	5.698	9.399	8.308	10.751	172.833	195.925	106.019	42.606	297.888	0.557	0.246	0.532	0.445	0.658	0.148	0.083	0.135	0.485	0.092	0.487	0.367	0.089	0.369	-473.635	-613.009	-338.181	0.001
500	1.059	0.986	0.886	0.338	1.674	-	-	-	-	-	9.393	1.221	9.323	8.565	10.198	154.728	151.041	101.718	45.313	267.107	0.517	0.077	0.507	0.447	0.587	0.134	0.065	0.125	0.492	0.074	0.493	0.374	0.069	0.376	-1193.499	-1499.660	-902.693	0.000
1000	0.968	0.691	0.870	0.356	1.581	-	-	-	-	-	9.331	0.643	9.303	8.711	9.946	145.190	128.584	99.003	46.636	251.411	0.502	0.058	0.496	0.448	0.556	0.129	0.055	0.121	0.493	0.063	0.494	0.378	0.059	0.380	-2395.839	-2969.115	-1861.763	0.000

Table C.113: exponential-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) RIOTINTO data.

n	$\hat{\lambda}_{1\,ave}$	$\pm\hat{\lambda}_{1\,ave}$	$\hat{\lambda}_{1\,median}$	$\hat{\lambda}_{1\,lower}$	$\hat{\lambda}_{1\,upper}$	-	-	-	-	-	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.107	3.550	2.795	0.512	8.243	-	-	-	-	-	15.741	536.878	9.377	7.361	13.324	1962.528	2575.535	1247.909	391.775	3187.737	0.662	0.671	0.589	0.420	0.853	0.166	0.103	0.147	0.399	0.145	0.404	0.435	0.158	0.424	-184.439	-256.979	-113.679	0.003
100	3.392	3.280	1.836	0.402	7.778	-	-	-	-	-	10.840	16.211	9.154	7.688	11.436	1707.050	1872.655	1193.001	478.099	2731.647	0.583	0.225	0.546	0.411	0.739	0.142	0.089	0.127	0.422	0.133	0.428	0.436	0.135	0.428	-373.537	-492.453	-256.597	0.001
200	2.651	2.986	1.157	0.327	7.168	-	-	-	-	-	9.658	9.139	8.928	7.851	10.353	1537.490	1331.066	1174.218	570.496	2409.339	0.551	0.160	0.522	0.410	0.681	0.120	0.073	0.107	0.446	0.116	0.451	0.435	0.113	0.427	-754.412	-948.000	-560.152	0.000
500	1.652	2.434	0.662	0.240	2.340	-	-	-	-	-	8.804	1.333	8.716	7.989	9.534	1487.653	1007.896	1210.379	684.795	2281.574	0.539	0.133	0.516	0.419	0.652	0.097	0.050	0.087	0.474	0.091	0.480	0.429	0.088	0.421	-1900.243	-2276.794	-1526.326	0.000
1000	1.142	1.943	0.530	0.213	1.365	-	-	-	-	-	8.639	0.610	8.622	8.058	9.216	1475.215	889.069	1205.017	760.361	2219.071	0.533	0.115	0.515	0.430	0.630	0.088	0.039	0.080	0.487	0.073	0.489	0.426	0.072	0.421	-3812.833	-4421.154	-3202.368	0.000

Table C.114: exponential-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) SSELN data.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.403	4.096	2.018	0.433	10.223	0.920	0.166	0.952	0.770	1.130	9.047	3.947	8.599	6.794	12.902	2323.328	3309.511	1409.963	480.910	6129.770	0.732	2.493	0.608	0.436	1.035	0.095	0.087	0.073	0.490	0.166	0.494	0.415	0.153	0.405	-182.945	-250.258	-81.157	0.095
100	3.409	3.649	2.200	0.627	7.078	0.973	0.121	0.983	0.895	1.084	8.694	1.926	8.525	7.124	10.532	1975.542	2291.242	1360.042	554.321	3492.826	0.580	0.377	0.551	0.414	0.762	0.084	0.071	0.066	0.493	0.139	0.495	0.423	0.131	0.416	-372.420	-489.244	-240.665	0.036
200	3.044	3.189	2.084	0.757	5.303	1.006	0.080	1.001	0.950	1.077	8.599	1.261	8.535	7.456	9.789	1762.657	1654.581	1307.229	626.791	2784.122	0.536	0.132	0.522	0.408	0.664	0.078	0.061	0.064	0.497	0.117	0.499	0.424	0.115	0.420	-745.634	-944.707	-543.857	0.007
500	2.528	2.484	1.949	0.931	3.511	1.027	0.050	1.018	0.985	1.077	8.546	0.807	8.504	7.798	9.305	1576.956	1166.377	1270.516	729.507	2351.349	0.508	0.098	0.501	0.407	0.604	0.073	0.046	0.063	0.501	0.091	0.503	0.426	0.092	0.424	-1873.058	-2265.378	-1488.904	0.000
1000	2.242	2.059	1.874	1.047	2.844	1.030	0.041	1.023	0.995	1.070	8.551	0.607	8.536	7.963	9.121	1476.670	903.019	1254.102	787.468	2125.246	0.495	0.084	0.494	0.408	0.579	0.069	0.036	0.064	0.503	0.072	0.504	0.428	0.075	0.426	-3760.587	-4421.067	-3113.759	0.000

Table C.115: gamma-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) ABFLN data.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	2.646	2.689	1.827	0.576	5.191	0.958	0.104	0.967	0.888	1.046	10.980	85.708	9.127	6.981	12.633	234.566	313.984	136.907	39.487	431.212	0.757	2.130	0.622	0.454	0.927	0.169	0.130	0.138	0.453	0.183	0.477	0.378	0.149	0.366	-130.307	-183.121	-72.585	0.019
100	2.333	2.216	1.737	0.624	3.731	0.984	0.068	0.982	0.931	1.041	10.157	23.369	9.143	7.561	11.458	195.784	222.644	127.641	46.067	333.316	0.592	1.859	0.543	0.432	0.704	0.156	0.111	0.132	0.450	0.168	0.478	0.394	0.131	0.381	-265.178	-356.755	-170.687	0.004
200	2.027	1.761	1.654	0.655	2.854	0.995	0.048	0.990	0.953	1.040	9.752	6.490	9.156	7.977	10.864	168.532	161.946	121.177	52.072	276.223	0.521	0.405	0.505	0.425	0.609	0.147	0.095	0.128	0.449	0.155	0.479	0.404	0.117	0.390	-536.084	-695.483	-369.981	0.002
500	1.720	1.195	1.595	0.673	2.395	1.000	0.037	0.995	0.966	1.037	9.537	2.290	9.141	8.327	10.365	149.199	116.460	116.737	59.265	234.174	0.487	0.065	0.483	0.423	0.550	0.141	0.081	0.124	0.449	0.144	0.481	0.410	0.106	0.396	-1348.327	-1693.472	-984.607	0.000
1000	1.589	0.878	1.601	0.674	2.245	1.001	0.033	0.998	0.969	1.034	9.497	1.896	9.133	8.493	10.125	141.003	94.854	115.586	64.247	215.190	0.476	0.052	0.473	0.424	0.527	0.139	0.073	0.123	0.449	0.138	0.483	0.413	0.099	0.397	-2705.351	-3306.636	-2022.287	0.000

Table C.116: gamma-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) BARC data.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.695	4.164	2.294	0.512	10.094	0.928	0.157	0.956	0.813	1.113	9.148	10.357	8.590	6.739	12.769	1454.860	1927.815	898.553	307.872	3635.982	0.703	2.014	0.588	0.416	1.002	0.101	0.095	0.076	0.481	0.165	0.487	0.418	0.146	0.408	-175.702	-234.898	-90.270	0.086
100	3.604	3.719	2.383	0.715	7.218	0.976	0.109	0.983	0.906	1.073	8.771	3.359	8.563	7.128	10.508	1227.695	1346.932	867.402	363.550	2134.013	0.565	0.561	0.527	0.397	0.725	0.088	0.079	0.068	0.486	0.137	0.492	0.426	0.122	0.420	-356.798	-455.545	-250.161	0.027
200	3.171	3.138	2.202	0.842	5.603	1.004	0.070	0.999	0.952	1.065	8.652	1.823	8.530	7.468	9.800	1066.717	909.418	833.658	421.323	1655.767	0.511	0.143	0.492	0.390	0.629	0.081	0.067	0.064	0.489	0.113	0.493	0.430	0.103	0.426	-716.052	-879.501	-557.905	0.005
500	2.639	2.451	2.009	0.981	3.927	1.020	0.045	1.012	0.983	1.064	8.620	1.051	8.556	7.853	9.382	931.927	577.791	792.889	482.360	1340.834	0.479	0.096	0.466	0.388	0.568	0.074	0.053	0.063	0.492	0.087	0.495	0.434	0.081	0.433	-1798.532	-2102.270	-1497.469	0.001
1000	2.248	1.663	1.910	1.050	2.986	1.026	0.038	1.018	0.993	1.064	8.604	1.046	8.559	8.026	9.147	883.277	460.243	783.025	523.972	1239.176	0.468	0.079	0.458	0.390	0.547	0.071	0.043	0.063	0.493	0.071	0.497	0.436	0.068	0.434	-3610.143	-4107.591	-3101.676	0.001

Table C.117: gamma-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) RRLN data.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	2.656	3.038	1.604	0.484	6.338	0.943	0.128	0.960	0.863	1.064	9.311	4.707	8.846	6.902	12.070	544.607	683.706	342.171	106.274	1104.039	0.840	9.973	0.613	0.442	0.936	0.127	0.100	0.104	0.502	0.145	0.505	0.371	0.125	0.365	-138.536	-190.110	-78.425	0.046
100	2.343	2.486	1.543	0.544	4.540	0.978	0.083	0.980	0.918	1.051	9.029	2.303	8.835	7.380	10.786	468.233	483.507	328.659	127.664	829.497	0.666	4.865	0.547	0.426	0.723	0.117	0.080	0.102	0.502	0.118	0.503	0.381	0.103	0.378	-282.993	-367.450	-195.005	0.020
200	1.957	1.927	1.427	0.576	3.014	0.994	0.056	0.990	0.948	1.048	8.958	1.288	8.870	7.797	10.200	414.651	352.655	317.670	147.392	700.809	0.575	7.200	0.511	0.422	0.631	0.111	0.066	0.102	0.501	0.095	0.501	0.387	0.086	0.387	-571.932	-714.142	-426.251	0.017
500	1.548	1.294	1.288	0.590	2.257	1.000	0.041	0.995	0.963	1.038	8.973	0.837	8.921	8.203	9.747	377.998	268.184	312.614	166.389	581.127	0.497	0.087	0.491	0.424	0.568	0.109	0.052	0.103	0.504	0.078	0.501	0.387	0.079	0.390	-1415.250	-1728.291	-1134.873	0.002
1000	1.402	1.052	1.247	0.610	2.070	1.001	0.034	0.997	0.969	1.032	8.974	0.601	8.948	8.392	9.549	359.458	215.404	308.784	176.583	540.529	0.485	0.062	0.482	0.424	0.543	0.107	0.041	0.103	0.505	0.064	0.501	0.388	0.066	0.391	-2834.991	-3386.332	-2326.003	0.000

Table C.118: gamma-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) VOD data.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	2.877	3.086	1.762	0.550	6.195	0.953	0.104	0.963	0.883	1.039	9.558	50.656	8.977	6.995	11.854	208.059	311.598	108.034	33.181	378.552	0.750	1.450	0.629	0.456	0.949	0.143	0.108	0.118	0.500	0.143	0.506	0.357	0.122	0.354	-115.010	-161.250	-65.335	0.020
100	2.503	2.630	1.589	0.582	4.718	0.978	0.066	0.979	0.927	1.032	9.203	2.681	8.984	7.528	10.814	173.411	228.395	99.294	36.098	292.403	0.588	0.792	0.548	0.435	0.716	0.128	0.089	0.111	0.498	0.120	0.501	0.374	0.101	0.373	-234.395	-314.063	-155.415	0.002
200	2.116	2.156	1.429	0.598	3.211	0.990	0.047	0.987	0.949	1.031	9.127	1.509	9.011	7.925	10.303	148.612	171.759	93.283	39.060	247.094	0.525	0.133	0.510	0.429	0.617	0.118	0.072	0.108	0.496	0.099	0.497	0.386	0.083	0.385	-473.965	-613.226	-337.996	0.000
500	1.709	1.576	1.291	0.616	2.413	0.996	0.036	0.992	0.963	1.030	9.131	5.001	9.055	8.299	9.908	130.248	123.990	89.482	42.548	215.315	0.493	0.068	0.487	0.428	0.559	0.112	0.055	0.106	0.494	0.077	0.494	0.394	0.066	0.393	-1194.598	-1500.733	-903.469	0.000
1000	1.512	1.213	1.230	0.623	2.199	0.998	0.032	0.993	0.967	1.029	9.104	0.643	9.071	8.475	9.727	122.859	104.457	88.312	44.790	202.616	0.482	0.054	0.479	0.430	0.535	0.111	0.046	0.106	0.492	0.064	0.493	0.397	0.056	0.396	-2397.946	-2972.811	-1862.574	0.000

Table C.119: gamma-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) RIOTINTO data.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.412	4.097	2.002	0.424	10.195	0.915	0.165	0.946	0.770	1.121	9.035	3.780	8.581	6.798	12.831	2041.271	2765.579	1289.241	434.446	5202.659	0.670	0.460	0.616	0.449	1.029	0.089	0.081	0.068	0.487	0.165	0.491	0.424	0.153	0.413	-183.948	-250.961	-85.423	0.094
100	3.399	3.706	2.192	0.601	6.878	0.968	0.119	0.979	0.891	1.073	8.737	4.629	8.531	7.130	10.468	1722.609	1920.137	1216.811	500.604	2962.665	0.581	0.220	0.555	0.425	0.750	0.077	0.063	0.061	0.491	0.138	0.495	0.433	0.131	0.425	-374.356	-490.607	-246.496	0.029
200	3.037	3.146	2.075	0.740	5.244	1.001	0.076	0.996	0.948	1.065	8.577	1.233	8.485	7.456	9.719	1511.965	1298.825	1176.586	578.709	2347.471	0.537	0.127	0.520	0.419	0.656	0.069	0.049	0.058	0.494	0.114	0.496	0.437	0.109	0.431	-754.450	-949.148	-557.983	0.003
500	2.454	2.194	1.916	0.912	3.560	1.021	0.048	1.011	0.984	1.066	8.539	0.785	8.502	7.789	9.300	1343.040	849.438	1137.653	663.976	1967.693	0.508	0.095	0.498	0.417	0.599	0.062	0.036	0.056	0.496	0.087	0.499	0.441	0.085	0.436	-1901.373	-2275.297	-1526.978	0.000
1000	2.191	1.831	1.860	1.020	2.937	1.027	0.039	1.018	0.993	1.065	8.530	0.587	8.512	7.987	9.092	1278.151	684.371	1112.198	711.472	1824.752	0.497	0.079	0.491	0.418	0.573	0.060	0.029	0.055	0.498	0.069	0.498	0.443	0.068	0.440	-3814.708	-4427.988	-3204.275	0.000

Table C.120: gamma-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) SSELN data.

n	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	30.239	941.623	6.187	1.000	48.987	9.540	24.148	1.394	0.766	19.456	9.420	12.675	8.592	6.713	12.294	3916.972	4302.970	2725.834	906.854	8331.099	1.419	5.262	0.956	0.623	1.935	0.200	0.139	0.168	0.485	0.157	0.494	0.316	0.128	0.311	-183.001	-249.562	-96.513	0.067
100	21.644	477.616	5.350	1.000	33.170	6.397	18.698	1.055	0.692	3.647	9.081	17.176	8.542	7.160	10.622	3664.149	3465.688	2751.075	999.068	6842.336	1.038	1.639	0.849	0.581	1.381	0.198	0.127	0.176	0.488	0.139	0.494	0.314	0.111	0.312	-368.611	-482.926	-238.825	0.039
200	16.701	240.035	4.797	1.002	26.113	5.179	16.661	0.876	0.654	1.959	8.749	2.492	8.575	7.562	9.885	3458.621	2757.066	2817.029	1112.255	5988.401	0.906	0.923	0.800	0.571	1.166	0.198	0.113	0.184	0.491	0.121	0.493	0.310	0.097	0.307	-736.713	-932.282	-532.842	0.019
500	9.910	13.400	4.326	1.126	20.741	3.594	13.493	0.750	0.624	1.167	8.701	0.827	8.645	7.920	9.478	3343.968	2265.994	2853.569	1275.731	5547.634	0.848	0.827	0.774	0.582	1.044	0.202	0.094	0.194	0.491	0.091	0.490	0.307	0.081	0.303	-1856.325	-2237.531	-1454.739	0.010
1000	8.638	10.469	4.364	1.284	17.438	1.904	8.611	0.710	0.614	0.953	8.730	0.629	8.674	8.150	9.320	3309.881	1949.035	2923.207	1413.533	5260.671	0.807	0.350	0.759	0.601	0.990	0.207	0.083	0.202	0.489	0.072	0.488	0.304	0.072	0.300	-3733.446	-4376.660	-3085.908	0.005

Table C.121: loglogistic-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) ABFLN data.

n	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.497	48.468	1.985	1.000	8.088	27.799	440.428	2.507	1.099	102.843	10.890	40.081	8.941	7.047	12.786	393.901	569.184	223.907	68.254	802.163	1.307	3.006	0.763	0.527	1.753	0.215	0.155	0.174	0.469	0.181	0.499	0.316	0.160	0.307	-128.001	-179.006	-65.609	0.046
100	3.785	27.545	1.833	1.000	5.282	27.485	43.573	1.961	0.970	95.855	16.324	2744.989	8.948	7.554	11.093	348.540	470.105	211.105	68.989	614.447	0.912	2.123	0.652	0.488	1.093	0.210	0.145	0.177	0.468	0.170	0.502	0.322	0.154	0.317	-257.454	-348.280	-159.695	0.012
200	2.900	20.212	1.714	1.000	3.892	28.015	42.308	1.704	0.926	92.541	9.632	7.464	8.969	7.943	10.459	298.606	340.187	198.725	72.785	514.765	0.703	0.674	0.593	0.473	0.856	0.201	0.131	0.175	0.467	0.158	0.502	0.333	0.147	0.327	-519.397	-679.105	-350.735	0.004
500	2.036	2.867	1.611	1.000	2.874	29.222	41.298	1.575	0.928	89.291	9.296	1.877	9.000	8.279	10.040	249.380	221.632	182.557	77.864	423.205	0.592	0.199	0.551	0.466	0.705	0.187	0.112	0.172	0.467	0.148	0.503	0.346	0.138	0.337	-1305.972	-1655.791	-937.810	0.001
1000	1.761	1.562	1.575	1.000	2.492	29.504	40.480	1.522	0.953	87.351	9.235	1.131	9.017	8.412	9.846	228.005	176.633	175.684	79.959	387.027	0.558	0.113	0.536	0.466	0.647	0.179	0.101	0.170	0.467	0.142	0.503	0.354	0.133	0.342	-2622.561	-3242.641	-1930.555	0.001

Table C.122: loglogistic-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) BARC data.

n	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	12.177	30.469	3.441	0.799	24.174	12.611	27.614	1.840	0.852	16.931	10.009	23.878	8.721	6.798	11.678	2579.177	3037.392	1665.641	535.625	4724.447	1.167	4.212	0.825	0.540	1.414	0.192	0.148	0.150	0.480	0.160	0.493	0.328	0.134	0.322	-172.955	-235.516	-108.082	0.019
100	11.827	99.945	3.612	1.000	21.585	8.858	22.501	1.234	0.748	4.546	9.342	17.094	8.599	7.196	10.484	2409.193	2505.618	1671.131	618.815	4219.227	0.880	1.761	0.745	0.513	1.113	0.192	0.137	0.161	0.483	0.139	0.492	0.325	0.117	0.321	-352.363	-453.062	-252.332	0.008
200	9.571	36.989	3.677	1.000	18.737	6.776	19.385	0.966	0.693	2.438	9.041	6.078	8.638	7.619	9.883	2259.197	1995.041	1681.961	708.244	3825.912	0.765	0.500	0.708	0.511	0.970	0.193	0.122	0.174	0.487	0.117	0.493	0.320	0.102	0.317	-709.163	-874.079	-554.589	0.002
500	7.543	9.541	3.592	1.193	15.040	4.687	15.778	0.813	0.665	1.437	8.814	1.089	8.694	8.006	9.528	2127.199	1509.348	1722.693	840.783	3468.712	0.703	0.275	0.676	0.522	0.862	0.198	0.106	0.189	0.485	0.089	0.491	0.316	0.084	0.314	-1788.357	-2094.354	-1488.031	0.001
1000	6.505	7.149	3.515	1.346	12.658	3.178	12.443	0.762	0.656	1.119	8.816	0.879	8.747	8.209	9.350	2063.711	1275.022	1728.409	911.890	3270.205	0.679	0.164	0.666	0.542	0.803	0.201	0.094	0.194	0.484	0.071	0.489	0.314	0.073	0.313	-3593.206	-4095.543	-3087.896	0.001

Table C.123: loglogistic-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) RRLN data.

n	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	6.221	33.721	1.905	0.864	10.670	22.000	39.585	2.338	0.955	86.798	10.056	30.134	8.805	6.966	11.780	904.922	1247.771	534.296	164.877	1660.059	1.227	4.262	0.766	0.522	1.446	0.179	0.132	0.140	0.511	0.141	0.519	0.310	0.132	0.307	-136.453	-188.815	-79.990	0.027
100	5.360	41.585	1.734	0.999	7.721	21.047	41.825	1.649	0.835	83.387	9.492	15.897	8.759	7.433	10.607	818.684	1028.457	512.269	182.488	1411.532	0.890	3.124	0.672	0.492	1.063	0.175	0.123	0.145	0.511	0.122	0.518	0.314	0.117	0.316	-278.001	-364.152	-190.300	0.014
200	3.727	13.842	1.570	1.000	5.364	21.410	50.960	1.318	0.787	82.194	9.145	7.596	8.761	7.806	10.053	727.073	774.413	491.000	197.652	1252.704	0.701	0.907	0.618	0.480	0.879	0.169	0.111	0.149	0.511	0.103	0.516	0.320	0.102	0.324	-561.985	-707.954	-414.535	0.014
500	2.387	5.219	1.399	0.999	3.239	23.186	35.975	1.176	0.772	80.248	8.946	1.689	8.809	8.135	9.656	634.564	570.831	462.459	205.322	1053.847	0.608	0.184	0.577	0.471	0.738	0.158	0.096	0.150	0.516	0.089	0.514	0.325	0.090	0.330	-1392.048	-1715.080	-1101.193	0.004
1000	1.914	2.293	1.306	0.999	2.625	23.904	35.643	1.124	0.784	78.667	8.923	0.987	8.846	8.286	9.509	590.151	483.709	441.718	208.475	976.993	0.578	0.154	0.561	0.471	0.682	0.152	0.088	0.151	0.517	0.074	0.513	0.331	0.078	0.334	-2788.664	-3364.670	-2252.625	0.001

Table C.124: loglogistic-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) VOD data.

n	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.826	43.124	1.477	0.139	9.880	23.042	42.164	2.302	1.039	111.670	8.986	17.877	8.537	6.296	12.849	327.834	528.325	153.600	48.378	946.147	1.213	2.533	0.733	0.516	2.194	0.198	0.142	0.157	0.491	0.144	0.507	0.311	0.136	0.312	-115.172	-158.333	-51.504	0.091
100	4.199	52.105	1.445	0.475	5.849	27.229	47.723	1.834	0.925	102.144	9.029	8.163	8.724	7.316	10.754	297.372	455.713	144.987	49.104	609.937	0.900	1.659	0.640	0.488	1.108	0.189	0.135	0.149	0.500	0.127	0.515	0.312	0.124	0.320	-230.449	-309.126	-139.867	0.038
200	3.011	25.516	1.385	0.999	4.120	31.960	45.556	1.683	0.904	99.077	9.003	5.515	8.802	7.838	10.040	256.651	345.792	135.703	51.300	478.258	0.698	0.804	0.587	0.476	0.822	0.172	0.119	0.137	0.509	0.105	0.520	0.319	0.109	0.330	-461.606	-605.754	-313.655	0.015
500	2.016	5.245	1.300	1.000	2.981	37.347	45.675	1.663	0.922	96.141	8.940	2.730	8.824	8.197	9.619	217.195	239.967	126.530	53.811	394.466	0.588	0.234	0.552	0.475	0.679	0.152	0.100	0.122	0.518	0.080	0.524	0.331	0.090	0.341	-1157.276	-1486.340	-846.282	0.005
1000	1.801	13.324	1.248	1.000	2.526	39.811	45.032	1.658	0.951	93.958	8.899	0.801	8.822	8.331	9.472	198.656	196.100	120.973	54.823	359.062	0.556	0.123	0.538	0.477	0.630	0.142	0.090	0.115	0.521	0.068	0.526	0.337	0.078	0.347	-2318.147	-2943.818	-1737.136	0.002

Table C.125: loglogistic-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) RIOTINTO data.

n	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	30.510	631.213	5.833	1.000	50.715	10.787	25.409	1.521	0.799	23.117	9.481	14.938	8.580	6.709	12.008	3343.795	3588.635	2399.078	738.660	6646.619	1.305	2.529	0.923	0.600	1.819	0.189	0.138	0.155	0.484	0.158	0.495	0.327	0.131	0.321	-183.392	-250.763	-101.871	0.056
100	24.580	391.722	5.260	1.000	37.947	7.537	20.385	1.104	0.707	4.318	8.970	7.066	8.506	7.124	10.510	3102.795	2950.305	2387.836	840.448	5554.779	1.055	1.782	0.815	0.551	1.355	0.187	0.129	0.161	0.487	0.138	0.494	0.326	0.115	0.320	-370.836	-483.678	-245.794	0.029
200	16.938	65.895	4.953	1.000	31.197	5.530	17.291	0.881	0.660	2.204	8.737	3.395	8.539	7.512	9.856	2901.014	2243.370	2414.125	969.903	4957.656	0.936	1.302	0.764	0.541	1.177	0.188	0.115	0.171	0.489	0.116	0.493	0.323	0.101	0.318	-747.392	-934.798	-544.416	0.018
500	12.427	23.489	4.742	1.094	25.071	4.317	15.034	0.751	0.629	1.357	8.627	0.820	8.563	7.897	9.368	2758.585	1738.482	2398.441	1152.462	4521.755	0.880	1.388	0.735	0.546	1.069	0.190	0.099	0.180	0.489	0.087	0.491	0.320	0.088	0.315	-1884.722	-2251.433	-1500.684	0.013
1000	10.728	18.389	5.193	1.244	21.657	3.015	12.048	0.704	0.615	1.022	8.627	0.544	8.606	8.089	9.176	2706.100	1482.094	2425.347	1239.105	4295.958	0.785	0.277	0.726	0.560	1.007	0.193	0.088	0.186	0.489	0.068	0.489	0.318	0.080	0.314	-3792.040	-4392.801	-3169.490	0.008

Table C.126: loglogistic-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) SSELN data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	25.876	78.747	7.404	1.447	69.842	1.973	4.794	1.019	0.569	4.813	9.368	7.684	8.669	6.834	13.631	3680.030	4042.521	2554.808	915.061	9392.235	1.246	3.339	0.895	0.614	2.052	0.197	0.132	0.165	0.476	0.156	0.480	0.327	0.129	0.322	-183.004	-249.033	-79.690	0.103
100	27.930	136.835	8.273	1.693	49.855	1.268	2.436	0.776	0.498	2.013	8.765	2.129	8.548	7.197	10.780	3556.005	3281.528	2685.168	1034.702	7015.746	0.976	1.746	0.815	0.577	1.359	0.188	0.119	0.166	0.487	0.136	0.490	0.325	0.110	0.324	-370.334	-483.237	-229.182	0.058
200	23.805	59.746	8.563	1.758	40.063	0.883	1.230	0.652	0.475	1.239	8.635	1.301	8.526	7.528	9.972	3358.529	2660.283	2734.857	1139.184	6066.118	0.849	0.656	0.774	0.562	1.132	0.186	0.104	0.172	0.489	0.116	0.490	0.325	0.094	0.324	-745.910	-934.864	-524.646	0.040
500	17.904	48.309	8.773	1.962	29.913	0.655	0.322	0.560	0.456	0.852	8.586	0.766	8.529	7.872	9.396	3189.397	2122.587	2759.033	1273.319	5228.459	0.783	0.687	0.745	0.570	0.973	0.187	0.084	0.180	0.488	0.091	0.487	0.325	0.077	0.324	-1871.306	-2248.627	-1451.579	0.024
1000	14.214	24.136	8.908	2.075	24.631	0.594	0.222	0.533	0.458	0.712	8.573	0.573	8.531	8.024	9.145	3084.632	1779.530	2777.432	1355.075	4882.781	0.739	0.166	0.731	0.578	0.907	0.187	0.071	0.185	0.488	0.073	0.487	0.325	0.065	0.322	-3748.182	-4396.292	-3076.942	0.012

Table C.127: weibull-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) ABFLN data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	7.356	16.451	3.308	1.464	32.857	1.830	3.858	1.113	0.662	5.845	11.512	56.256	9.132	7.194	21.287	408.236	523.717	254.595	93.337	1493.408	1.500	3.375	0.841	0.581	5.308	0.257	0.158	0.226	0.453	0.182	0.481	0.289	0.154	0.277	-131.063	-178.071	-36.334	0.130
100	6.703	18.488	3.090	1.533	22.903	1.357	2.490	0.899	0.597	2.831	10.718	32.202	9.147	7.731	14.797	383.687	470.807	252.245	101.909	1097.886	1.052	2.116	0.705	0.527	2.206	0.252	0.146	0.229	0.451	0.170	0.481	0.297	0.148	0.290	-265.991	-347.376	-118.257	0.114
200	5.536	17.411	2.832	1.583	22.027	1.132	14.472	0.796	0.582	2.552	10.200	15.717	9.183	8.139	14.680	339.529	376.434	239.486	108.377	993.351	0.786	1.004	0.628	0.502	1.575	0.245	0.130	0.227	0.447	0.158	0.479	0.308	0.140	0.302	-538.982	-677.359	-259.609	0.125
500	3.947	6.222	2.557	1.663	58.022	0.937	0.551	0.749	0.593	3.302	9.855	10.999	9.210	8.509	31.628	277.933	234.367	218.181	115.193	1727.155	0.623	0.258	0.571	0.486	2.625	0.232	0.108	0.219	0.442	0.148	0.476	0.326	0.129	0.315	-1363.836	-1639.289	-292.540	0.158
1000	3.161	3.817	2.428	1.748	-	0.893	0.478	0.736	0.612	-	9.703	2.080	9.234	8.700	-	247.327	164.690	208.530	120.194	-	0.573	0.170	0.547	0.482	-	0.225	0.093	0.215	0.438	0.143	0.473	0.337	0.122	0.321	-2745.949	-3194.102	-	0.189

Table C.128: weibull-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) BARC data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	21.074	58.006	6.325	1.513	55.749	2.026	4.447	1.058	0.591	4.583	9.701	26.833	8.687	6.814	13.464	2479.574	2764.860	1660.558	603.977	6254.498	1.126	3.249	0.827	0.568	1.791	0.201	0.142	0.164	0.472	0.158	0.480	0.327	0.130	0.322	-175.281	-233.944	-86.602	0.095
100	20.837	56.978	6.612	1.771	40.760	1.342	2.580	0.808	0.520	2.136	8.985	7.438	8.578	7.244	10.845	2371.059	2414.372	1696.900	690.901	4686.754	0.871	1.205	0.741	0.533	1.182	0.191	0.128	0.164	0.481	0.135	0.488	0.328	0.113	0.326	-356.067	-452.180	-238.631	0.054
200	17.933	40.582	6.666	1.896	32.516	0.919	1.319	0.673	0.492	1.320	8.810	2.929	8.599	7.607	10.016	2208.031	1865.926	1696.646	764.475	4029.120	0.751	0.469	0.696	0.523	0.975	0.190	0.113	0.171	0.482	0.113	0.486	0.328	0.097	0.327	-716.743	-873.839	-540.178	0.036
500	13.318	30.523	6.577	1.987	24.343	0.693	0.344	0.590	0.480	0.894	8.867	6.277	8.633	7.936	9.518	2019.236	1383.266	1677.935	844.533	3397.169	0.677	0.225	0.659	0.520	0.842	0.190	0.099	0.178	0.480	0.088	0.485	0.330	0.080	0.329	-1798.898	-2094.602	-1469.702	0.028
1000	10.580	11.787	6.789	2.117	20.081	0.630	0.261	0.559	0.476	0.747	8.712	1.239	8.622	8.087	9.244	1929.250	1113.585	1683.478	885.967	3077.886	0.650	0.127	0.647	0.532	0.775	0.190	0.088	0.184	0.479	0.073	0.483	0.331	0.069	0.331	-3603.199	-4098.225	-3073.635	0.017

Table C.129: weibull-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) RRLN data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	10.772	29.799	3.438	1.270	64.052	2.360	128.683	1.047	0.613	9.156	9.569	7.091	8.866	7.098	17.132	914.644	1125.880	582.851	221.877	3829.202	1.294	3.292	0.813	0.573	4.690	0.205	0.130	0.176	0.500	0.140	0.506	0.295	0.129	0.291	-138.660	-186.232	-35.167	0.138
100	9.888	24.693	3.187	1.349	45.322	1.453	5.049	0.813	0.553	3.495	9.081	2.938	8.826	7.528	12.442	860.816	993.851	581.006	250.452	2781.158	0.973	2.580	0.707	0.530	1.947	0.197	0.116	0.177	0.502	0.117	0.505	0.301	0.114	0.303	-283.748	-361.902	-137.170	0.124
200	7.975	19.926	2.808	1.381	48.538	1.058	4.142	0.704	0.533	2.648	8.955	1.348	8.843	7.917	11.551	781.261	773.170	562.337	271.557	2769.318	0.747	1.109	0.643	0.510	1.490	0.191	0.100	0.177	0.499	0.095	0.500	0.310	0.100	0.314	-577.289	-706.172	-298.640	0.137
500	5.187	10.486	2.452	1.430	-	0.827	0.798	0.654	0.534	-	8.929	0.781	8.879	8.267	-	685.840	551.329	537.897	297.004	-	0.627	0.208	0.596	0.497	-	0.183	0.082	0.175	0.496	0.077	0.495	0.321	0.083	0.324	-1456.417	-1714.941	-	0.176
1000	3.806	5.675	2.258	1.461	-	0.776	0.407	0.643	0.543	-	8.941	0.582	8.910	8.466	-	635.525	436.969	519.350	310.950	-	0.587	0.120	0.573	0.491	-	0.178	0.071	0.175	0.494	0.066	0.491	0.328	0.072	0.329	-2919.347	-3365.227	-	0.208

Table C.130: weibull-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) VOD data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	7.894	20.661	3.460	1.385	59.905	1.970	4.085	1.126	0.666	13.399	9.647	11.724	8.937	7.202	21.817	370.803	521.206	203.227	74.182	2262.621	1.640	3.610	0.852	0.583	13.734	0.227	0.140	0.198	0.506	0.141	0.515	0.267	0.132	0.264	-115.403	-157.086	-16.017	0.149
100	7.538	19.690	3.247	1.451	46.799	1.436	2.923	0.889	0.598	5.853	9.237	4.446	8.943	7.689	13.960	358.924	479.435	207.532	79.306	1738.180	1.174	2.418	0.705	0.528	6.002	0.221	0.131	0.197	0.505	0.120	0.511	0.274	0.125	0.280	-235.863	-307.861	-77.134	0.143
200	6.525	14.117	2.972	1.512	-	1.105	1.729	0.777	0.576	-	9.093	1.508	8.965	8.064	-	331.165	405.920	207.683	84.838	-	0.896	1.587	0.627	0.506	-	0.215	0.120	0.194	0.502	0.101	0.506	0.283	0.117	0.295	-480.627	-604.698	-	0.168
500	4.827	7.887	2.670	1.637	-	0.909	0.793	0.723	0.587	-	9.077	0.817	9.013	8.450	-	283.510	288.846	205.378	97.018	-	0.669	0.541	0.572	0.496	-	0.204	0.102	0.187	0.496	0.082	0.500	0.300	0.102	0.311	-1231.167	-1486.197	-	0.238
1000	3.831	5.311	2.520	1.782	-	0.833	0.414	0.712	0.612	-	9.090	0.601	9.046	8.650	-	257.037	212.770	207.911	114.289	-	0.590	0.228	0.547	0.493	-	0.198	0.088	0.186	0.491	0.072	0.492	0.312	0.088	0.320	-2515.045	-2951.594	-	0.308

Table C.131: weibull-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) RIOTINTO data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	25.463	240.895	6.101	1.292	80.914	2.239	4.482	1.129	0.608	6.484	9.435	7.740	8.687	6.851	14.315	3095.121	3441.366	2178.585	764.812	8382.939	1.144	2.053	0.851	0.588	2.129	0.183	0.129	0.148	0.474	0.157	0.480	0.343	0.135	0.336	-183.818	-248.922	-76.713	0.113
100	26.469	77.142	6.527	1.539	54.959	1.446	2.848	0.836	0.520	2.466	8.859	5.485	8.564	7.228	10.894	2940.424	2796.857	2231.097	876.369	5811.995	0.959	1.331	0.772	0.547	1.353	0.173	0.118	0.146	0.486	0.136	0.490	0.341	0.117	0.336	-372.487	-483.657	-229.345	0.065
200	24.896	63.409	7.051	1.629	46.558	0.955	1.306	0.682	0.484	1.469	8.628	1.199	8.529	7.569	9.918	2778.924	2135.243	2280.517	977.805	5022.647	0.848	0.983	0.731	0.532	1.131	0.171	0.106	0.151	0.488	0.113	0.490	0.340	0.101	0.337	-752.311	-936.782	-529.813	0.043
500	20.660	45.682	8.637	1.764	36.918	0.703	0.503	0.574	0.462	0.930	8.548	0.737	8.496	7.849	9.336	2632.085	1579.336	2346.127	1119.122	4362.377	0.786	0.921	0.710	0.532	0.979	0.174	0.089	0.162	0.487	0.086	0.488	0.340	0.083	0.336	-1898.083	-2261.741	-1500.780	0.026
1000	16.584	28.577	10.181	1.891	30.489	0.620	0.269	0.536	0.458	0.770	8.512	0.519	8.484	8.015	9.062	2570.881	1355.086	2393.361	1210.504	3992.545	0.723	0.212	0.712	0.541	0.906	0.175	0.076	0.172	0.486	0.068	0.487	0.339	0.072	0.335	-3807.590	-4398.889	-3154.710	0.020

Table C.132: weibull-exponential-weibull mixture on limit order arrival zero inflated (exponentially distributed) SSELN data.

	AIC			BIC		
exponential/exponential/weibull	0.2420	0.3582	0.5923	0.3890	-	-
gamma/exponential/weibull	0.1565	0.2340	0.3979	0.1207	0.2235	0.3886
loglogistic/exponential/weibull	0.3207	-	-	0.2621	0.4112	-
weibull/exponential/weibull	0.2719	0.3987	-	0.2193	0.3531	0.5817

Table C.133: ABFLN, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2247	0.3834	-	0.3816	-	-
gamma/exponential/weibull	0.1666	0.2845	0.4863	0.1243	0.2356	0.4863
loglogistic/exponential/weibull	0.3963	-	-	0.3335	0.4782	-
weibull/exponential/weibull	0.2094	0.3283	0.5050	0.1575	0.2824	0.5050

Table C.134: BARC, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2284	0.3546	0.5711	0.3796	-	-
gamma/exponential/weibull	0.1634	0.2549	0.4205	0.1254	0.2214	0.4095
loglogistic/exponential/weibull	0.3433	-	-	0.2811	0.4366	-
weibull/exponential/weibull	0.2586	0.3828	-	0.2074	0.3350	0.5649

Table C.135: RRLN, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2326	0.3826	-	0.3772	-	-
gamma/exponential/weibull	0.1680	0.2787	0.4743	0.1284	0.2343	0.4743
loglogistic/exponential/weibull	0.3756	-	-	0.3186	0.4654	-
weibull/exponential/weibull	0.2175	0.3264	0.5030	0.1696	0.2928	0.5030

Table C.136: VOD, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2544	0.3868	-	0.4038	-	-
gamma/exponential/weibull	0.1910	0.2925	0.5053	0.1445	0.2789	0.5053
loglogistic/exponential/weibull	0.3297	-	-	0.2765	0.4142	-
weibull/exponential/weibull	0.2229	0.3184	0.4873	0.1731	0.3034	0.4873

Table C.137: RIOTINTO, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2308	0.3560	0.5832	0.3795	-	-
gamma/exponential/weibull	0.1640	0.2487	0.4151	0.1256	0.2272	0.4088
loglogistic/exponential/weibull	0.3381	-	-	0.2794	0.4279	-
weibull/exponential/weibull	0.2662	0.3942	-	0.2146	0.3415	0.5673

Table C.138: SSELN, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2238	0.3299	0.5574	0.3502	-	-
gamma/exponential/weibull	0.1671	0.2504	0.4340	0.1330	0.2291	0.3948
loglogistic/exponential/weibull	0.3192	-	-	0.2723	0.3993	-
weibull/exponential/weibull	0.2822	0.4117	-	0.2368	0.3625	0.5919

Table C.139: ABFLN, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2017	0.3721	-	0.3325	0.5455	-
gamma/exponential/weibull	0.1605	0.3006	0.5018	0.1248	0.2134	0.5018
loglogistic/exponential/weibull	0.4390	-	-	0.3858	-	-
weibull/exponential/weibull	0.1972	0.3252	0.4953	0.1552	0.2390	0.4953

Table C.140: BARC, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2174	0.3285	0.5371	0.3497	-	-
gamma/exponential/weibull	0.1828	0.2764	0.4557	0.1471	0.2444	0.4301
loglogistic/exponential/weibull	0.3259	-	-	0.2766	0.4075	-
weibull/exponential/weibull	0.2680	0.3884	-	0.2207	0.3420	0.5592

Table C.141: RRLN, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2164	0.3696	-	0.3374	0.5177	-
gamma/exponential/weibull	0.1702	0.2952	0.4910	0.1359	0.2210	0.4910
loglogistic/exponential/weibull	0.3934	-	-	0.3465	-	-
weibull/exponential/weibull	0.2116	0.3213	0.4923	0.1719	0.2474	0.4923

Table C.142: VOD, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2225	0.3792	-	0.3417	0.5300	-
gamma/exponential/weibull	0.1779	0.3079	0.5247	0.1410	0.2292	0.5247
loglogistic/exponential/weibull	0.3961	-	-	0.3531	-	-
weibull/exponential/weibull	0.2029	0.3122	0.4743	0.1636	0.2401	0.4743

Table C.143: RIOTINTO, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2168	0.3272	0.5489	0.3453	-	-
gamma/exponential/weibull	0.1743	0.2634	0.4506	0.1396	0.2375	0.4148
loglogistic/exponential/weibull	0.3275	-	-	0.2790	0.4067	-
weibull/exponential/weibull	0.2811	0.4092	-	0.2358	0.3542	0.5785

Table C.144: SSELN, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2124	0.3225	0.5549	0.3188	-	-
gamma/exponential/weibull	0.1638	0.2485	0.4431	0.1386	0.2190	0.3929
loglogistic/exponential/weibull	0.3331	-	-	0.2931	0.4136	-
weibull/exponential/weibull	0.2902	0.4277	-	0.2491	0.3666	0.6044

Table C.145: ABFLN, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1853	0.3669	-	0.2845	0.5135	-
gamma/exponential/weibull	0.1516	0.3124	0.5155	0.1249	0.2368	0.5155
loglogistic/exponential/weibull	0.4756	-	-	0.4352	-	-
weibull/exponential/weibull	0.1857	0.3187	0.4822	0.1536	0.2477	0.4822

Table C.146: BARC, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2116	0.3234	0.5347	0.3177	-	-
gamma/exponential/weibull	0.1729	0.2730	0.4627	0.1459	0.2330	0.4201
loglogistic/exponential/weibull	0.3406	-	-	0.2993	0.4133	-
weibull/exponential/weibull	0.2732	0.4012	-	0.2354	0.3519	0.5773

Table C.147: RRLN, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2047	0.3668	-	0.2984	0.4898	-
gamma/exponential/weibull	0.1640	0.3046	0.5040	0.1380	0.2413	0.5040
loglogistic/exponential/weibull	0.4166	-	-	0.3812	-	-
weibull/exponential/weibull	0.2029	0.3135	0.4798	0.1706	0.2538	0.4798

Table C.148: VOD, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1915	0.3791	-	0.2814	0.5064	-
gamma/exponential/weibull	0.1590	0.3207	0.5448	0.1316	0.2519	0.5448
loglogistic/exponential/weibull	0.4656	-	-	0.4329	-	-
weibull/exponential/weibull	0.1837	0.2999	0.4550	0.1540	0.2415	0.4550

Table C.149: RIOTINTO, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2055	0.3095	0.5319	0.3080	-	-
gamma/exponential/weibull	0.1710	0.2639	0.4680	0.1444	0.2314	0.4098
loglogistic/exponential/weibull	0.3379	-	-	0.2984	0.4119	-
weibull/exponential/weibull	0.2855	0.4265	-	0.2492	0.3565	0.5897

Table C.150: SSELN, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1929	0.3067	0.5661	0.2671	0.3979	0.6524
gamma/exponential/weibull	0.1412	0.2252	0.4337	0.1240	0.1894	0.3474
loglogistic/exponential/weibull	0.3546	-	-	0.3267	-	-
weibull/exponential/weibull	0.3111	0.4679	-	0.2820	0.4125	-

Table C.151: ABFLN, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1668	0.3670	-	0.2341	0.4787	-
gamma/exponential/weibull	0.1400	0.3242	0.5389	0.1214	0.2647	0.5389
loglogistic/exponential/weibull	0.5214	-	-	0.4956	-	-
weibull/exponential/weibull	0.1716	0.3086	0.4608	0.1488	0.2563	0.4608

Table C.152: BARC, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2026	0.3116	0.5364	0.2755	0.4031	-
gamma/exponential/weibull	0.1672	0.2639	0.4631	0.1471	0.2248	0.4020
loglogistic/exponential/weibull	0.3490	-	-	0.3212	-	-
weibull/exponential/weibull	0.2808	0.4241	-	0.2557	0.3716	0.5974

Table C.153: RRLN, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1836	0.3748	-	0.2498	0.4683	-
gamma/exponential/weibull	0.1575	0.3187	0.5440	0.1385	0.2682	0.5440
loglogistic/exponential/weibull	0.4677	-	-	0.4414	-	-
weibull/exponential/weibull	0.1899	0.3043	0.4536	0.1689	0.2613	0.4536

Table C.154: VOD, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1614	0.3919	-	0.2219	0.4897	-
gamma/exponential/weibull	0.1363	0.3339	0.5833	0.1178	0.2780	0.5833
loglogistic/exponential/weibull	0.5425	-	-	0.5211	-	-
weibull/exponential/weibull	0.1599	0.2743	0.4167	0.1393	0.2323	0.4167

Table C.155: RIOTINTO, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1903	0.2979	0.5308	0.2738	0.3888	0.6238
gamma/exponential/weibull	0.1669	0.2535	0.4690	0.1463	0.2161	0.3760
loglogistic/exponential/weibull	0.3415	-	-	0.3085	-	-
weibull/exponential/weibull	0.3010	0.4483	-	0.2713	0.3948	-

Table C.156: SSELN, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1866	0.2971	0.5872	0.2393	0.3620	0.6502
gamma/exponential/weibull	0.1293	0.2022	0.4123	0.1185	0.1791	0.3493
loglogistic/exponential/weibull	0.3634	-	-	0.3437	-	-
weibull/exponential/weibull	0.3202	0.5002	-	0.2981	0.4584	-

Table C.157: ABFLN, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1603	0.3651	-	0.2128	0.4562	-
gamma/exponential/weibull	0.1300	0.3306	0.5519	0.1148	0.2796	0.5519
loglogistic/exponential/weibull	0.5397	-	-	0.5199	-	-
weibull/exponential/weibull	0.1699	0.3041	0.4479	0.1523	0.2641	0.4479

Table C.158: BARC, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1981	0.3108	0.5728	0.2589	0.3854	0.6400
gamma/exponential/weibull	0.1469	0.2389	0.4266	0.1355	0.2128	0.3593
loglogistic/exponential/weibull	0.3533	-	-	0.3282	-	-
weibull/exponential/weibull	0.3011	0.4496	-	0.2767	0.4011	-

Table C.159: RRLN, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1683	0.3693	-	0.2194	0.4443	-
gamma/exponential/weibull	0.1496	0.3257	0.5582	0.1352	0.2850	0.5582
loglogistic/exponential/weibull	0.4963	-	-	0.4759	-	-
weibull/exponential/weibull	0.1855	0.3047	0.4415	0.1692	0.2705	0.4415

Table C.160: VOD, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1479	0.4027	-	0.1871	0.4767	-
gamma/exponential/weibull	0.1211	0.3475	0.6177	0.1102	0.3020	0.6177
loglogistic/exponential/weibull	0.5848	-	-	0.5697	-	-
weibull/exponential/weibull	0.1462	0.2498	0.3823	0.1329	0.2213	0.3823

Table C.161: RIOTINTO, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1739	0.2882	0.5379	0.2256	0.3570	0.6113
gamma/exponential/weibull	0.1530	0.2444	0.4621	0.1401	0.2177	0.3887
loglogistic/exponential/weibull	0.3611	-	-	0.3403	-	-
weibull/exponential/weibull	0.3119	0.4675	-	0.2940	0.4254	-

Table C.162: SSELN, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04224	0.51720	0.00044	6.17884	3.01940	6.33080	0.42141	0.15519	0.42007	0.57859	0.15519	0.57993	-193.47063	-269.66948	-120.92763	0.00000
100	0.00562	0.19052	0.00042	5.71996	2.33371	6.04054	0.42072	0.13395	0.41996	0.57928	0.13395	0.58004	-394.97806	-522.37250	-275.95868	0.00000
200	0.00053	0.00072	0.00041	5.54223	1.93550	5.90788	0.42329	0.11374	0.42212	0.57671	0.11374	0.57788	-801.15058	-1012.05915	-600.16127	0.00000
500	0.00047	0.00032	0.00039	5.41670	1.57438	5.72645	0.42620	0.08839	0.42657	0.57380	0.08839	0.57343	-2027.19553	-2435.10081	-1620.41763	0.00000
1000	0.00045	0.00026	0.00038	5.36931	1.37100	5.67315	0.42798	0.07179	0.42846	0.57202	0.07179	0.57154	-4078.53666	-4767.50732	-3388.97813	0.00000

Table C.163: 2-component exponential mixture on limit order arrival ABFLN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.03413	0.37296	0.00456	5.14064	3.58199	4.98311	0.37949	0.13212	0.37284	0.62051	0.13212	0.62716	-139.49738	-195.84216	-83.71608	0.00001
100	0.01877	0.26967	0.00411	4.49593	2.70410	4.67423	0.37277	0.11057	0.36887	0.62723	0.11057	0.63113	-287.34362	-384.77984	-191.74625	0.00000
200	0.00756	0.11396	0.00381	4.15122	2.16371	4.53845	0.37057	0.09485	0.36721	0.62943	0.09485	0.63279	-585.60133	-758.24213	-415.70564	0.00000
500	0.00435	0.00317	0.00358	3.93425	3.48873	4.43607	0.37110	0.08013	0.36779	0.62890	0.08013	0.63221	-1486.32475	-1868.67972	-1109.09630	0.00002
1000	0.00406	0.00254	0.00347	3.87449	1.76310	4.40151	0.37200	0.07169	0.36726	0.62800	0.07169	0.63274	-2993.64110	-3682.77178	-2277.71842	0.00000

Table C.164: 2-component exponential mixture on limit order arrival BARC data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.03928	0.48903	0.00063	6.08302	4.01143	6.21093	0.42057	0.14582	0.42008	0.57943	0.14582	0.57992	-186.90553	-255.02318	-122.74159	0.00002
100	0.01104	0.24520	0.00059	5.61012	2.42722	5.95434	0.42001	0.12163	0.42024	0.57999	0.12163	0.57976	-382.61559	-492.57210	-280.72677	0.00003
200	0.00154	0.05873	0.00057	5.40496	1.98895	5.80964	0.42187	0.10011	0.42362	0.57813	0.10011	0.57638	-777.33182	-954.29992	-613.18284	0.00000
500	0.00071	0.00187	0.00055	5.27212	1.65666	5.68280	0.42382	0.07714	0.42703	0.57618	0.07714	0.57297	-1968.38981	-2297.17536	-1642.76562	0.00000
1000	0.00065	0.00146	0.00054	5.21635	1.49798	5.57153	0.42520	0.06410	0.42763	0.57480	0.06410	0.57237	-3961.04570	-4513.55895	-3411.36577	0.00000

Table C.165: 2-component exponential mixture on limit order arrival RRLN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.09844	0.72477	0.00187	5.72951	4.33104	5.56266	0.37556	0.12283	0.37248	0.62444	0.12283	0.62752	-146.97173	-204.60169	-94.12285	0.00000
100	0.06812	0.58824	0.00172	5.22057	2.43933	5.27890	0.37397	0.10247	0.37381	0.62603	0.10247	0.62619	-301.54951	-397.37192	-216.44842	0.00000
200	0.01968	0.26293	0.00160	4.93935	1.72548	5.11716	0.37268	0.09015	0.37618	0.62732	0.09015	0.62382	-612.86939	-778.45250	-470.18626	0.00000
500	0.00186	0.00166	0.00152	4.77994	1.34556	5.00487	0.37287	0.07618	0.37804	0.62713	0.07618	0.62196	-1552.82924	-1904.02895	-1249.46755	0.00000
1000	0.00175	0.00125	0.00149	4.72731	1.17763	4.95211	0.37361	0.06278	0.37753	0.62639	0.06278	0.62247	-3126.00974	-3754.77941	-2550.65736	0.00000

Table C.166: 2-component exponential mixture on limit order arrival VOD data.

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.02379	0.27498	0.00591	5.86049	5.08986	5.54588	0.36982	0.11520	0.36483	0.63018	0.11520	0.63517	-124.24442	-174.08267	-75.15751	0.00000
100	0.01187	0.14678	0.00534	5.26967	6.75633	5.18525	0.36590	0.09479	0.36307	0.63410	0.09479	0.63693	-256.19674	-341.73392	-172.89864	0.00000
200	0.00731	0.05405	0.00504	4.91087	1.75073	5.01224	0.36462	0.07876	0.36260	0.63538	0.07876	0.63740	-522.32932	-674.84768	-375.66001	0.00000
500	0.00602	0.00467	0.00480	4.71237	1.49299	4.90685	0.36516	0.06312	0.36338	0.63484	0.06312	0.63662	-1325.38241	-1670.12506	-1002.73671	0.00000
1000	0.00570	0.00405	0.00468	4.64375	1.19441	4.85706	0.36594	0.05479	0.36354	0.63406	0.05479	0.63646	-2668.73761	-3326.02894	-2062.02674	0.00000

Table C.167: 2-component exponential mixture on limit order arrival RIOTINTO data.

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.00562	0.17969	0.00049	6.30879	2.93172	6.43302	0.43311	0.15206	0.42113	0.56689	0.15206	0.57887	-195.78899	-270.57171	-123.80922	0.00000
100	0.00214	0.09338	0.00047	5.93213	2.23372	6.19180	0.43472	0.12739	0.43004	0.56528	0.12739	0.56996	-399.73997	-522.55735	-279.26005	0.00000
200	0.00118	0.06700	0.00046	5.76262	1.80208	6.06916	0.43742	0.10558	0.43466	0.56258	0.10558	0.56534	-810.82914	-1012.42800	-612.71654	0.00000
500	0.00051	0.00029	0.00045	5.68643	1.41157	6.00612	0.44027	0.08142	0.43617	0.55973	0.08142	0.56383	-2050.44710	-2441.15069	-1659.71731	0.00000
1000	0.00048	0.00021	0.00044	5.68378	1.17362	5.94928	0.44190	0.06558	0.44032	0.55810	0.06558	0.55968	-4122.48648	-4759.04746	-3501.64135	0.00000

Table C.168: 2-component exponential mixture on limit order arrival SSELN data.

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{3\ \text{ave}}$	$\pm\hat{\lambda}_{3\ \text{ave}}$	$\hat{\lambda}_{3\ \text{median}}$	$\hat{\lambda}_{4\ \text{ave}}$	$\pm\hat{\lambda}_{4\ \text{ave}}$	$\hat{\lambda}_{4\ \text{median}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	$\hat{\pi}_{4\ \text{ave}}$	$\pm\hat{\pi}_{4\ \text{ave}}$	$\hat{\pi}_{4\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04309	0.53883	0.00034	1.28396	2.47888	0.13678	4.62057	3.54112	3.94852	14.93565	99.81546	9.55739	0.32120	0.13572	0.32005	0.13464	0.08553	0.11323	0.14541	0.09337	0.12669	0.39876	0.13435	0.41023	-180.97595	-253.82751	-108.99576	0.00897
100	0.00496	0.17263	0.00030	0.59446	1.59524	0.04442	3.75514	3.34430	2.17633	11.70477	30.29928	9.28713	0.30444	0.11550	0.30130	0.13284	0.07438	0.12017	0.13811	0.08621	0.12130	0.42462	0.12377	0.43522	-367.05709	-487.24990	-245.56410	0.01689
200	0.00036	0.00035	0.00028	0.22020	0.85076	0.02798	2.48793	2.86160	1.09324	9.69509	10.89654	9.03319	0.28923	0.09852	0.28929	0.13198	0.06318	0.12348	0.12192	0.07048	0.11015	0.45688	0.10871	0.46241	-740.89376	-934.76226	-528.38869	0.03127
500	0.00032	0.00021	0.00026	0.03928	0.16313	0.02070	1.13590	1.61825	0.62881	8.92416	0.98195	8.84458	0.27776	0.07823	0.27928	0.12982	0.05065	0.12360	0.10374	0.05149	0.09484	0.48868	0.08575	0.49307	-1865.21021	-2223.38207	-1371.80614	0.06678
1000	0.00031	0.00018	0.00026	0.02298	0.06067	0.01906	0.71435	0.75420	0.53633	8.79902	0.75799	8.75857	0.27440	0.06522	0.27593	0.12946	0.04245	0.12361	0.09769	0.04274	0.09076	0.49845	0.07033	0.49978	-3739.03143	-4317.33885	-2655.86488	0.11848

Table C.169: 4-component exponential mixture on limit order arrival ABFLN data.

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{3\ \text{ave}}$	$\pm\hat{\lambda}_{3\ \text{ave}}$	$\hat{\lambda}_{3\ \text{median}}$	$\hat{\lambda}_{4\ \text{ave}}$	$\pm\hat{\lambda}_{4\ \text{ave}}$	$\hat{\lambda}_{4\ \text{median}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	$\hat{\pi}_{4\ \text{ave}}$	$\pm\hat{\pi}_{4\ \text{ave}}$	$\hat{\pi}_{4\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.02998	0.38940	0.00283	0.49703	1.18234	0.13931	2.83978	3.02980	1.46666	16.07222	234.86039	10.06810	0.23900	0.12918	0.22608	0.18046	0.09700	0.16503	0.17883	0.11514	0.15157	0.40172	0.16313	0.43415	-129.14818	-183.62245	-74.44368	0.00004
100	0.01544	0.26810	0.00232	0.22557	0.63270	0.07656	2.08935	2.49102	1.13813	12.80196	84.55765	9.85461	0.21279	0.10831	0.20122	0.17974	0.08086	0.16929	0.18445	0.11190	0.15935	0.42302	0.15604	0.45710	-263.60518	-356.58086	-170.12925	0.00008
200	0.00497	0.11446	0.00200	0.10216	0.32763	0.04592	1.35430	1.72541	0.85523	11.80192	298.52004	9.66260	0.19003	0.08971	0.18025	0.17771	0.06730	0.16957	0.19021	0.10531	0.16763	0.44205	0.15051	0.47598	-533.68889	-695.22141	-368.78767	0.00008
500	0.00211	0.00152	0.00177	0.04657	0.10416	0.03121	0.84629	0.86469	0.63146	10.26019	25.26063	9.47419	0.17175	0.07217	0.16368	0.17568	0.05508	0.16900	0.19782	0.09612	0.17478	0.45475	0.14574	0.49062	-1347.30019	-1693.76091	-982.25405	0.00017
1000	0.00193	0.00127	0.00167	0.03482	0.03671	0.02707	0.71186	0.52768	0.55049	10.03039	4.21464	9.39380	0.16453	0.06428	0.15611	0.17529	0.04948	0.16871	0.20134	0.09110	0.17866	0.45884	0.14261	0.49685	-2705.85656	-3312.19432	-2020.66859	0.00030

Table C.170: 4-component exponential mixture on limit order arrival BARC data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.04093	0.51825	0.00046	1.05619	2.25069	0.11405	4.40592	3.53412	3.31647	14.93230	91.55959	9.56859	0.30419	0.12923	0.30028	0.14916	0.09829	0.12425	0.14863	0.09802	0.12816	0.39801	0.13547	0.41295	-173.55613	-237.87440	-112.27408	0.00246
100	0.01144	0.26596	0.00040	0.43176	1.31406	0.04314	3.45740	3.28218	1.86591	12.26969	87.59614	9.27203	0.28417	0.10910	0.28076	0.14898	0.08319	0.13469	0.14043	0.09045	0.12160	0.42643	0.12259	0.43867	-352.74702	-454.97681	-254.99053	0.00482
200	0.00160	0.08835	0.00035	0.13735	0.62803	0.02527	2.15968	2.68869	0.92005	10.45494	66.15270	9.03117	0.26425	0.09255	0.26376	0.14951	0.07054	0.14010	0.12762	0.07747	0.11289	0.45861	0.10787	0.46564	-712.74106	-875.54152	-554.57800	0.00926
500	0.00045	0.00165	0.00032	0.03718	0.24428	0.01754	0.92492	1.43950	0.50122	8.99121	3.92247	8.78504	0.24747	0.07538	0.24869	0.14917	0.05932	0.14167	0.11555	0.05964	0.10492	0.48781	0.08548	0.49381	-1796.62560	-2093.68682	-1474.64337	0.02160
1000	0.00039	0.00109	0.00031	0.01900	0.02844	0.01514	0.56233	0.58324	0.41858	8.80644	1.33362	8.70302	0.24074	0.06538	0.24346	0.14960	0.05344	0.14354	0.11314	0.05157	0.10343	0.49651	0.07207	0.50149	-3603.21925	-4093.17868	-3019.08857	0.04048

Table C.171: 4-component exponential mixture on limit order arrival RRLN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.09991	0.74832	0.00128	0.77882	1.68590	0.14014	3.66345	3.28806	2.12499	13.72140	120.45308	9.82349	0.25765	0.12277	0.25398	0.14480	0.08030	0.12935	0.15970	0.09745	0.13879	0.43785	0.12705	0.44755	-135.84184	-191.61040	-84.45656	0.00010
100	0.06867	0.60297	0.00108	0.35925	0.99321	0.06166	2.67926	2.81756	1.51844	11.15957	52.82459	9.57065	0.23521	0.10223	0.23296	0.14616	0.06889	0.13678	0.15284	0.08387	0.13765	0.46579	0.10857	0.47134	-276.69927	-367.78725	-195.37774	0.00021
200	0.01810	0.25167	0.00094	0.16172	0.58249	0.03208	1.65744	2.02103	1.04297	9.70330	8.50251	9.33956	0.21365	0.08413	0.21342	0.14771	0.05880	0.14108	0.14579	0.06661	0.13654	0.49285	0.09045	0.49285	-559.33059	-714.69886	-425.60764	0.00041
500	0.00103	0.00111	0.00084	0.05913	0.24538	0.02160	0.98467	1.09828	0.72589	9.21378	0.92179	9.16263	0.19704	0.06350	0.19883	0.14880	0.04907	0.14469	0.14295	0.04963	0.13717	0.51121	0.07211	0.50777	-1411.46807	-1729.11306	-1135.34916	0.00103
1000	0.00095	0.00080	0.00080	0.03010	0.10534	0.01895	0.80022	0.66924	0.63035	9.10311	0.65337	9.08399	0.19173	0.05284	0.19330	0.14859	0.04212	0.14544	0.14235	0.04270	0.13708	0.51734	0.06210	0.51306	-2837.06645	-3387.31657	-2322.38264	0.00206

Table C.172: 4-component exponential mixture on limit order arrival VOD data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.01819	0.27873	0.00342	0.54221	1.22405	0.14717	3.18492	3.10141	1.77745	14.22170	447.42477	9.92052	0.22432	0.11457	0.21653	0.16800	0.09067	0.15455	0.16133	0.10178	0.14005	0.44635	0.12556	0.46119	-114.33523	-161.80582	-67.50581	0.00003
100	0.00777	0.15265	0.00272	0.24836	0.64654	0.07742	2.32363	2.56551	1.36620	11.05952	47.61064	9.68998	0.19628	0.09790	0.18733	0.17350	0.07682	0.16639	0.15963	0.08965	0.14398	0.47060	0.10730	0.48030	-233.59620	-313.74074	-155.14279	0.00004
200	0.00365	0.05126	0.00235	0.10589	0.30858	0.04348	1.49147	1.79349	1.02636	9.83166	11.46461	9.48939	0.17056	0.07966	0.16240	0.17674	0.06311	0.17232	0.16004	0.07441	0.15077	0.49266	0.09011	0.49709	-472.89517	-612.84339	-336.84624	0.00003
500	0.00264	0.00215	0.00213	0.04416	0.08692	0.02995	0.90345	0.82828	0.74816	9.38595	1.06096	9.32036	0.14949	0.05767	0.14451	0.17863	0.04880	0.17532	0.16431	0.05999	0.15704	0.50756	0.07374	0.50959	-1193.52503	-1500.30191	-901.71783	0.00006
1000	0.00243	0.00181	0.00203	0.03362	0.03509	0.02687	0.76745	0.52200	0.65277	9.28121	0.68151	9.24907	0.14213	0.04660	0.13856	0.17925	0.04160	0.17648	0.16680	0.05262	0.15968	0.51183	0.06345	0.51315	-2397.55782	-2970.87062	-1862.39515	0.00013

Table C.173: 4-component exponential mixture on limit order arrival RIOTINTO data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.00535	0.18345	0.00037	1.30277	2.50252	0.14315	4.77943	3.56469	4.54952	14.40244	65.60386	9.49961	0.33153	0.13561	0.32128	0.13370	0.08780	0.10990	0.13967	0.08913	0.12329	0.39510	0.13260	0.40880	-184.03844	-255.55169	-113.42008	0.00572
100	0.00240	0.12143	0.00034	0.57904	1.57273	0.04291	3.86022	3.36542	2.37817	11.97558	36.36737	9.23780	0.31115	0.11568	0.30507	0.13577	0.07795	0.12262	0.13309	0.08307	0.11822	0.41999	0.12268	0.43184	-373.63606	-489.11595	-254.99221	0.01104
200	0.00097	0.06550	0.00031	0.18144	0.73274	0.02350	2.50679	2.90031	1.06375	10.33404	51.75970	8.97953	0.29102	0.09994	0.28748	0.13889	0.06619	0.13171	0.11552	0.06790	0.10515	0.45457	0.10994	0.46127	-753.80646	-941.27756	-549.81846	0.02097
500	0.00032	0.00018	0.00029	0.03533	0.18120	0.01615	1.06600	1.63116	0.54827	8.80816	0.94810	8.73697	0.27437	0.08130	0.27277	0.14101	0.05241	0.13653	0.09619	0.04427	0.08993	0.48843	0.08484	0.49180	-1897.33824	-2254.10068	-1463.79378	0.04855
1000	0.00031	0.00014	0.00029	0.01693	0.02584	0.01471	0.61401	0.65649	0.46855	8.66975	0.58503	8.65028	0.26950	0.06882	0.27084	0.14151	0.04316	0.13758	0.09084	0.03518	0.08508	0.49816	0.06787	0.49807	-3804.09855	-4359.82437	-2958.12188	0.08882

Table C.174: 4-component exponential mixture on limit order arrival SSELN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04293	0.42561	1.02618	4.08596	6.25802	27.87504	0.28325	0.10979	0.05488	0.08217	0.11853	0.35139	-179.08709	-252.66261	-108.87311	0.00005
100	0.00535	0.13769	0.64214	3.63036	6.43653	13.21987	0.27294	0.11494	0.05881	0.07742	0.11246	0.36344	-364.32706	-485.72147	-249.40120	0.00009
200	0.00032	0.04095	0.43121	3.06564	6.76099	11.03227	0.26226	0.11935	0.06251	0.07295	0.10582	0.37711	-737.30744	-937.50200	-543.07777	0.00009
500	0.00027	0.01466	0.26294	2.25178	7.36392	9.51150	0.24804	0.12361	0.06690	0.06942	0.09869	0.39334	-1860.62030	-2246.82814	-1479.27508	0.00023
1000	0.00025	0.01090	0.19669	1.68605	7.76236	9.04698	0.23415	0.12776	0.07112	0.06979	0.09640	0.40078	-3738.47849	-4397.33223	-3091.26624	0.00047

Table C.175: 6-component exponential mixture on limit order arrival ABFLN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.02930	0.12509	0.39150	2.27136	4.35217	22.98642	0.19108	0.13210	0.08839	0.09884	0.12262	0.36697	-128.45945	-182.79798	-73.97341	0.00002
100	0.01515	0.07331	0.28778	1.79972	4.21927	16.45713	0.17370	0.13970	0.09603	0.09529	0.11197	0.38331	-262.56093	-355.10328	-169.32387	0.00003
200	0.00479	0.04471	0.22412	1.42958	4.11062	12.58396	0.15423	0.14490	0.10526	0.09551	0.10150	0.39860	-532.11056	-693.20259	-367.36941	0.00007
500	0.00170	0.01964	0.15281	1.11347	3.93793	10.51970	0.12758	0.14842	0.11637	0.10182	0.09320	0.41260	-1343.42722	-1689.22459	-979.90349	0.00010
1000	0.00146	0.01199	0.11224	0.97613	3.75846	10.32437	0.11035	0.14989	0.12083	0.10756	0.09255	0.41882	-2698.03162	-3304.22431	-2014.24654	0.00015

Table C.176: 6-component exponential mixture on limit order arrival BARC data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04003	0.31779	0.83332	3.85392	6.10554	18.51025	0.26181	0.12332	0.06381	0.08263	0.11861	0.34982	-172.00156	-236.26900	-110.93673	0.00008
100	0.01140	0.09484	0.47914	3.33584	6.30560	13.57579	0.24616	0.13053	0.07082	0.07709	0.11147	0.36392	-350.56385	-453.26422	-254.02019	0.00017
200	0.00156	0.03397	0.30213	2.71136	6.66432	11.25291	0.22839	0.13701	0.07821	0.07406	0.10439	0.37794	-709.90405	-873.86767	-556.01891	0.00026
500	0.00035	0.01283	0.18606	1.86401	7.21690	9.55220	0.20343	0.14420	0.08669	0.07431	0.09640	0.39497	-1791.21096	-2093.83172	-1490.66432	0.00066
1000	0.00029	0.00757	0.12575	1.32805	7.58024	9.08773	0.18374	0.14869	0.09372	0.07809	0.09402	0.40174	-3596.08044	-4095.59555	-3096.58056	0.00100

Table C.177: 6-component exponential mixture on limit order arrival RRLN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.09945	0.22005	0.60616	3.05467	5.39777	19.09556	0.21091	0.11740	0.06642	0.08906	0.12228	0.39393	-134.75717	-190.31447	-83.49907	0.00001
100	0.06833	0.13286	0.41003	2.41091	5.39104	17.50074	0.19481	0.12527	0.07325	0.08371	0.11105	0.41191	-275.12603	-366.05563	-194.19616	0.00001
200	0.01861	0.07836	0.30154	1.86520	5.49867	10.69882	0.17669	0.13193	0.08041	0.08227	0.09894	0.42976	-556.99863	-711.78173	-424.14976	0.00002
500	0.00084	0.02153	0.18338	1.35554	5.67205	9.56936	0.15166	0.13825	0.08853	0.08781	0.08864	0.44512	-1406.16840	-1724.32070	-1130.78467	0.00006
1000	0.00074	0.00928	0.11349	1.14416	5.81089	9.37787	0.13565	0.14119	0.09259	0.09359	0.08644	0.45054	-2826.07404	-3380.61059	-2315.98176	0.00011

Table C.178: 6-component exponential mixture on limit order arrival VOD data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.01749	0.10838	0.38480	2.51490	4.91967	16.04070	0.16903	0.13201	0.08065	0.09074	0.12109	0.40649	-113.62657	-160.90279	-67.02135	0.00001
100	0.00709	0.05880	0.27576	1.98848	4.79329	12.56748	0.15117	0.14136	0.08834	0.08576	0.10776	0.42561	-232.46302	-312.55285	-154.32249	0.00001
200	0.00310	0.03537	0.21382	1.57645	4.68307	10.67029	0.13332	0.14708	0.09729	0.08437	0.09519	0.44274	-471.32263	-611.04395	-335.93602	0.00002
500	0.00212	0.01903	0.15105	1.24360	4.53162	9.70778	0.11294	0.14927	0.10749	0.08828	0.08415	0.45787	-1190.13149	-1496.23716	-898.43129	0.00002
1000	0.00190	0.01422	0.11939	1.11038	4.31150	9.53891	0.10261	0.14896	0.11123	0.09187	0.08024	0.46508	-2390.16145	-2960.99551	-1856.67922	0.00004

Table C.179: 6-component exponential mixture on limit order arrival RIOTINTO data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.00502	0.38346	1.01266	4.19960	6.40544	19.26184	0.28795	0.11621	0.05316	0.07997	0.11673	0.34598	-182.06637	-254.05104	-112.94702	0.00004
100	0.00228	0.09501	0.61155	3.73300	6.65781	14.23348	0.27362	0.12427	0.05823	0.07491	0.11138	0.35759	-370.41521	-487.99552	-254.19600	0.00004
200	0.00096	0.02663	0.36665	3.15836	7.04804	11.34825	0.25974	0.13110	0.06327	0.07022	0.10545	0.37021	-749.49234	-941.70897	-557.31058	0.00008
500	0.00028	0.01234	0.23121	2.34269	7.69994	9.66091	0.24373	0.13506	0.06945	0.06686	0.09986	0.38504	-1891.96222	-2263.62908	-1522.70182	0.00021
1000	0.00026	0.00870	0.16965	1.81764	8.16441	8.93669	0.22994	0.13830	0.07496	0.06709	0.09724	0.39247	-3798.65547	-4409.13971	-3200.68269	0.00042

Table C.180: 6-component exponential mixture on limit order arrival SSELN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\lambda}_{7\text{ ave}}$	$\hat{\lambda}_{8\text{ ave}}$	$\hat{\lambda}_{9\text{ ave}}$	$\hat{\lambda}_{10\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	$\hat{\pi}_{7\text{ ave}}$	$\hat{\pi}_{8\text{ ave}}$	$\hat{\pi}_{9\text{ ave}}$	$\hat{\pi}_{10\text{ ave}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04159	0.39965	0.76057	2.27494	3.31130	4.86181	6.73473	7.96942	9.60770	35.17256	0.28102	0.10547	0.04066	0.03095	0.03443	0.04334	0.06140	0.08017	0.15146	0.17110	-178.93621	-252.36505	-108.86632	0.00005
100	0.00503	0.12601	0.38090	1.72020	2.89747	4.67653	6.99593	8.28130	9.27796	20.41961	0.26967	0.11090	0.04558	0.03148	0.03336	0.04138	0.06004	0.08048	0.15434	0.17277	-364.20155	-486.66979	-249.35846	0.00009
200	0.00031	0.03249	0.24595	1.28571	2.43823	4.54856	7.41276	8.61755	9.11972	16.08943	0.25683	0.11566	0.05133	0.03300	0.03282	0.03921	0.05875	0.08084	0.15571	0.17586	-736.85688	-937.12984	-542.61945	0.00009
500	0.00027	0.01248	0.14312	0.90220	1.82202	4.22227	8.05458	8.90532	9.04448	10.99005	0.23744	0.12073	0.06113	0.03546	0.03312	0.03652	0.05762	0.08090	0.15568	0.18139	-1859.96452	-2248.38057	-1477.49490	0.00023
1000	0.00024	0.00877	0.10676	0.69814	1.45031	3.98843	8.46646	8.94600	8.98748	10.41241	0.22184	0.12416	0.06855	0.03755	0.03457	0.03498	0.05771	0.08131	0.15563	0.18369	-3737.30029	-4385.18059	-3089.61839	0.00047

Table C.181: 10-component exponential mixture on limit order arrival ABFLN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\lambda}_{7\text{ ave}}$	$\hat{\lambda}_{8\text{ ave}}$	$\hat{\lambda}_{9\text{ ave}}$	$\hat{\lambda}_{10\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	$\hat{\pi}_{7\text{ ave}}$	$\hat{\pi}_{8\text{ ave}}$	$\hat{\pi}_{9\text{ ave}}$	$\hat{\pi}_{10\text{ ave}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.02940	0.11588	0.28110	1.06126	1.73925	2.91689	4.97921	6.76868	10.26250	42.16893	0.18851	0.12611	0.06611	0.04867	0.04810	0.05121	0.06170	0.07767	0.15466	0.17725	-128.32281	-182.52133	-73.65309	0.00052
100	0.01534	0.06909	0.20929	0.84188	1.41631	2.47410	5.01668	7.20746	10.10208	34.37364	0.17060	0.13459	0.07379	0.04964	0.04839	0.04986	0.05849	0.07627	0.15790	0.18047	-262.28164	-354.60498	-168.99956	0.00083
200	0.00460	0.04092	0.16737	0.72933	1.19482	2.01596	5.15060	7.66971	10.11495	19.91667	0.15041	0.14084	0.08446	0.05123	0.04942	0.04950	0.05541	0.07534	0.15830	0.18508	-531.70872	-692.11054	-366.49899	0.00048
500	0.00166	0.01798	0.11963	0.65160	1.05827	1.52852	5.32124	8.09218	10.19384	13.16869	0.12279	0.14562	0.09968	0.05489	0.05183	0.05057	0.05205	0.07481	0.15740	0.19036	-1343.06282	-1688.04561	-978.84875	0.00012
1000	0.00142	0.01095	0.09144	0.60937	1.00914	1.28981	5.43168	8.24653	10.21141	10.57059	0.10542	0.14702	0.10651	0.05929	0.05424	0.05240	0.05051	0.07506	0.15728	0.19227	-2697.26955	-3305.82595	-2013.70371	0.00015

Table C.182: 10-component exponential mixture on limit order arrival BARC data.

n	$\hat\lambda_{1\text{ ave}}$	$\hat\lambda_{2\text{ ave}}$	$\hat\lambda_{3\text{ ave}}$	$\hat\lambda_{4\text{ ave}}$	$\hat\lambda_{5\text{ ave}}$	$\hat\lambda_{6\text{ ave}}$	$\hat\lambda_{7\text{ ave}}$	$\hat\lambda_{8\text{ ave}}$	$\hat\lambda_{9\text{ ave}}$	$\hat\lambda_{10\text{ ave}}$	$\hat\pi_{1\text{ ave}}$	$\hat\pi_{2\text{ ave}}$	$\hat\pi_{3\text{ ave}}$	$\hat\pi_{4\text{ ave}}$	$\hat\pi_{5\text{ ave}}$	$\hat\pi_{6\text{ ave}}$	$\hat\pi_{7\text{ ave}}$	$\hat\pi_{8\text{ ave}}$	$\hat\pi_{9\text{ ave}}$	$\hat\pi_{10\text{ ave}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.04068	0.29894	0.60289	1.98851	3.06710	4.65491	6.62543	7.86894	9.64053	66.18949	0.25918	0.11888	0.04884	0.03304	0.03518	0.04334	0.06093	0.07971	0.15075	0.17015	-171.88512	-236.14108	-111.09483	0.00015
100	0.01108	0.08896	0.29189	1.43586	2.61699	4.47707	6.90393	8.22561	9.29376	39.48442	0.24191	0.12659	0.05669	0.03402	0.03424	0.04121	0.05960	0.08021	0.15404	0.17150	-350.41063	-453.01362	-254.03328	0.00036
200	0.00147	0.02902	0.17862	1.05364	2.16639	4.24874	7.37136	8.56634	9.17349	19.08725	0.22157	0.13290	0.06721	0.03650	0.03408	0.03904	0.05816	0.08025	0.15525	0.17505	-709.48422	-873.95084	-555.84442	0.00079
500	0.00033	0.00996	0.10617	0.73274	1.55302	3.85097	8.01183	8.83882	9.11183	11.71340	0.19048	0.14137	0.08175	0.04162	0.03554	0.03661	0.05680	0.08030	0.15543	0.18011	-1790.13284	-2094.85394	-1489.29828	0.00083
1000	0.00028	0.00568	0.07303	0.55869	1.28097	3.44200	8.39090	8.93222	9.08111	9.51368	0.16682	0.14789	0.08969	0.04655	0.03782	0.03572	0.05626	0.08015	0.15480	0.18431	-3594.55297	-4096.63525	-3093.23789	0.00100

Table C.183: 10-component exponential mixture on limit order arrival RRLN data.

n	$\hat\lambda_{1\text{ ave}}$	$\hat\lambda_{2\text{ ave}}$	$\hat\lambda_{3\text{ ave}}$	$\hat\lambda_{4\text{ ave}}$	$\hat\lambda_{5\text{ ave}}$	$\hat\lambda_{6\text{ ave}}$	$\hat\lambda_{7\text{ ave}}$	$\hat\lambda_{8\text{ ave}}$	$\hat\lambda_{9\text{ ave}}$	$\hat\lambda_{10\text{ ave}}$	$\hat\pi_{1\text{ ave}}$	$\hat\pi_{2\text{ ave}}$	$\hat\pi_{3\text{ ave}}$	$\hat\pi_{4\text{ ave}}$	$\hat\pi_{5\text{ ave}}$	$\hat\pi_{6\text{ ave}}$	$\hat\pi_{7\text{ ave}}$	$\hat\pi_{8\text{ ave}}$	$\hat\pi_{9\text{ ave}}$	$\hat\pi_{10\text{ ave}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.09983	0.20779	0.43445	1.55441	2.40926	3.80132	5.98220	7.72973	9.79536	68.87051	0.20831	0.11320	0.04937	0.03738	0.04015	0.04768	0.06380	0.08353	0.16719	0.18940	-134.69017	-190.14595	-83.32849	0.00002
100	0.06828	0.12681	0.28941	1.17713	1.93494	3.32481	6.19840	8.33231	9.57225	21.40563	0.19132	0.12187	0.05689	0.03817	0.03995	0.04548	0.06056	0.08276	0.17044	0.19255	-274.96152	-365.75960	-194.01291	0.00001
200	0.01841	0.07414	0.21580	0.98900	1.56629	2.80945	6.55377	8.86982	9.46386	17.20054	0.17197	0.12913	0.06660	0.03968	0.04068	0.04427	0.05759	0.08257	0.17089	0.19663	-556.88761	-711.12088	-424.14019	0.00002
500	0.00082	0.01884	0.13231	0.86541	1.30595	2.14238	7.18442	9.23765	9.41954	10.43586	0.14499	0.13673	0.07942	0.04298	0.04280	0.04418	0.05450	0.08261	0.16951	0.20227	-1406.05880	-1722.92798	-1131.75176	0.00006
1000	0.00072	0.00848	0.08601	0.79941	1.22152	1.73367	7.62243	9.34011	9.40176	9.56764	0.12885	0.13964	0.08537	0.04617	0.04493	0.04565	0.05326	0.08288	0.16917	0.20409	-2824.86957	-3374.75142	-2316.76519	0.00011

Table C.184: 10-component exponential mixture on limit order arrival VOD data.

n	$\hat\lambda_{1\text{ ave}}$	$\hat\lambda_{2\text{ ave}}$	$\hat\lambda_{3\text{ ave}}$	$\hat\lambda_{4\text{ ave}}$	$\hat\lambda_{5\text{ ave}}$	$\hat\lambda_{6\text{ ave}}$	$\hat\lambda_{7\text{ ave}}$	$\hat\lambda_{8\text{ ave}}$	$\hat\lambda_{9\text{ ave}}$	$\hat\lambda_{10\text{ ave}}$	$\hat\pi_{1\text{ ave}}$	$\hat\pi_{2\text{ ave}}$	$\hat\pi_{3\text{ ave}}$	$\hat\pi_{4\text{ ave}}$	$\hat\pi_{5\text{ ave}}$	$\hat\pi_{6\text{ ave}}$	$\hat\pi_{7\text{ ave}}$	$\hat\pi_{8\text{ ave}}$	$\hat\pi_{9\text{ ave}}$	$\hat\pi_{10\text{ ave}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.01742	0.09878	0.26621	1.13116	1.90584	3.25976	5.62052	7.59785	9.93545	84.28970	0.16655	0.12662	0.06060	0.04243	0.04299	0.04844	0.06325	0.08325	0.17124	0.19462	-113.54733	-160.79358	-66.89900	0.00006
100	0.00702	0.05373	0.19130	0.88476	1.55182	2.75001	5.70186	8.15833	9.69531	19.77145	0.14839	0.13665	0.06812	0.04310	0.04296	0.04685	0.05957	0.08188	0.17449	0.19800	-232.37902	-312.39527	-154.09426	0.00007
200	0.00299	0.03174	0.15460	0.76299	1.31896	2.24195	5.84089	8.73379	9.62031	14.81260	0.13015	0.14305	0.07814	0.04401	0.04366	0.04633	0.05605	0.08096	0.17486	0.20279	-471.14496	-610.84082	-335.53296	0.00005
500	0.00208	0.01769	0.11952	0.67272	1.18072	1.71992	6.06544	9.20562	9.56540	10.23154	0.10933	0.14600	0.09127	0.04608	0.04543	0.04707	0.05211	0.08058	0.17368	0.20845	-1189.95953	-1495.77963	-898.82778	0.00004
1000	0.00187	0.01329	0.10013	0.62692	1.13470	1.44790	6.26827	9.36647	9.53126	9.59100	0.09863	0.14551	0.09696	0.04853	0.04724	0.04848	0.04988	0.08086	0.17348	0.21044	-2389.92857	-2960.63252	-1855.18143	0.00004

Table C.185: 10-component exponential mixture on limit order arrival RIOTINTO data.

n	$\hat\lambda_{1\text{ ave}}$	$\hat\lambda_{2\text{ ave}}$	$\hat\lambda_{3\text{ ave}}$	$\hat\lambda_{4\text{ ave}}$	$\hat\lambda_{5\text{ ave}}$	$\hat\lambda_{6\text{ ave}}$	$\hat\lambda_{7\text{ ave}}$	$\hat\lambda_{8\text{ ave}}$	$\hat\lambda_{9\text{ ave}}$	$\hat\lambda_{10\text{ ave}}$	$\hat\pi_{1\text{ ave}}$	$\hat\pi_{2\text{ ave}}$	$\hat\pi_{3\text{ ave}}$	$\hat\pi_{4\text{ ave}}$	$\hat\pi_{5\text{ ave}}$	$\hat\pi_{6\text{ ave}}$	$\hat\pi_{7\text{ ave}}$	$\hat\pi_{8\text{ ave}}$	$\hat\pi_{9\text{ ave}}$	$\hat\pi_{10\text{ ave}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.00531	0.35657	0.73548	2.30397	3.40407	4.99147	6.85914	8.03501	9.55858	29.85897	0.28540	0.11260	0.03951	0.02957	0.03303	0.04215	0.06056	0.07951	0.14940	0.16827	-181.96890	-254.04734	-112.65238	0.00002
100	0.00193	0.08491	0.34998	1.74253	2.98171	4.83821	7.18432	8.36497	9.21075	22.52546	0.27019	0.12081	0.04551	0.03018	0.03198	0.04022	0.05966	0.08017	0.15226	0.16900	-370.36380	-487.98337	-254.54599	0.00008
200	0.00061	0.02301	0.20174	1.28008	2.51132	4.76790	7.66339	8.67081	9.04359	18.02501	0.25410	0.12755	0.05291	0.03164	0.03124	0.03799	0.05885	0.08085	0.15343	0.17145	-749.19645	-941.64317	-557.05015	0.00008
500	0.00027	0.01096	0.13088	0.83872	1.87683	4.67677	8.31961	8.86645	8.95099	12.10769	0.23275	0.13216	0.06446	0.03488	0.03133	0.03519	0.05811	0.08098	0.15317	0.17696	-1890.58542	-2264.52435	-1518.00835	0.00021
1000	0.00025	0.00677	0.09828	0.66680	1.52617	4.54010	8.67214	8.88036	8.90746	9.48545	0.21415	0.13699	0.07250	0.03763	0.03287	0.03348	0.05839	0.08110	0.15292	0.17996	-3797.42393	-4399.51273	-3190.90673	0.00042

Table C.186: 10-component exponential mixture on limit order arrival SSELN data.

	AIC			BIC		
2-comp-exponential	0.1830	0.3021	0.5865	0.6281	-	-
4-comp-exponential	0.5128	-	-	0.3646	0.9483	-
6-comp-exponential	0.2784	0.6234	-	0.0073	0.0517	0.9999
10-comp-exponential	0.0259	0.0745	0.4135	0.0000	0.0000	0.0001

Table C.187: ABFLN, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.1960	0.3743	0.7211	0.7338	-	-
4-comp-exponential	0.5579	-	-	0.2639	0.9810	-
6-comp-exponential	0.2274	0.5633	-	0.0023	0.0190	0.9999
10-comp-exponential	0.0188	0.0624	0.2789	0.0000	0.0000	0.0000

Table C.188: BARC, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.1696	0.2918	0.5779	0.6177	-	-
4-comp-exponential	0.5316	-	-	0.3747	0.9600	-
6-comp-exponential	0.2745	0.6333	-	0.0075	0.0399	0.9999
10-comp-exponential	0.0243	0.0749	0.4221	0.0000	0.0000	0.0001

Table C.189: RRLN, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.1753	0.3277	0.6651	0.6896	-	-
4-comp-exponential	0.5408	-	-	0.3068	0.9709	-
6-comp-exponential	0.2605	0.6002	-	0.0035	0.0291	0.9999
10-comp-exponential	0.0234	0.0721	0.3349	0.0000	0.0000	0.0001

Table C.190: VOD, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.1941	0.3786	0.7340	0.7472	-	-
4-comp-exponential	0.5457	-	-	0.2508	0.9780	-
6-comp-exponential	0.2398	0.5572	-	0.0020	0.0220	0.9999
10-comp-exponential	0.0203	0.0643	0.2660	0.0000	0.0000	0.0000

Table C.191: RIOTINTO, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.1828	0.3098	0.6096	0.6561	-	-
4-comp-exponential	0.5118	-	-	0.3370	0.9506	-
6-comp-exponential	0.2806	0.6199	-	0.0069	0.0494	0.9999
10-comp-exponential	0.0247	0.0704	0.3904	0.0000	0.0000	0.0001

Table C.192: SSELN, N = 50: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0637	0.1091	0.2528	0.3450	0.6492	-
4-comp-exponential	0.5043	-	-	0.6190	-	-
6-comp-exponential	0.3564	0.7122	-	0.0359	0.3508	0.9995
10-comp-exponential	0.0755	0.1786	0.7471	0.0000	0.0000	0.0004

Table C.193: ABFLN, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0513	0.1108	0.2975	0.3810	0.7938	-
4-comp-exponential	0.5742	-	-	0.6013	-	-
6-comp-exponential	0.3147	0.7253	-	0.0177	0.2062	0.9998
10-comp-exponential	0.0597	0.1639	0.7025	0.0000	0.0000	0.0002

Table C.194: BARC, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0549	0.0974	0.2298	0.3143	0.6323	-
4-comp-exponential	0.5158	-	-	0.6521	-	-
6-comp-exponential	0.3564	0.7244	-	0.0336	0.3677	0.9995
10-comp-exponential	0.0728	0.1781	0.7702	0.0000	0.0000	0.0004

Table C.195: RRLN, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0551	0.1055	0.2705	0.3598	0.7484	-
4-comp-exponential	0.5316	-	-	0.6164	-	-
6-comp-exponential	0.3427	0.7128	-	0.0238	0.2516	0.9996
10-comp-exponential	0.0706	0.1817	0.7295	0.0000	0.0000	0.0003

Table C.196: VOD, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0540	0.1146	0.3130	0.4048	0.8177	-
4-comp-exponential	0.5484	-	-	0.5778	-	-
6-comp-exponential	0.3315	0.7085	-	0.0174	0.1823	0.9997
10-comp-exponential	0.0661	0.1769	0.6870	0.0000	0.0000	0.0003

Table C.197: RIOTINTO, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0678	0.1140	0.2607	0.3657	0.6775	-
4-comp-exponential	0.4927	-	-	0.5988	-	-
6-comp-exponential	0.3628	0.7116	-	0.0354	0.3225	0.9996
10-comp-exponential	0.0767	0.1743	0.7393	0.0000	0.0000	0.0003

Table C.198: SSELN, N = 100: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0177	0.0290	0.0738	0.1334	0.3010	0.7310
4-comp-exponential	0.4566	-	-	0.7667	-	-
6-comp-exponential	0.3861	0.6944	-	0.0996	0.6979	-
10-comp-exponential	0.1396	0.2764	0.9261	0.0002	0.0009	0.2689

Table C.199: ABFLN, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0070	0.0168	0.0644	0.1127	0.3618	0.8749
4-comp-exponential	0.5232	-	-	0.8249	-	-
6-comp-exponential	0.3567	0.7261	-	0.0625	0.6378	-
10-comp-exponential	0.1131	0.2571	0.9356	0.0000	0.0004	0.1251

Table C.200: BARC, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0130	0.0214	0.0572	0.0982	0.2536	0.7071
4-comp-exponential	0.4666	-	-	0.8103	-	-
6-comp-exponential	0.3847	0.6979	-	0.0914	0.7452	-
10-comp-exponential	0.1357	0.2806	0.9428	0.0001	0.0012	0.2928

Table C.201: RRLN, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0128	0.0242	0.0676	0.1125	0.3395	0.8416
4-comp-exponential	0.4748	-	-	0.8097	-	-
6-comp-exponential	0.3804	0.6975	-	0.0778	0.6599	-
10-comp-exponential	0.1321	0.2783	0.9324	0.0000	0.0006	0.1584

Table C.202: VOD, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0080	0.0181	0.0680	0.1144	0.3970	0.8982
4-comp-exponential	0.4985	-	-	0.8175	-	-
6-comp-exponential	0.3683	0.7030	-	0.0680	0.6026	-
10-comp-exponential	0.1253	0.2789	0.9320	0.0000	0.0004	0.1018

Table C.203: RIOTINTO, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0142	0.0259	0.0736	0.1311	0.3082	0.7737
4-comp-exponential	0.4510	-	-	0.7694	-	-
6-comp-exponential	0.3978	0.7006	-	0.0994	0.6913	-
10-comp-exponential	0.1370	0.2735	0.9264	0.0001	0.0005	0.2263

Table C.204: SSELN, N = 200: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0014	0.0028	0.0056	0.0114	0.0400	0.2001
4-comp-exponential	0.3602	0.5686	-	0.7597	-	-
6-comp-exponential	0.4160	-	-	0.2268	0.9437	-
10-comp-exponential	0.2222	0.4283	0.9942	0.0019	0.0161	0.7997

Table C.205: ABFLN, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0002	0.0006	0.0023	0.0060	0.0409	0.2482
4-comp-exponential	0.4123	-	-	0.8385	-	-
6-comp-exponential	0.3973	0.6611	-	0.1553	0.9487	-
10-comp-exponential	0.1901	0.3383	0.9977	0.0002	0.0103	0.7517

Table C.206: BARC, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0027	0.0035	0.0063	0.0090	0.0268	0.1462
4-comp-exponential	0.3536	0.5749	-	0.7802	-	-
6-comp-exponential	0.4078	-	-	0.2099	0.9578	-
10-comp-exponential	0.2356	0.4212	0.9934	0.0007	0.0151	0.8534

Table C.207: RRLN, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0014	0.0022	0.0066	0.0094	0.0354	0.2376
4-comp-exponential	0.3775	0.6149	-	0.7985	-	-
6-comp-exponential	0.4072	-	-	0.1916	0.9527	-
10-comp-exponential	0.2139	0.3829	0.9934	0.0005	0.0119	0.7624

Table C.208: VOD, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0003	0.0005	0.0023	0.0046	0.0329	0.2876
4-comp-exponential	0.4119	-	-	0.8275	-	-
6-comp-exponential	0.3870	0.6399	-	0.1676	0.9558	-
10-comp-exponential	0.2009	0.3596	0.9977	0.0003	0.0112	0.7124

Table C.209: RIOTINTO, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0006	0.0008	0.0039	0.0112	0.0355	0.2039
4-comp-exponential	0.3640	0.5853	-	0.7692	-	-
6-comp-exponential	0.4025	-	-	0.2186	0.9506	-
10-comp-exponential	0.2329	0.4138	0.9961	0.0010	0.0138	0.7961

Table C.210: SSELN, N = 500: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0000	0.0000	0.1180	0.0009	0.0024	0.0216
4-comp-exponential	0.2793	0.4584	0.8815	0.6305	-	-
6-comp-exponential	0.4292	-	-	0.3611	0.9520	-
10-comp-exponential	0.2910	0.5411	-	0.0071	0.0451	0.9779

Table C.211: ABFLN, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0001	0.0001	0.0002	0.0001	0.0013	0.0328
4-comp-exponential	0.3284	0.5462	-	0.7511	-	-
6-comp-exponential	0.4173	-	-	0.2455	0.9628	-
10-comp-exponential	0.2541	0.4536	0.9997	0.0033	0.0358	0.9671

Table C.212: BARC, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0003	0.0003	0.0401	0.0013	0.0050	0.0184
4-comp-exponential	0.2693	0.4580	0.9592	0.6497	-	-
6-comp-exponential	0.4329	-	-	0.3406	0.9491	-
10-comp-exponential	0.2968	0.5410	-	0.0077	0.0452	0.9809

Table C.213: RRLN, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0002	0.0003	0.0029	0.0011	0.0024	0.0204
4-comp-exponential	0.2887	0.4912	0.9971	0.6775	-	-
6-comp-exponential	0.4249	-	-	0.3172	0.9520	-
10-comp-exponential	0.2862	0.5085	-	0.0041	0.0456	0.9796

Table C.214: VOD, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0000	0.0000	0.0001	0.0001	0.0008	0.0191
4-comp-exponential	0.3406	0.5578	-	0.7395	-	-
6-comp-exponential	0.4034	-	-	0.2568	0.9584	-
10-comp-exponential	0.2559	0.4422	0.9999	0.0036	0.0409	0.9809

Table C.215: RIOTINTO, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0004	0.0004	0.0888	0.0008	0.0029	0.0200
4-comp-exponential	0.2948	0.4775	0.9112	0.6468	-	-
6-comp-exponential	0.4274	-	-	0.3394	0.9545	-
10-comp-exponential	0.2773	0.5221	-	0.0129	0.0425	0.9800

Table C.216: SSELN, N = 1000: limit order arrival times, zero inflated (exponentially distributed).

n	$\hat\lambda_{1\ ave}$	$\pm\hat\lambda_{1\ ave}$	$\hat\lambda_{1\ median}$	$\hat\lambda_{1\ lower}$	$\hat\lambda_{1\ upper}$	$\hat\lambda_{2\ ave}$	$\pm\hat\lambda_{2\ ave}$	$\hat\lambda_{2\ median}$	$\hat\lambda_{2\ lower}$	$\hat\lambda_{2\ upper}$	$\hat\lambda_{3\ ave}$	$\pm\hat\lambda_{3\ ave}$	$\hat\lambda_{3\ median}$	$\hat\lambda_{3\ lower}$	$\hat\lambda_{3\ upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	$\hat\pi_{3\ ave}$	$\pm\hat\pi_{3\ ave}$	$\hat\pi_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.094711	1.238391	0.000319	0.000148	0.000807	1.758	4.862	0.072	0.023	0.671	13.358	5.754	16.670	4.278	17.046	0.307	0.127	0.303	0.170	0.098	0.155	0.524	0.139	0.534	-210.255	-276.144	-147.872	0.000
100	0.011833	0.448610	0.000301	0.000154	0.000638	0.771	3.208	0.054	0.023	0.308	13.564	6.136	17.426	4.393	17.758	0.305	0.107	0.302	0.161	0.086	0.149	0.535	0.125	0.540	-426.762	-535.911	-324.255	0.000
200	0.000370	0.000331	0.000287	0.000161	0.000549	0.301	1.803	0.045	0.023	0.186	13.578	6.571	18.179	4.528	18.481	0.303	0.090	0.302	0.157	0.073	0.150	0.540	0.110	0.543	-862.715	-1040.596	-691.105	0.000
500	0.000334	0.000228	0.000275	0.000166	0.000493	0.076	0.320	0.041	0.025	0.108	13.371	7.149	19.154	4.672	19.445	0.302	0.071	0.303	0.156	0.059	0.149	0.543	0.089	0.544	-2176.615	-2522.078	-1835.548	0.000
1000	0.000320	0.000198	0.000267	0.000173	0.000466	0.059	0.064	0.040	0.027	0.090	13.027	7.513	19.836	4.769	20.174	0.302	0.060	0.301	0.156	0.051	0.149	0.543	0.073	0.544	-4373.001	-4957.164	-3799.173	0.000

Table C.217: 3-component exponential mixture on limit order arrival ABFLN data with censoring on region [0, 0.5].

n	$\hat\lambda_{1\ ave}$	$\pm\hat\lambda_{1\ ave}$	$\hat\lambda_{1\ median}$	$\hat\lambda_{1\ lower}$	$\hat\lambda_{1\ upper}$	$\hat\lambda_{2\ ave}$	$\pm\hat\lambda_{2\ ave}$	$\hat\lambda_{2\ median}$	$\hat\lambda_{2\ lower}$	$\hat\lambda_{2\ upper}$	$\hat\lambda_{3\ ave}$	$\pm\hat\lambda_{3\ ave}$	$\hat\lambda_{3\ median}$	$\hat\lambda_{3\ lower}$	$\hat\lambda_{3\ upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	$\hat\pi_{3\ ave}$	$\pm\hat\pi_{3\ ave}$	$\hat\pi_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.057738	0.907880	0.002703	0.001099	0.007626	0.443	2.041	0.115	0.035	0.346	10.855	6.995	16.508	2.293	16.985	0.223	0.120	0.212	0.221	0.109	0.208	0.556	0.137	0.569	-158.579	-206.070	-111.111	0.000
100	0.031003	0.675184	0.002423	0.001118	0.005360	0.241	1.200	0.108	0.036	0.275	11.002	7.315	17.259	2.540	17.714	0.220	0.101	0.212	0.223	0.091	0.213	0.557	0.120	0.568	-322.990	-404.408	-242.145	0.000
200	0.008333	0.302377	0.002264	0.001144	0.004378	0.163	0.657	0.105	0.037	0.237	11.163	7.638	18.002	2.786	18.455	0.219	0.085	0.213	0.224	0.077	0.216	0.558	0.106	0.567	-653.735	-797.110	-511.542	0.000
500	0.002511	0.001620	0.002136	0.001203	0.003718	0.126	0.144	0.104	0.040	0.210	11.357	8.059	6.276	3.117	19.440	0.219	0.070	0.214	0.224	0.063	0.216	0.557	0.092	0.566	-1650.158	-1964.768	-1336.372	0.000
1000	0.002359	0.001341	0.002075	0.001240	0.003428	0.120	0.080	0.104	0.042	0.200	11.578	8.385	5.497	3.325	20.186	0.219	0.062	0.214	0.225	0.056	0.217	0.556	0.084	0.563	-3314.489	-3878.578	-2726.988	0.000

Table C.218: 3-component exponential mixture on limit order arrival BARC data with censoring on region [0, 0.5].

n	$\hat\lambda_{1\ ave}$	$\pm\hat\lambda_{1\ ave}$	$\hat\lambda_{1\ median}$	$\hat\lambda_{1\ lower}$	$\hat\lambda_{1\ upper}$	$\hat\lambda_{2\ ave}$	$\pm\hat\lambda_{2\ ave}$	$\hat\lambda_{2\ median}$	$\hat\lambda_{2\ lower}$	$\hat\lambda_{2\ upper}$	$\hat\lambda_{3\ ave}$	$\pm\hat\lambda_{3\ ave}$	$\hat\lambda_{3\ median}$	$\hat\lambda_{3\ lower}$	$\hat\lambda_{3\ upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	$\hat\pi_{3\ ave}$	$\pm\hat\pi_{3\ ave}$	$\hat\pi_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.088671	1.186608	0.000437	0.000195	0.001112	1.376	4.305	0.075	0.022	0.501	13.098	5.955	16.663	4.056	17.032	0.290	0.120	0.285	0.183	0.110	0.164	0.527	0.133	0.535	-202.931	-260.453	-149.043	0.000
100	0.027138	0.675298	0.000404	0.000204	0.000853	0.507	2.512	0.056	0.021	0.242	13.317	6.302	17.422	4.288	17.737	0.287	0.100	0.285	0.178	0.095	0.164	0.535	0.116	0.540	-412.642	-504.277	-327.071	0.000
200	0.003120	0.219023	0.000383	0.000210	0.000718	0.172	1.159	0.046	0.022	0.158	13.373	6.695	18.181	4.457	18.460	0.285	0.083	0.284	0.177	0.080	0.167	0.538	0.098	0.540	-834.803	-981.610	-697.174	0.000
500	0.000461	0.000998	0.000366	0.000222	0.000622	0.079	0.450	0.042	0.024	0.114	13.315	7.207	19.183	4.647	19.421	0.285	0.065	0.284	0.177	0.067	0.169	0.538	0.076	0.540	-2106.833	-2378.236	-1837.396	0.000
1000	0.000419	0.000336	0.000359	0.000231	0.000580	0.060	0.051	0.040	0.026	0.103	13.129	7.574	19.904	4.756	20.159	0.284	0.056	0.284	0.178	0.060	0.171	0.538	0.062	0.539	-4231.984	-4686.409	-3777.856	0.000

Table C.219: 3-component exponential mixture on limit order arrival RRLN data with censoring on region [0, 0.5].

n	$\hat\lambda_{1\ ave}$	$\pm\hat\lambda_{1\ ave}$	$\hat\lambda_{1\ median}$	$\hat\lambda_{1\ lower}$	$\hat\lambda_{1\ upper}$	$\hat\lambda_{2\ ave}$	$\pm\hat\lambda_{2\ ave}$	$\hat\lambda_{2\ median}$	$\hat\lambda_{2\ lower}$	$\hat\lambda_{2\ upper}$	$\hat\lambda_{3\ ave}$	$\pm\hat\lambda_{3\ ave}$	$\hat\lambda_{3\ median}$	$\hat\lambda_{3\ lower}$	$\hat\lambda_{3\ upper}$	$\hat\pi_{1\ ave}$	$\pm\hat\pi_{1\ ave}$	$\hat\pi_{1\ median}$	$\hat\pi_{2\ ave}$	$\pm\hat\pi_{2\ ave}$	$\hat\pi_{2\ median}$	$\hat\pi_{3\ ave}$	$\pm\hat\pi_{3\ ave}$	$\hat\pi_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.193174	1.688270	0.001225	0.000493	0.003276	0.807	3.069	0.103	0.027	0.439	12.155	6.346	16.665	3.537	17.032	0.242	0.113	0.241	0.184	0.096	0.172	0.573	0.119	0.577	-167.962	-217.430	-122.326	0.000
100	0.133440	1.423538	0.001115	0.000502	0.002493	0.388	1.917	0.087	0.026	0.289	12.022	6.733	17.398	3.732	17.744	0.240	0.094	0.241	0.182	0.080	0.173	0.578	0.102	0.579	-341.830	-423.445	-269.313	0.000
200	0.026780	0.556579	0.001035	0.000509	0.002068	0.209	1.052	0.077	0.027	0.223	11.716	7.095	18.089	3.878	18.471	0.239	0.079	0.242	0.180	0.067	0.174	0.581	0.091	0.580	-691.509	-830.963	-571.511	0.000
500	0.001179	0.000964	0.000986	0.000523	0.001759	0.117	0.248	0.065	0.028	0.176	10.940	7.434	5.414	4.029	19.443	0.238	0.063	0.240	0.177	0.053	0.173	0.585	0.078	0.582	-1746.010	-2034.743	-1495.157	0.000
1000	0.001109	0.000693	0.000961	0.000535	0.001636	0.088	0.097	0.059	0.029	0.151	10.141	7.498	5.116	4.113	20.168	0.237	0.053	0.237	0.175	0.046	0.172	0.588	0.069	0.585	-3509.509	-4014.018	-3040.974	0.000

Table C.220: 3-component exponential mixture on limit order arrival VOD data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.033375	0.671456	0.003178	0.001171	0.010221	0.429	1.980	0.112	0.034	0.353	11.937	6.473	16.660	3.370	17.021	0.205	0.105	0.197	0.213	0.103	0.203	0.583	0.116	0.590	-146.753	-188.463	-105.532	0.000
100	0.013044	0.392782	0.002837	0.001189	0.007072	0.227	1.097	0.105	0.036	0.276	12.064	6.816	17.413	3.565	17.749	0.201	0.091	0.196	0.216	0.086	0.209	0.583	0.101	0.589	-298.920	-369.859	-229.362	0.000
200	0.004605	0.135164	0.002683	0.001220	0.005794	0.154	0.596	0.100	0.037	0.230	12.064	7.175	18.148	3.719	18.489	0.199	0.078	0.195	0.217	0.071	0.212	0.583	0.089	0.589	-605.067	-729.845	-484.163	0.000
500	0.003199	0.002367	0.002577	0.001259	0.005043	0.119	0.155	0.096	0.041	0.196	12.029	7.630	19.055	3.866	19.473	0.199	0.065	0.195	0.218	0.057	0.213	0.583	0.076	0.590	-1527.083	-1803.484	-1264.798	0.000
1000	0.003024	0.002050	0.002515	0.001286	0.004678	0.110	0.072	0.094	0.043	0.181	11.988	7.958	5.544	3.962	20.217	0.198	0.059	0.195	0.218	0.050	0.213	0.583	0.070	0.591	-3067.359	-3591.461	-2584.212	0.000

Table C.221: 3-component exponential mixture on limit order arrival RIOTINTO data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.011354	0.427137	0.000352	0.000170	0.000853	1.730	4.839	0.071	0.020	0.667	13.473	5.639	16.671	4.475	17.045	0.315	0.126	0.306	0.167	0.098	0.153	0.518	0.140	0.529	-212.947	-276.949	-151.217	0.000
100	0.005004	0.286894	0.000336	0.000179	0.000664	0.708	3.087	0.050	0.019	0.289	13.686	6.015	17.423	4.611	17.760	0.311	0.106	0.306	0.160	0.086	0.149	0.529	0.124	0.536	-432.405	-536.714	-329.988	0.000
200	0.001961	0.172651	0.000322	0.000187	0.000566	0.203	1.386	0.039	0.019	0.158	13.645	6.469	18.164	4.703	18.478	0.308	0.090	0.304	0.158	0.073	0.151	0.533	0.106	0.538	-874.058	-1044.726	-703.590	0.000
500	0.000351	0.000202	0.000310	0.000196	0.000486	0.064	0.301	0.034	0.020	0.091	13.382	7.032	19.126	4.879	19.434	0.305	0.072	0.302	0.160	0.059	0.156	0.534	0.083	0.538	-2204.740	-2532.347	-1875.727	0.000
1000	0.000334	0.000153	0.000306	0.000204	0.000449	0.048	0.047	0.033	0.022	0.071	13.288	7.400	19.859	5.002	20.169	0.304	0.060	0.300	0.161	0.050	0.158	0.535	0.068	0.537	-4427.614	-4968.043	-3893.835	0.000

Table C.222: 3-component exponential mixture on limit order arrival SSELN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.098630	1.250105	0.000318	0.000147	0.000800	1.857	4.896	0.066	0.022	0.693	10.528	5.486	8.229	4.771	16.953	0.305	0.127	0.302	0.167	0.098	0.152	0.527	0.139	0.537	-204.363	-268.863	-143.322	0.000
100	0.011976	0.449590	0.000298	0.000152	0.000629	0.854	3.300	0.047	0.021	0.246	9.158	4.829	7.381	4.915	17.300	0.302	0.108	0.300	0.158	0.085	0.145	0.541	0.124	0.548	-413.717	-520.154	-313.532	0.000
200	0.000365	0.000414	0.000283	0.000158	0.000539	0.344	1.935	0.039	0.020	0.120	7.869	3.616	6.898	5.052	10.532	0.299	0.090	0.298	0.152	0.072	0.144	0.549	0.110	0.553	-834.453	-1007.957	-667.713	0.000
500	0.000327	0.000221	0.000272	0.000164	0.000484	0.074	0.531	0.035	0.022	0.068	6.860	2.088	6.502	5.250	8.258	0.297	0.071	0.298	0.149	0.058	0.143	0.554	0.089	0.555	-2100.533	-2438.853	-1765.616	0.000
1000	0.000314	0.000191	0.000263	0.000170	0.000460	0.046	0.226	0.034	0.023	0.057	6.523	1.368	6.380	5.353	7.613	0.297	0.059	0.296	0.149	0.049	0.143	0.555	0.073	0.555	-4215.643	-4790.327	-3662.217	0.001

Table C.223: 3-component exponential mixture on limit order arrival ABFLN data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.061833	0.924435	0.002671	0.001087	0.007531	0.462	2.087	0.106	0.033	0.309	7.311	4.927	5.873	2.885	13.857	0.221	0.120	0.209	0.215	0.110	0.202	0.564	0.138	0.578	-150.609	-197.546	-103.298	0.000
100	0.033722	0.690553	0.002376	0.001095	0.005245	0.239	1.252	0.094	0.033	0.235	6.351	3.925	5.533	3.152	9.723	0.215	0.100	0.207	0.215	0.092	0.205	0.569	0.120	0.583	-306.340	-386.088	-226.650	0.000
200	0.009016	0.311741	0.002199	0.001112	0.004245	0.152	0.744	0.086	0.033	0.185	5.624	2.975	5.339	3.399	7.551	0.212	0.084	0.206	0.214	0.076	0.206	0.574	0.106	0.586	-618.777	-758.505	-480.285	0.000
500	0.002414	0.001550	0.002059	0.001160	0.003581	0.097	0.201	0.079	0.035	0.150	5.061	2.048	5.199	3.699	6.484	0.210	0.068	0.204	0.211	0.061	0.204	0.579	0.091	0.590	-1557.975	-1864.624	-1255.683	0.000
1000	0.002261	0.001276	0.001995	0.001187	0.003292	0.087	0.096	0.077	0.036	0.136	4.867	1.689	5.151	3.877	6.115	0.209	0.060	0.204	0.210	0.052	0.204	0.581	0.082	0.591	-3126.244	-3678.732	-2561.884	0.000

Table C.224: 3-component exponential mixture on limit order arrival BARC data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.093448	1.201197	0.000434	0.000194	0.001104	1.457	4.347	0.068	0.021	0.406	10.133	5.475	7.862	4.612	16.932	0.288	0.120	0.284	0.181	0.111	0.162	0.531	0.132	0.539	-197.145	-253.725	-143.964	0.000
100	0.028071	0.680355	0.000399	0.000201	0.000841	0.564	2.620	0.048	0.020	0.193	8.723	4.637	7.183	4.839	15.330	0.284	0.100	0.282	0.174	0.095	0.160	0.542	0.114	0.549	-399.572	-489.735	-315.272	0.000
200	0.003108	0.219052	0.000376	0.000207	0.000702	0.182	1.245	0.039	0.020	0.114	7.496	3.342	6.686	5.001	9.892	0.280	0.083	0.279	0.172	0.081	0.161	0.548	0.098	0.550	-806.382	-950.063	-671.399	0.000
500	0.000449	0.001000	0.000359	0.000218	0.000602	0.067	0.518	0.035	0.021	0.076	6.640	1.940	6.385	5.196	8.020	0.279	0.064	0.279	0.171	0.067	0.163	0.550	0.076	0.552	-2030.828	-2293.600	-1768.950	0.000
1000	0.000410	0.000325	0.000353	0.000227	0.000564	0.044	0.031	0.034	0.022	0.066	6.361	1.340	6.280	5.314	7.466	0.279	0.054	0.278	0.171	0.060	0.164	0.551	0.061	0.553	-4075.492	-4514.352	-3636.603	0.000

Table C.225: 3-component exponential mixture on limit order arrival RRLN data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.236418	1.782255	0.001213	0.000489	0.003240	0.887	3.118	0.092	0.025	0.376	8.465	4.888	6.696	4.160	16.567	0.240	0.113	0.238	0.180	0.098	0.167	0.580	0.116	0.588	-162.036	-210.922	-116.693	0.000
100	0.168263	1.513460	0.001095	0.000494	0.002452	0.442	2.038	0.072	0.024	0.232	7.150	3.647	6.064	4.346	9.860	0.236	0.093	0.236	0.177	0.082	0.167	0.587	0.096	0.592	-328.857	-409.142	-256.947	0.000
200	0.043349	0.664070	0.001011	0.000498	0.002005	0.263	1.340	0.060	0.024	0.158	6.242	2.374	5.761	4.491	7.681	0.233	0.078	0.235	0.175	0.068	0.167	0.593	0.082	0.595	-663.438	-800.431	-544.280	0.000
500	0.001139	0.000932	0.000958	0.000508	0.001693	0.112	0.543	0.050	0.025	0.110	5.674	1.261	5.551	4.650	6.622	0.230	0.061	0.232	0.169	0.054	0.166	0.600	0.075	0.597	-1670.248	-1952.487	-1420.188	0.000
1000	0.001072	0.000664	0.000934	0.000519	0.001576	0.066	0.244	0.047	0.026	0.092	5.514	0.870	5.479	4.751	6.274	0.229	0.051	0.230	0.168	0.046	0.165	0.603	0.066	0.600	-3352.233	-3846.655	-2889.575	0.000

Table C.226: 3-component exponential mixture on limit order arrival VOD data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.033416	0.671769	0.003128	0.001156	0.010045	0.439	2.031	0.101	0.032	0.298	7.927	4.586	6.248	3.988	14.122	0.201	0.105	0.193	0.205	0.103	0.196	0.593	0.116	0.601	-140.149	-181.984	-98.210	0.000
100	0.012911	0.392834	0.002759	0.001162	0.006847	0.214	1.139	0.088	0.032	0.227	6.852	3.486	5.813	4.171	9.738	0.195	0.090	0.189	0.207	0.085	0.200	0.598	0.100	0.605	-285.352	-355.694	-215.587	0.000
200	0.004452	0.135168	0.002588	0.001182	0.005530	0.129	0.628	0.078	0.033	0.171	6.080	2.446	5.536	4.308	7.538	0.191	0.077	0.186	0.208	0.071	0.202	0.601	0.086	0.608	-576.193	-698.750	-456.773	0.000
500	0.003029	0.002206	0.002477	0.001203	0.004738	0.087	0.156	0.070	0.036	0.133	5.534	1.431	5.342	4.446	6.514	0.188	0.063	0.184	0.207	0.057	0.203	0.605	0.072	0.611	-1449.839	-1716.711	-1193.052	0.000
1000	0.002857	0.001898	0.002422	0.001215	0.004400	0.077	0.049	0.067	0.038	0.118	5.361	1.018	5.265	4.528	6.154	0.187	0.056	0.183	0.208	0.048	0.203	0.606	0.064	0.613	-2908.162	-3410.219	-2436.304	0.000

Table C.227: 3-component exponential mixture on limit order arrival RIOTINTO data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.011366	0.427163	0.000351	0.000170	0.000845	1.825	4.871	0.064	0.019	0.659	10.728	5.427	8.708	4.954	16.969	0.313	0.126	0.304	0.165	0.099	0.150	0.521	0.140	0.534	-207.354	-269.757	-146.915	0.000
100	0.004998	0.286900	0.000332	0.000177	0.000653	0.780	3.187	0.043	0.017	0.227	9.322	4.779	7.575	5.125	17.315	0.308	0.107	0.302	0.157	0.086	0.145	0.535	0.124	0.544	-419.903	-521.616	-319.977	0.000
200	0.001953	0.172651	0.000317	0.000184	0.000551	0.235	1.558	0.033	0.017	0.105	7.958	3.473	7.023	5.268	10.444	0.303	0.090	0.299	0.155	0.073	0.147	0.542	0.106	0.547	-846.464	-1013.139	-680.922	0.000
500	0.000343	0.000195	0.000305	0.000193	0.000473	0.056	0.409	0.029	0.017	0.057	7.000	1.970	6.669	5.454	8.331	0.299	0.072	0.296	0.156	0.058	0.152	0.545	0.083	0.549	-2129.973	-2449.126	-1809.818	0.000
1000	0.000326	0.000148	0.000301	0.000199	0.000443	0.034	0.026	0.028	0.018	0.046	6.702	1.292	6.598	5.599	7.730	0.298	0.059	0.295	0.156	0.049	0.153	0.546	0.067	0.548	-4272.500	-4800.333	-3751.137	0.000

Table C.228: 3-component exponential mixture on limit order arrival SSELN data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\lambda}_{3\ ave}$	$\pm\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{3\ median}$	$\hat{\lambda}_{3\ lower}$	$\hat{\lambda}_{3\ upper}$	$\hat{\lambda}_{4\ ave}$	$\pm\hat{\lambda}_{4\ ave}$	$\hat{\lambda}_{4\ median}$	$\hat{\lambda}_{4\ lower}$	$\hat{\lambda}_{4\ upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\hat{\pi}_{4\ ave}$	$\pm\hat{\pi}_{4\ ave}$	$\hat{\pi}_{4\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.095727	1.245590	0.000301	0.000140	0.000787	1.074	3.896	0.036	0.013	0.282	5.522	7.344	1.101	0.235	16.952	16.084	2.989	16.758	16.292	17.093	0.291	0.129	0.286	0.127	0.081	0.112	0.112	0.078	0.097	0.470	0.126	0.479	-209.032	-274.455	-143.750	0.012
100	0.011959	0.451751	0.000278	0.000143	0.000605	0.454	2.512	0.028	0.009	0.122	4.009	6.672	0.803	0.255	17.522	17.194	2.172	17.528	17.185	17.811	0.284	0.113	0.281	0.127	0.071	0.116	0.103	0.071	0.089	0.486	0.115	0.494	-423.816	-532.340	-315.638	0.014
200	0.000335	0.000306	0.000263	0.000145	0.000515	0.121	1.126	0.022	0.007	0.056	2.818	5.821	0.596	0.250	1.903	18.148	1.507	18.300	18.040	18.534	0.277	0.098	0.276	0.128	0.062	0.120	0.097	0.062	0.085	0.498	0.102	0.504	-855.353	-1032.025	-673.079	0.015
500	0.000298	0.000203	0.000251	0.000150	0.000452	0.029	0.307	0.019	0.006	0.034	1.342	3.829	0.455	0.239	1.020	19.278	0.691	19.317	19.124	19.493	0.271	0.079	0.272	0.129	0.051	0.123	0.092	0.050	0.083	0.509	0.085	0.512	-2157.373	-2499.645	-1796.647	0.014
1000	0.000283	0.000173	0.000242	0.000151	0.000426	0.019	0.017	0.017	0.005	0.028	0.700	2.117	0.415	0.237	0.783	20.073	0.174	20.083	19.931	20.225	0.267	0.067	0.268	0.130	0.044	0.124	0.090	0.042	0.084	0.513	0.071	0.514	-4332.045	-4914.767	-3722.126	0.016

Table C.229: 4-component exponential mixture on limit order arrival ABFLN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\lambda}_{3\ ave}$	$\pm\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{3\ median}$	$\hat{\lambda}_{3\ lower}$	$\hat{\lambda}_{3\ upper}$	$\hat{\lambda}_{4\ ave}$	$\pm\hat{\lambda}_{4\ ave}$	$\hat{\lambda}_{4\ median}$	$\hat{\lambda}_{4\ lower}$	$\hat{\lambda}_{4\ upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\hat{\pi}_{4\ ave}$	$\pm\hat{\pi}_{4\ ave}$	$\hat{\pi}_{4\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.057820	0.912494	0.002454	0.000976	0.007415	0.266	1.657	0.056	0.017	0.185	1.870	4.199	0.572	0.172	1.964	13.904	5.931	16.724	3.251	17.048	0.199	0.120	0.185	0.156	0.084	0.144	0.151	0.098	0.134	0.495	0.137	0.510	-157.550	-204.430	-108.389	0.011
100	0.030827	0.678999	0.002089	0.000913	0.004864	0.132	1.020	0.041	0.014	0.132	1.208	2.974	0.549	0.198	1.422	15.065	5.799	17.508	16.271	17.769	0.185	0.100	0.174	0.162	0.073	0.154	0.158	0.092	0.143	0.495	0.124	0.509	-320.255	-400.249	-236.464	0.012
200	0.007911	0.304341	0.001864	0.000858	0.003812	0.069	0.532	0.031	0.011	0.094	0.863	2.068	0.503	0.208	1.218	16.055	5.762	18.287	17.807	18.497	0.171	0.084	0.163	0.167	0.063	0.160	0.164	0.082	0.153	0.498	0.110	0.510	-647.063	-787.056	-499.866	0.013
500	0.001983	0.001429	0.001670	0.000804	0.003156	0.036	0.115	0.023	0.008	0.066	0.614	1.134	0.430	0.203	1.036	17.162	5.873	19.312	19.011	19.471	0.157	0.066	0.152	0.168	0.052	0.164	0.171	0.070	0.161	0.503	0.093	0.514	-1629.823	-1932.111	-1305.586	0.015
1000	0.001810	0.001184	0.001575	0.000784	0.002854	0.028	0.031	0.020	0.007	0.052	0.519	0.609	0.385	0.199	0.944	17.909	6.050	20.084	19.848	20.221	0.150	0.055	0.146	0.169	0.046	0.165	0.174	0.063	0.166	0.507	0.081	0.516	-3270.261	-3808.014	-2661.660	0.017

Table C.230: 4-component exponential mixture on limit order arrival BARC data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\lambda}_{3\ ave}$	$\pm\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{3\ median}$	$\hat{\lambda}_{3\ lower}$	$\hat{\lambda}_{3\ upper}$	$\hat{\lambda}_{4\ ave}$	$\pm\hat{\lambda}_{4\ ave}$	$\hat{\lambda}_{4\ median}$	$\hat{\lambda}_{4\ lower}$	$\hat{\lambda}_{4\ upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\hat{\pi}_{4\ ave}$	$\pm\hat{\pi}_{4\ ave}$	$\hat{\pi}_{4\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.089567	1.193799	0.000406	0.000182	0.001080	0.835	3.421	0.036	0.011	0.227	4.857	7.056	0.887	0.192	16.918	15.862	3.502	16.754	16.277	17.085	0.271	0.122	0.265	0.141	0.093	0.121	0.115	0.081	0.100	0.474	0.123	0.484	-201.650	-258.724	-145.153	0.013
100	0.027404	0.679754	0.000364	0.000181	0.000802	0.277	1.885	0.026	0.008	0.109	3.316	6.145	0.660	0.210	17.207	17.015	2.779	17.526	17.189	17.788	0.260	0.105	0.258	0.143	0.079	0.129	0.108	0.076	0.093	0.489	0.110	0.498	-409.307	-500.382	-319.438	0.014
200	0.003095	0.220479	0.000335	0.000183	0.000651	0.074	0.751	0.020	0.005	0.056	2.122	5.014	0.495	0.200	1.468	18.005	2.203	18.295	18.053	18.506	0.249	0.091	0.250	0.146	0.068	0.136	0.106	0.068	0.093	0.498	0.096	0.505	-827.303	-972.493	-684.089	0.013
500	0.000395	0.000996	0.000309	0.000185	0.000548	0.028	0.365	0.015	0.004	0.031	0.908	2.923	0.365	0.174	0.841	19.142	1.752	19.312	19.141	19.465	0.238	0.076	0.241	0.148	0.058	0.140	0.107	0.058	0.098	0.506	0.078	0.511	-2085.074	-2354.506	-1800.071	0.016
1000	0.000351	0.000318	0.000299	0.000188	0.000506	0.016	0.017	0.013	0.003	0.026	0.461	1.234	0.327	0.169	0.639	19.957	1.487	20.078	19.948	20.200	0.234	0.065	0.236	0.149	0.051	0.143	0.108	0.051	0.100	0.509	0.065	0.512	-4188.834	-4634.694	-3703.786	0.016

Table C.231: 4-component exponential mixture on limit order arrival RRLN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\lambda}_{3\ ave}$	$\pm\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{3\ median}$	$\hat{\lambda}_{3\ lower}$	$\hat{\lambda}_{3\ upper}$	$\hat{\lambda}_{4\ ave}$	$\pm\hat{\lambda}_{4\ ave}$	$\hat{\lambda}_{4\ median}$	$\hat{\lambda}_{4\ lower}$	$\hat{\lambda}_{4\ upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\hat{\pi}_{4\ ave}$	$\pm\hat{\pi}_{4\ ave}$	$\hat{\pi}_{4\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.194041	1.696017	0.001119	0.000444	0.003153	0.505	2.489	0.045	0.013	0.208	3.266	5.717	0.937	0.234	2.672	15.767	3.692	16.790	16.379	17.084	0.220	0.112	0.216	0.138	0.079	0.126	0.128	0.084	0.113	0.514	0.115	0.519	-166.722	-215.709	-119.329	0.011
100	0.134368	1.431325	0.000982	0.000426	0.002317	0.252	1.691	0.031	0.010	0.125	2.014	4.319	0.781	0.265	1.902	17.072	2.688	17.560	17.285	17.795	0.209	0.094	0.208	0.143	0.069	0.133	0.128	0.076	0.117	0.520	0.100	0.524	-338.580	-419.020	-262.923	0.012
200	0.026872	0.559745	0.000886	0.000409	0.001868	0.117	0.924	0.023	0.008	0.079	1.320	3.167	0.650	0.264	1.454	18.133	1.821	18.331	18.132	18.517	0.199	0.079	0.200	0.145	0.060	0.138	0.130	0.065	0.121	0.526	0.087	0.526	-683.794	-821.029	-559.332	0.012
500	0.000986	0.000884	0.000814	0.000399	0.001559	0.042	0.220	0.018	0.007	0.046	0.836	1.987	0.516	0.250	1.163	19.318	0.857	19.350	19.210	19.494	0.190	0.064	0.192	0.147	0.049	0.143	0.131	0.052	0.124	0.532	0.074	0.530	-1724.005	-2005.247	-1462.870	0.013
1000	0.000917	0.000620	0.000781	0.000397	0.001444	0.024	0.061	0.016	0.006	0.035	0.633	1.136	0.463	0.249	1.024	20.104	0.639	20.120	20.003	20.244	0.185	0.054	0.186	0.147	0.042	0.144	0.131	0.045	0.126	0.536	0.065	0.532	-3463.307	-3957.550	-2976.366	0.014

Table C.232: 4-component exponential mixture on limit order arrival VOD data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\lambda}_{3\ ave}$	$\pm\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{3\ median}$	$\hat{\lambda}_{3\ lower}$	$\hat{\lambda}_{3\ upper}$	$\hat{\lambda}_{4\ ave}$	$\pm\hat{\lambda}_{4\ ave}$	$\hat{\lambda}_{4\ median}$	$\hat{\lambda}_{4\ lower}$	$\hat{\lambda}_{4\ upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\hat{\pi}_{4\ ave}$	$\pm\hat{\pi}_{4\ ave}$	$\hat{\pi}_{4\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.033256	0.675137	0.002842	0.001026	0.010047	0.244	1.586	0.050	0.017	0.174	1.962	4.269	0.645	0.175	1.987	15.327	4.361	16.798	16.318	17.079	0.177	0.103	0.166	0.155	0.083	0.144	0.140	0.091	0.125	0.528	0.114	0.536	-145.682	-186.856	-103.181	0.011
100	0.012540	0.394957	0.002374	0.000956	0.006231	0.109	0.875	0.037	0.014	0.123	1.266	2.995	0.617	0.212	1.483	16.641	3.627	17.570	17.258	17.799	0.163	0.087	0.155	0.164	0.072	0.157	0.145	0.082	0.133	0.528	0.102	0.534	-296.119	-366.013	-224.732	0.011
200	0.003920	0.135908	0.002159	0.000902	0.004800	0.058	0.456	0.030	0.012	0.086	0.904	2.013	0.571	0.236	1.269	17.808	2.836	18.339	18.111	18.526	0.150	0.073	0.145	0.170	0.060	0.166	0.151	0.073	0.142	0.528	0.090	0.533	-598.320	-720.777	-474.232	0.011
500	0.002477	0.001954	0.002021	0.000849	0.004060	0.034	0.034	0.025	0.010	0.056	0.658	0.999	0.493	0.244	1.093	19.158	1.760	19.355	19.189	19.499	0.140	0.056	0.136	0.174	0.048	0.171	0.156	0.061	0.149	0.530	0.075	0.533	-1507.617	-1777.445	-1238.864	0.013
1000	0.002313	0.001689	0.001956	0.000818	0.003793	0.029	0.027	0.023	0.009	0.045	0.579	0.581	0.445	0.246	0.996	20.024	1.219	20.121	19.983	20.245	0.135	0.047	0.132	0.175	0.041	0.173	0.158	0.054	0.151	0.531	0.066	0.534	-3025.659	-3534.333	-2532.415	0.012

Table C.233: 4-component exponential mixture on limit order arrival RIOTINTO data with censoring on region [0, 0.5].

n	$\hat\lambda_{1\,ave}$	$\pm\hat\lambda_{1\,ave}$	$\hat\lambda_{1\,median}$	$\hat\lambda_{1\,lower}$	$\hat\lambda_{1\,upper}$	$\hat\lambda_{2\,ave}$	$\pm\hat\lambda_{2\,ave}$	$\hat\lambda_{2\,median}$	$\hat\lambda_{2\,lower}$	$\hat\lambda_{2\,upper}$	$\hat\lambda_{3\,ave}$	$\pm\hat\lambda_{3\,ave}$	$\hat\lambda_{3\,median}$	$\hat\lambda_{3\,lower}$	$\hat\lambda_{3\,upper}$	$\hat\lambda_{4\,ave}$	$\pm\hat\lambda_{4\,ave}$	$\hat\lambda_{4\,median}$	$\hat\lambda_{4\,lower}$	$\hat\lambda_{4\,upper}$	$\hat\pi_{1\,ave}$	$\pm\hat\pi_{1\,ave}$	$\hat\pi_{1\,median}$	$\hat\pi_{2\,ave}$	$\pm\hat\pi_{2\,ave}$	$\hat\pi_{2\,median}$	$\hat\pi_{3\,ave}$	$\pm\hat\pi_{3\,ave}$	$\hat\pi_{3\,median}$	$\hat\pi_{4\,ave}$	$\pm\hat\pi_{4\,ave}$	$\hat\pi_{4\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.011440	0.429791	0.000331	0.000160	0.000819	1.022	3.819	0.033	0.011	0.262	5.698	7.426	1.121	0.229	16.961	16.183	2.739	16.759	16.295	17.094	0.296	0.130	0.287	0.131	0.086	0.114	0.107	0.073	0.095	0.466	0.127	0.477	-211.702	-275.358	-147.297	0.012
100	0.005017	0.288796	0.000310	0.000165	0.000616	0.373	2.288	0.023	0.008	0.113	4.126	6.775	0.791	0.247	17.530	17.279	1.836	17.527	17.182	17.809	0.286	0.113	0.281	0.134	0.076	0.122	0.097	0.064	0.086	0.483	0.114	0.492	-429.205	-532.843	-321.977	0.013
200	0.001939	0.173878	0.000289	0.000167	0.000515	0.068	0.734	0.018	0.005	0.048	2.780	5.817	0.568	0.226	1.842	18.197	1.163	18.298	18.027	18.525	0.276	0.099	0.273	0.137	0.066	0.130	0.090	0.054	0.081	0.496	0.101	0.502	-866.591	-1035.568	-687.218	0.014
500	0.000307	0.000173	0.000274	0.000169	0.000442	0.024	0.292	0.014	0.004	0.027	1.217	3.695	0.405	0.204	0.875	19.287	0.522	19.317	19.113	19.487	0.267	0.082	0.266	0.140	0.052	0.135	0.086	0.043	0.079	0.507	0.082	0.511	-2183.878	-2512.008	-1840.520	0.014
1000	0.000291	0.000136	0.000268	0.000175	0.000411	0.015	0.017	0.013	0.004	0.023	0.687	2.326	0.361	0.201	0.681	20.066	0.159	20.075	19.924	20.212	0.263	0.070	0.266	0.142	0.044	0.138	0.085	0.036	0.079	0.510	0.067	0.511	-4387.108	-4917.306	-3819.638	0.015

Table C.234: 4-component exponential mixture on limit order arrival SSELN data with censoring on region [0, 0.5].

n	$\hat\lambda_{1\,ave}$	$\pm\hat\lambda_{1\,ave}$	$\hat\lambda_{1\,median}$	$\hat\lambda_{1\,lower}$	$\hat\lambda_{1\,upper}$	$\hat\lambda_{2\,ave}$	$\pm\hat\lambda_{2\,ave}$	$\hat\lambda_{2\,median}$	$\hat\lambda_{2\,lower}$	$\hat\lambda_{2\,upper}$	$\hat\lambda_{3\,ave}$	$\pm\hat\lambda_{3\,ave}$	$\hat\lambda_{3\,median}$	$\hat\lambda_{3\,lower}$	$\hat\lambda_{3\,upper}$	$\hat\lambda_{4\,ave}$	$\pm\hat\lambda_{4\,ave}$	$\hat\lambda_{4\,median}$	$\hat\lambda_{4\,lower}$	$\hat\lambda_{4\,upper}$	$\hat\pi_{1\,ave}$	$\pm\hat\pi_{1\,ave}$	$\hat\pi_{1\,median}$	$\hat\pi_{2\,ave}$	$\pm\hat\pi_{2\,ave}$	$\hat\pi_{2\,median}$	$\hat\pi_{3\,ave}$	$\pm\hat\pi_{3\,ave}$	$\hat\pi_{3\,median}$	$\hat\pi_{4\,ave}$	$\pm\hat\pi_{4\,ave}$	$\hat\pi_{4\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.099743	1.257481	0.000302	0.000140	0.000787	1.129	3.916	0.037	0.013	0.282	5.845	6.890	1.474	0.196	16.907	11.568	5.023	12.435	5.752	17.035	0.290	0.129	0.285	0.127	0.081	0.112	0.109	0.076	0.096	0.474	0.120	0.488	-202.805	-266.524	-139.277	0.012
100	0.012098	0.452681	0.000278	0.000143	0.000603	0.501	2.561	0.028	0.009	0.126	4.424	6.016	0.838	0.204	12.806	10.731	4.592	9.492	5.982	17.451	0.283	0.113	0.281	0.126	0.071	0.115	0.101	0.067	0.090	0.489	0.106	0.501	-410.582	-515.949	-305.344	0.014
200	0.000332	0.000396	0.000262	0.000144	0.000509	0.147	1.227	0.022	0.006	0.057	2.934	4.611	0.526	0.178	7.428	9.872	3.893	8.957	6.156	14.702	0.275	0.099	0.275	0.126	0.062	0.117	0.095	0.058	0.085	0.504	0.095	0.514	-827.468	-998.354	-651.519	0.015
500	0.000292	0.000195	0.000246	0.000145	0.000443	0.029	0.326	0.017	0.004	0.033	1.472	2.876	0.364	0.135	3.663	9.074	3.198	8.255	6.195	12.867	0.265	0.082	0.267	0.127	0.052	0.120	0.087	0.046	0.080	0.521	0.083	0.525	-2083.650	-2418.581	-1732.307	0.016
1000	0.000274	0.000171	0.000235	0.000144	0.000413	0.017	0.018	0.014	0.003	0.027	0.770	1.795	0.305	0.104	0.603	8.697	2.871	7.852	6.260	11.866	0.257	0.073	0.261	0.130	0.048	0.122	0.084	0.039	0.079	0.529	0.073	0.531	-4183.673	-4752.639	-3590.819	0.016

Table C.235: 4-component exponential mixture on limit order arrival ABFLN data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat\lambda_{1\,ave}$	$\pm\hat\lambda_{1\,ave}$	$\hat\lambda_{1\,median}$	$\hat\lambda_{1\,lower}$	$\hat\lambda_{1\,upper}$	$\hat\lambda_{2\,ave}$	$\pm\hat\lambda_{2\,ave}$	$\hat\lambda_{2\,median}$	$\hat\lambda_{2\,lower}$	$\hat\lambda_{2\,upper}$	$\hat\lambda_{3\,ave}$	$\pm\hat\lambda_{3\,ave}$	$\hat\lambda_{3\,median}$	$\hat\lambda_{3\,lower}$	$\hat\lambda_{3\,upper}$	$\hat\lambda_{4\,ave}$	$\pm\hat\lambda_{4\,ave}$	$\hat\lambda_{4\,median}$	$\hat\lambda_{4\,lower}$	$\hat\lambda_{4\,upper}$	$\hat\pi_{1\,ave}$	$\pm\hat\pi_{1\,ave}$	$\hat\pi_{1\,median}$	$\hat\pi_{2\,ave}$	$\pm\hat\pi_{2\,ave}$	$\hat\pi_{2\,median}$	$\hat\pi_{3\,ave}$	$\pm\hat\pi_{3\,ave}$	$\hat\pi_{3\,median}$	$\hat\pi_{4\,ave}$	$\pm\hat\pi_{4\,ave}$	$\hat\pi_{4\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.062195	0.929835	0.002447	0.000973	0.007466	0.280	1.666	0.055	0.017	0.200	2.137	4.087	0.471	0.155	4.390	8.425	5.065	7.347	3.588	14.901	0.198	0.120	0.184	0.152	0.084	0.139	0.137	0.097	0.116	0.512	0.139	0.535	-149.627	-195.968	-100.356	0.013
100	0.033711	0.695196	0.002060	0.000897	0.004865	0.143	1.062	0.038	0.013	0.137	1.340	2.800	0.408	0.160	1.877	7.820	4.406	6.887	3.928	13.202	0.182	0.101	0.172	0.157	0.074	0.146	0.141	0.092	0.124	0.520	0.128	0.543	-305.191	-384.125	-221.498	0.015
200	0.008615	0.313846	0.001796	0.000818	0.003704	0.075	0.617	0.025	0.009	0.088	0.830	1.790	0.344	0.150	0.959	7.259	3.805	6.537	4.185	11.607	0.165	0.086	0.156	0.159	0.064	0.150	0.149	0.085	0.137	0.527	0.118	0.549	-618.163	-758.109	-471.960	0.015
500	0.001833	0.001344	0.001537	0.000740	0.002908	0.029	0.158	0.016	0.006	0.045	0.447	0.901	0.251	0.128	0.544	6.540	3.089	6.130	4.439	9.682	0.144	0.070	0.137	0.159	0.054	0.153	0.158	0.071	0.149	0.539	0.106	0.557	-1562.295	-1868.476	-1241.181	0.015
1000	0.001623	0.001060	0.001414	0.000718	0.002544	0.019	0.049	0.013	0.006	0.030	0.295	0.473	0.204	0.118	0.427	6.093	2.688	5.879	4.551	8.191	0.133	0.059	0.126	0.159	0.048	0.153	0.161	0.062	0.154	0.548	0.096	0.563	-3140.044	-3692.264	-2542.069	0.015

Table C.236: 4-component exponential mixture on limit order arrival BARC data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat\lambda_{1\,ave}$	$\pm\hat\lambda_{1\,ave}$	$\hat\lambda_{1\,median}$	$\hat\lambda_{1\,lower}$	$\hat\lambda_{1\,upper}$	$\hat\lambda_{2\,ave}$	$\pm\hat\lambda_{2\,ave}$	$\hat\lambda_{2\,median}$	$\hat\lambda_{2\,lower}$	$\hat\lambda_{2\,upper}$	$\hat\lambda_{3\,ave}$	$\pm\hat\lambda_{3\,ave}$	$\hat\lambda_{3\,median}$	$\hat\lambda_{3\,lower}$	$\hat\lambda_{3\,upper}$	$\hat\lambda_{4\,ave}$	$\pm\hat\lambda_{4\,ave}$	$\hat\lambda_{4\,median}$	$\hat\lambda_{4\,lower}$	$\hat\lambda_{4\,upper}$	$\hat\pi_{1\,ave}$	$\pm\hat\pi_{1\,ave}$	$\hat\pi_{1\,median}$	$\hat\pi_{2\,ave}$	$\pm\hat\pi_{2\,ave}$	$\hat\pi_{2\,median}$	$\hat\pi_{3\,ave}$	$\pm\hat\pi_{3\,ave}$	$\hat\pi_{3\,median}$	$\hat\pi_{4\,ave}$	$\pm\hat\pi_{4\,ave}$	$\hat\pi_{4\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.094604	1.209130	0.000406	0.000182	0.001086	0.858	3.399	0.037	0.011	0.231	5.142	6.621	0.937	0.164	16.857	11.248	5.090	11.572	5.553	17.027	0.270	0.122	0.265	0.139	0.092	0.119	0.112	0.080	0.097	0.479	0.118	0.492	-195.594	-251.471	-139.795	0.014
100	0.028368	0.684664	0.000362	0.000181	0.000798	0.315	1.965	0.025	0.007	0.112	3.651	5.528	0.603	0.163	8.319	10.411	4.535	9.395	5.864	16.429	0.259	0.105	0.258	0.142	0.079	0.127	0.105	0.073	0.091	0.494	0.103	0.507	-396.695	-485.820	-308.407	0.013
200	0.003090	0.220657	0.000329	0.000179	0.000644	0.089	0.830	0.019	0.004	0.057	2.275	4.059	0.407	0.133	5.842	9.585	3.839	8.698	6.012	14.307	0.246	0.093	0.248	0.143	0.068	0.133	0.103	0.068	0.090	0.508	0.091	0.517	-800.913	-942.074	-658.451	0.015
500	0.000378	0.000994	0.000296	0.000178	0.000524	0.026	0.370	0.012	0.002	0.030	0.997	2.297	0.270	0.091	0.749	8.747	3.120	7.960	6.026	12.406	0.229	0.081	0.233	0.146	0.058	0.138	0.103	0.057	0.093	0.522	0.078	0.529	-2019.926	-2283.041	-1741.458	0.016
1000	0.000330	0.000310	0.000281	0.000174	0.000472	0.013	0.017	0.009	0.002	0.023	0.461	1.190	0.193	0.069	0.466	8.234	2.757	7.426	6.036	11.260	0.217	0.072	0.220	0.148	0.053	0.141	0.107	0.052	0.098	0.529	0.068	0.534	-4054.469	-4493.472	-3579.007	0.015

Table C.237: 4-component exponential mixture on limit order arrival RRLN data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat\lambda_{1\,ave}$	$\pm\hat\lambda_{1\,ave}$	$\hat\lambda_{1\,median}$	$\hat\lambda_{1\,lower}$	$\hat\lambda_{1\,upper}$	$\hat\lambda_{2\,ave}$	$\pm\hat\lambda_{2\,ave}$	$\hat\lambda_{2\,median}$	$\hat\lambda_{2\,lower}$	$\hat\lambda_{2\,upper}$	$\hat\lambda_{3\,ave}$	$\pm\hat\lambda_{3\,ave}$	$\hat\lambda_{3\,median}$	$\hat\lambda_{3\,lower}$	$\hat\lambda_{3\,upper}$	$\hat\lambda_{4\,ave}$	$\pm\hat\lambda_{4\,ave}$	$\hat\lambda_{4\,median}$	$\hat\lambda_{4\,lower}$	$\hat\lambda_{4\,upper}$	$\hat\pi_{1\,ave}$	$\pm\hat\pi_{1\,ave}$	$\hat\pi_{1\,median}$	$\hat\pi_{2\,ave}$	$\pm\hat\pi_{2\,ave}$	$\hat\pi_{2\,median}$	$\hat\pi_{3\,ave}$	$\pm\hat\pi_{3\,ave}$	$\hat\pi_{3\,median}$	$\hat\pi_{4\,ave}$	$\pm\hat\pi_{4\,ave}$	$\hat\pi_{4\,median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.238190	1.790403	0.001116	0.000443	0.003172	0.571	2.549	0.045	0.013	0.225	3.683	5.309	0.870	0.192	6.916	9.614	4.762	8.066	4.818	16.709	0.220	0.112	0.215	0.136	0.079	0.125	0.117	0.081	0.101	0.527	0.106	0.538	-160.905	-209.187	-113.563	0.013
100	0.169776	1.521737	0.000972	0.000421	0.002317	0.319	1.834	0.029	0.010	0.133	2.374	3.860	0.593	0.189	5.034	8.732	3.935	7.613	5.085	13.625	0.207	0.095	0.206	0.140	0.068	0.131	0.113	0.072	0.099	0.540	0.090	0.549	-326.886	-406.364	-251.023	0.014
200	0.043337	0.665420	0.000856	0.000396	0.001828	0.186	1.215	0.020	0.006	0.076	1.432	2.532	0.421	0.157	3.517	7.959	3.139	7.079	5.246	11.398	0.193	0.082	0.194	0.142	0.059	0.133	0.114	0.064	0.103	0.552	0.079	0.559	-660.364	-796.171	-535.792	0.012
500	0.000921	0.000857	0.000751	0.000371	0.001467	0.050	0.439	0.013	0.004	0.034	0.721	1.411	0.269	0.120	0.726	7.138	2.320	6.485	5.296	9.198	0.174	0.067	0.175	0.143	0.051	0.138	0.115	0.053	0.109	0.568	0.072	0.571	-1665.979	-1946.812	-1405.041	0.013
1000	0.000828	0.000566	0.000698	0.000361	0.001325	0.017	0.145	0.010	0.003	0.023	0.413	0.863	0.198	0.105	0.475	6.672	1.845	6.205	5.323	7.987	0.163	0.058	0.163	0.144	0.044	0.140	0.115	0.045	0.110	0.578	0.068	0.578	-3348.542	-3836.037	-2858.012	0.017

Table C.238: 4-component exponential mixture on limit order arrival VOD data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\lambda}_{4\text{ lower}}$	$\hat{\lambda}_{4\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.033436	0.676081	0.002831	0.001022	0.010109	0.251	1.579	0.049	0.017	0.187	2.274	4.177	0.493	0.150	4.565	9.088	4.628	7.805	4.566	15.077	0.177	0.103	0.165	0.151	0.082	0.139	0.122	0.088	0.104	0.550	0.113	0.563	-139.275	-180.630	-95.712	0.013
100	0.012527	0.395774	0.002333	0.000938	0.006245	0.111	0.882	0.034	0.013	0.127	1.403	2.852	0.409	0.157	2.304	8.377	3.886	7.244	4.824	13.191	0.160	0.088	0.152	0.158	0.071	0.149	0.124	0.081	0.109	0.558	0.101	0.570	-284.558	-354.124	-211.308	0.015
200	0.003806	0.136187	0.002051	0.000866	0.004637	0.056	0.472	0.024	0.010	0.078	0.846	1.804	0.334	0.149	1.027	7.740	3.205	6.760	4.963	11.438	0.144	0.074	0.137	0.161	0.060	0.155	0.130	0.074	0.121	0.566	0.091	0.576	-576.565	-697.153	-451.612	0.015
500	0.002251	0.001783	0.001839	0.000803	0.003641	0.026	0.063	0.018	0.008	0.037	0.435	0.865	0.249	0.137	0.535	6.985	2.435	6.247	5.057	9.352	0.126	0.058	0.121	0.161	0.047	0.158	0.138	0.063	0.131	0.574	0.080	0.584	-1454.992	-1716.857	-1190.249	0.013
1000	0.002042	0.001435	0.001752	0.000777	0.003330	0.019	0.019	0.015	0.007	0.028	0.291	0.440	0.211	0.135	0.411	6.519	1.960	5.995	5.084	7.918	0.118	0.048	0.114	0.161	0.039	0.158	0.140	0.055	0.133	0.580	0.072	0.589	-2920.518	-3409.643	-2435.875	0.014

Table C.239: 4-component exponential mixture on limit order arrival RIOTINTO data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\lambda}_{4\text{ lower}}$	$\hat{\lambda}_{4\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.011462	0.429876	0.000331	0.000160	0.000821	1.062	3.831	0.034	0.011	0.275	6.003	6.943	1.630	0.190	16.917	11.730	4.934	12.727	5.951	17.043	0.296	0.130	0.287	0.130	0.085	0.113	0.104	0.071	0.093	0.470	0.121	0.484	-205.939	-267.488	-143.143	0.013
100	0.005025	0.289049	0.000309	0.000165	0.000616	0.406	2.333	0.023	0.007	0.116	4.574	6.106	0.858	0.196	14.359	10.894	4.505	9.681	6.246	17.458	0.286	0.113	0.281	0.133	0.075	0.121	0.096	0.060	0.087	0.485	0.106	0.498	-417.122	-516.701	-312.574	0.015
200	0.001939	0.174009	0.000287	0.000166	0.000514	0.078	0.803	0.018	0.005	0.051	2.976	4.643	0.518	0.161	7.579	9.997	3.771	9.068	6.382	14.725	0.275	0.100	0.273	0.136	0.065	0.128	0.090	0.051	0.082	0.500	0.094	0.509	-840.575	-1002.522	-666.633	0.016
500	0.000300	0.000166	0.000271	0.000167	0.000433	0.023	0.292	0.013	0.003	0.027	1.406	2.806	0.329	0.111	3.704	9.209	3.072	8.438	6.455	12.829	0.262	0.086	0.262	0.139	0.053	0.133	0.084	0.042	0.077	0.515	0.079	0.522	-2115.165	-2431.414	-1786.966	0.015
1000	0.000281	0.000131	0.000259	0.000167	0.000395	0.013	0.018	0.011	0.002	0.022	0.752	1.846	0.261	0.086	0.550	8.899	2.832	8.047	6.443	12.202	0.253	0.076	0.256	0.140	0.046	0.135	0.083	0.036	0.078	0.523	0.067	0.527	-4244.031	-4766.884	-3696.834	0.016

Table C.240: 4-component exponential mixture on limit order arrival SSELN data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	3.933	6.897	0.571	0.159	17.931	18.082	0.758	18.263	17.927	18.514	2187.487	1889.132	1610.941	850.999	3764.034	0.616	0.191	0.577	0.460	0.788	0.108	0.072	0.092	0.493	0.109	0.495	0.399	0.109	0.395	-866.651	-1039.147	-675.438	0.032
500	2.526	5.858	0.382	0.116	1.281	19.017	0.867	19.246	18.947	19.437	2078.613	1477.532	1629.702	940.443	3337.376	0.597	0.147	0.574	0.465	0.733	0.105	0.060	0.091	0.504	0.086	0.506	0.391	0.085	0.388	-2168.479	-2511.948	-1810.120	0.011
1000	1.636	4.730	0.328	0.087	0.798	19.718	0.881	19.959	19.616	20.145	2064.876	1304.009	1669.501	1008.077	3255.719	0.592	0.130	0.571	0.472	0.713	0.103	0.054	0.091	0.510	0.071	0.510	0.387	0.071	0.386	-4347.124	-4930.889	-3755.680	0.008
2000	1.021	3.567	0.304	0.074	0.590	20.280	1.085	20.605	20.127	20.834	2057.015	1218.420	1672.684	1052.137	3244.796	0.586	0.111	0.567	0.484	0.695	0.102	0.048	0.093	0.513	0.059	0.515	0.385	0.061	0.383	-8702.283	-9668.035	-7733.437	0.004
5000	0.637	2.564	0.290	0.065	0.458	20.942	1.074	21.244	20.556	21.662	2031.866	1102.857	1702.325	1090.409	3079.744	0.577	0.093	0.565	0.484	0.665	0.102	0.042	0.093	0.514	0.046	0.513	0.384	0.049	0.386	-21811.183	-23948.086	-19753.975	0.003

Table C.241: exponential-exponential-weibull mixture on limit order arrival ABFLN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	2.104	4.856	0.533	0.250	1.493	17.320	1.576	18.149	15.462	18.438	236.706	193.446	182.743	88.358	390.093	0.634	1.098	0.604	0.524	0.721	0.199	0.118	0.171	0.448	0.164	0.487	0.353	0.118	0.344	-653.065	-790.550	-506.820	0.012
500	1.280	3.522	0.458	0.261	1.147	17.944	1.818	19.103	15.588	19.372	220.464	148.075	181.203	96.473	343.621	0.590	0.073	0.582	0.525	0.653	0.187	0.108	0.160	0.459	0.161	0.502	0.354	0.108	0.342	-1635.256	-1942.541	-1321.633	0.002
1000	0.906	2.548	0.429	0.271	1.042	18.585	1.966	19.821	15.993	20.078	213.917	127.830	182.244	102.021	324.518	0.577	0.055	0.572	0.527	0.625	0.182	0.102	0.157	0.463	0.159	0.507	0.355	0.102	0.342	-3277.639	-3827.197	-2693.234	0.000
2000	0.695	1.697	0.416	0.283	0.982	19.372	1.975	20.464	16.865	20.728	209.623	115.898	183.786	105.641	312.441	0.568	0.045	0.565	0.528	0.608	0.179	0.099	0.156	0.464	0.157	0.510	0.356	0.098	0.342	-6566.997	-7566.787	-5468.960	0.000
5000	0.567	0.819	0.401	0.293	0.942	20.281	1.811	21.086	18.334	21.416	205.947	104.801	185.190	108.373	302.192	0.561	0.036	0.559	0.527	0.594	0.177	0.095	0.156	0.466	0.155	0.511	0.357	0.095	0.342	-16441.322	-18602.094	-13869.131	0.000

Table C.242: exponential-exponential-weibull mixture on limit order arrival BARC data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	5.547	7.874	0.706	0.223	18.102	17.911	0.976	18.231	17.723	18.486	1248.445	981.514	976.856	550.214	2066.042	0.557	0.135	0.532	0.434	0.707	0.113	0.064	0.102	0.473	0.098	0.474	0.414	0.103	0.410	-837.111	-977.592	-680.751	0.034
500	4.231	7.438	0.446	0.182	18.638	18.644	1.275	19.171	17.687	19.388	1174.424	708.363	991.809	632.022	1720.552	0.541	0.109	0.525	0.437	0.651	0.107	0.055	0.097	0.486	0.079	0.488	0.407	0.082	0.403	-2091.211	-2362.721	-1807.637	0.014
1000	3.154	6.678	0.358	0.169	1.434	19.149	1.462	19.814	17.425	20.089	1164.742	603.647	1011.746	687.101	1655.253	0.535	0.092	0.524	0.444	0.631	0.103	0.048	0.094	0.494	0.065	0.497	0.403	0.067	0.399	-4191.456	-4636.958	-3722.588	0.010
2000	2.203	5.684	0.317	0.165	0.706	19.566	1.648	20.335	17.445	20.740	1170.185	562.269	1026.648	732.917	1603.659	0.532	0.081	0.525	0.451	0.611	0.100	0.041	0.092	0.500	0.053	0.504	0.400	0.057	0.398	-8395.166	-9140.156	-7641.620	0.009
5000	1.548	4.761	0.292	0.173	0.482	20.185	1.655	20.815	18.243	21.403	1151.189	452.508	1048.362	775.926	1537.699	0.525	0.070	0.521	0.454	0.598	0.097	0.034	0.091	0.503	0.043	0.506	0.399	0.046	0.394	-21041.401	-22495.876	-19469.530	0.009

Table C.243: exponential-exponential-weibull mixture on limit order arrival RRLN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	1.333	3.295	0.564	0.228	1.523	17.962	0.987	18.276	17.934	18.482	549.001	454.374	422.388	211.442	952.344	0.895	5.183	0.585	0.503	0.722	0.126	0.066	0.117	0.521	0.089	0.521	0.354	0.085	0.354	-698.084	-822.830	-562.938	0.034
500	0.738	1.394	0.499	0.245	1.200	18.832	1.227	19.267	18.926	19.432	508.450	348.071	418.591	226.485	801.896	0.572	0.089	0.564	0.502	0.644	0.119	0.053	0.113	0.528	0.077	0.525	0.353	0.073	0.356	-1734.317	-2006.501	-1474.411	0.012
1000	0.627	0.587	0.484	0.267	1.079	19.503	1.333	19.970	19.625	20.138	482.834	289.646	411.784	231.203	739.148	0.557	0.059	0.553	0.500	0.613	0.116	0.046	0.110	0.531	0.067	0.527	0.354	0.062	0.358	-3464.811	-3954.962	-2997.715	0.005
2000	0.599	0.341	0.478	0.286	1.017	20.172	1.291	20.573	20.218	20.768	466.241	251.702	406.529	237.640	701.478	0.548	0.050	0.546	0.500	0.594	0.113	0.041	0.108	0.531	0.055	0.529	0.356	0.051	0.358	-6939.321	-7801.272	-6109.029	0.001
5000	0.583	0.309	0.461	0.304	0.970	20.888	1.036	21.123	20.768	21.409	454.169	222.067	407.177	242.556	674.932	0.540	0.041	0.540	0.499	0.578	0.111	0.035	0.106	0.532	0.045	0.530	0.358	0.041	0.359	-17387.747	-19263.254	-15632.716	0.001

Table C.244: exponential-exponential-weibull mixture on limit order arrival VOD data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	1.566	3.812	0.588	0.237	1.464	17.596	1.340	18.223	15.962	18.463	212.752	209.740	143.517	65.362	372.429	0.634	0.126	0.615	0.536	0.730	0.150	0.082	0.138	0.517	0.093	0.522	0.333	0.086	0.335	-602.906	-722.958	-479.197	0.015
500	0.849	2.040	0.486	0.243	1.169	18.267	1.689	19.203	15.926	19.409	196.504	164.329	142.047	67.964	331.604	0.601	0.073	0.594	0.536	0.664	0.143	0.068	0.134	0.522	0.077	0.525	0.335	0.069	0.337	-1510.902	-1780.315	-1250.726	0.003
1000	0.665	1.147	0.451	0.257	1.076	18.950	1.836	19.918	16.392	20.117	188.882	144.839	140.773	69.174	318.840	0.587	0.057	0.583	0.535	0.638	0.139	0.059	0.131	0.524	0.067	0.526	0.337	0.060	0.340	-3028.250	-3538.620	-2553.351	0.001
2000	0.596	0.586	0.427	0.268	1.025	19.731	1.820	20.548	17.629	20.775	184.253	133.213	140.131	69.415	309.293	0.579	0.048	0.577	0.535	0.622	0.136	0.052	0.128	0.525	0.059	0.526	0.339	0.053	0.342	-6065.720	-7029.961	-5180.502	0.000
5000	0.571	0.325	0.415	0.281	0.993	20.631	1.591	21.169	20.596	21.463	180.284	123.341	139.393	70.174	300.475	0.572	0.040	0.569	0.534	0.610	0.134	0.046	0.127	0.525	0.051	0.525	0.341	0.047	0.344	-15189.731	-17473.506	-13096.162	0.000

Table C.245: exponential-exponential-weibull mixture on limit order arrival RIOTINTO data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	6.184	8.110	0.876	0.255	18.120	18.020	0.814	18.252	17.825	18.507	1654.049	1349.850	1281.605	698.127	2695.150	0.576	0.160	0.546	0.446	0.717	0.103	0.054	0.097	0.468	0.108	0.470	0.429	0.111	0.421	-875.150	-1043.167	-688.716	0.025
500	4.694	7.741	0.514	0.195	18.747	18.850	1.052	19.202	18.741	19.411	1555.730	953.128	1294.105	798.650	2332.826	0.561	0.125	0.539	0.450	0.669	0.095	0.048	0.088	0.482	0.087	0.485	0.423	0.086	0.415	-2193.748	-2523.479	-1854.404	0.009
1000	3.806	7.256	0.417	0.186	18.770	19.400	1.242	19.863	19.118	20.099	1520.239	815.119	1294.952	861.750	2209.729	0.552	0.104	0.538	0.459	0.646	0.089	0.040	0.082	0.490	0.073	0.493	0.421	0.073	0.416	-4395.220	-4929.031	-3840.910	0.005
2000	2.954	6.554	0.357	0.168	0.892	19.858	1.420	20.448	19.057	20.746	1526.720	755.000	1328.667	911.506	2155.617	0.550	0.096	0.539	0.462	0.638	0.086	0.036	0.078	0.496	0.058	0.501	0.418	0.060	0.414	-8810.017	-9656.981	-7901.494	0.000
5000	1.773	5.130	0.319	0.143	0.552	20.448	1.474	20.966	19.542	21.457	1555.505	665.826	1377.278	991.106	2164.319	0.549	0.078	0.542	0.480	0.616	0.084	0.032	0.076	0.503	0.045	0.503	0.413	0.047	0.412	-22076.929	-23729.312	-20269.572	0.000

Table C.246: exponential-exponential-weibull mixture on limit order arrival SSELN data with censoring on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.1732	0.3677	0.5785	-
4-comp exponential	0.4485	-	0.0440	0.2762
exponential/exponential/weibull	0.3783	0.6323	0.3775	0.7237

Table C.247: ABFLN, N = 200: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.1620	0.2421	0.5998	-
4-comp exponential	0.4375	-	0.0064	0.0638
exponential/exponential/weibull	0.4005	0.7579	0.3938	0.9361

Table C.248: BARC, N = 200: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.1447	0.2380	0.5160	-
4-comp exponential	0.4258	0.7619	0.0365	0.2297
exponential/exponential/weibull	0.4295	-	0.4475	0.7697

Table C.249: RRLN, N = 200: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0770	0.1553	0.4534	0.7900
4-comp exponential	0.4080	0.8447	0.0132	0.2099
exponential/exponential/weibull	0.5150	-	0.5334	-

Table C.250: VOD, N = 200: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.1033	0.1665	0.5585	-
4-comp exponential	0.4826	-	0.0072	0.0503
exponential/exponential/weibull	0.4141	0.8335	0.4344	0.9495

Table C.251: RIOTINTO, N = 200: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.1654	0.3716	0.5607	-
4-comp exponential	0.4670	-	0.0466	0.3023
exponential/exponential/weibull	0.3676	0.6284	0.3927	0.6974

Table C.252: SSELN, N = 200: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0449	0.1954	0.2220	0.3583
4-comp exponential	0.5442	-	0.2208	0.6415
exponential/exponential/weibull	0.4109	0.8046	0.5572	-

Table C.253: ABFLN, N = 500: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0344	0.0567	0.1872	0.3335
4-comp exponential	0.4739	0.9433	0.0612	0.6665
exponential/exponential/weibull	0.4917	-	0.7516	-

Table C.254: BARC, N = 500: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0259	0.0613	0.1484	0.2768
4-comp exponential	0.4345	0.9385	0.1472	0.7230
exponential/exponential/weibull	0.5395	-	0.7044	-

Table C.255: RRLN, N = 500: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0105	0.0343	0.0600	0.1826
4-comp exponential	0.3451	0.9656	0.0630	0.8174
exponential/exponential/weibull	0.6443	-	0.8770	-

Table C.256: VOD, N = 500: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0071	0.0220	0.1021	0.2508
4-comp exponential	0.5353	-	0.0627	0.7491
exponential/exponential/weibull	0.4576	0.9780	0.8352	-

Table C.257: RIOTINTO, N = 500: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0306	0.2058	0.2136	0.3215
4-comp exponential	0.5440	-	0.2165	0.6783
exponential/exponential/weibull	0.4254	0.7942	0.5698	-

Table C.258: SSELN, N = 500: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0075	0.1128	0.0611	0.1166
4-comp exponential	0.5628	-	0.3620	0.8829
exponential/exponential/weibull	0.4297	0.8872	0.5769	-

Table C.259: ABFLN, N = 1000: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0124	0.0283	0.0705	0.1203
4-comp exponential	0.4061	0.9717	0.1054	0.8796
exponential/exponential/weibull	0.5816	-	0.8241	-

Table C.260: BARC, N = 1000: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0070	0.0268	0.0281	0.0743
4-comp exponential	0.3630	0.9729	0.2071	0.9254
exponential/exponential/weibull	0.6300	-	0.7648	-

Table C.261: RRLN, N = 1000: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0023	0.0186	0.0075	0.0385
4-comp exponential	0.2512	0.9813	0.0715	0.9614
exponential/exponential/weibull	0.7465	-	0.9210	-

Table C.262: VOD, N = 1000: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0013	0.0150	0.0134	0.0553
4-comp exponential	0.4859	0.9850	0.1133	0.9446
exponential/exponential/weibull	0.5128	-	0.8732	-

Table C.263: RIOTINTO, N = 1000: limit order arrival times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0083	0.1380	0.0446	0.0863
4-comp exponential	0.5438	-	0.3716	0.9133
exponential/exponential/weibull	0.4479	0.8620	0.5838	-

Table C.264: SSELN, N = 1000: limit order arrival times, censored on region [0, 0.5].

Appendix D

Mid-Price Waiting Time Parameter Estimates

Chapter 14 presented a representative sample of mixed distribution parameter estimates and model selection results for mid-price waiting time data. The full set are provided in this appendix.

D.1 Removal of "zero inflated" Data

- Exponential/Weibull mixtures: Tables [D.1 - D.6]
- Gamma/Weibull mixtures: Tables [D.7 - D.12]
- Loglogistic/Weibull mixtures: Tables [D.13 - D.18]
- Weibull/Weibull mixtures: Tables [D.19 - D.24]
- Model Selection via AIC and BIC Results: Tables [D.25 - D.54]

D.2 Distributing "zero inflated" Data Uniformly

- Exponential/Uniform/Weibull mixtures: Tables [D.55 - D.60]
- Gamma/Uniform/Weibull mixtures: Tables [D.61 - D.66]
- Loglogistic/Uniform/Weibull mixtures: Tables [D.67 - D.72]
- Weibull/Uniform/Weibull mixtures: Tables [D.73 - D.78]
- Model Selection via AIC and BIC Results: Tables [D.79 - D.108]

D.3 Distributing "zero inflated" Data Exponentially

- Exponential/Exponential/Weibull mixtures: Tables [D.109 - D.114]
- Gamma/Exponential/Weibull mixtures: Tables [D.115 - D.120]
- Loglogistic/Exponential/Weibull mixtures: Tables [D.121 - D.126]
- Weibull/Exponential/Weibull mixtures: Tables [D.127 - D.132]
- Model Selection via AIC and BIC Results: Tables [D.133 - D.162]

D.4 g-component Exponential Mixture

- 2-component Exponential mixtures: Tables [D.163 - D.168]
- 4-component Exponential mixtures: Tables [D.169 - D.174]
- 6-component Exponential mixtures: Tables [D.175 - D.180]
- 10-component Exponential mixtures: Tables [D.181 - D.186]
- Model Selection via AIC and BIC Results: Tables [D.187 - D.216]

D.5 Censoring "zero inflated" Data

- 3-component Exponential mixtures with censoring [0, 0.5]: Tables [D.217 - D.222]
- 3-component Exponential mixtures with censoring [0, 0.5, 1.5, 2.5, 10]: Tables [D.223 - D.228]
- 4-component Exponential mixtures with censoring [0, 0.5]: Tables [D.229 - D.234]
- 4-component Exponential mixtures with censoring [0, 0.5, 1.5, 2.5, 10]: Tables [D.235 - D.240]
- Exponential/Exponential/Weibull mixtures with censoring [0, 0.5]: Tables [D.241 - D.246]
- Model Selection via AIC and BIC Results: Tables [D.247 - D.264]

n	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	-	-	-	-	-	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.159	0.153	0.109	0.037	0.278	-	-	-	-	-	15049.870	16238.485	10352.264	3667.868	25962.966	0.516	1.422	0.429	0.348	0.566	0.310	0.166	0.283	0.690	0.166	0.717	-426.031	-490.132	-364.271	0.001
100	0.146	0.123	0.113	0.046	0.247	-	-	-	-	-	12851.622	11260.778	9776.985	4065.876	21281.435	0.447	0.367	0.408	0.347	0.506	0.297	0.147	0.275	0.703	0.147	0.725	-854.792	-964.447	-751.042	0.000
200	0.139	0.097	0.114	0.054	0.223	-	-	-	-	-	11223.287	7956.719	9199.937	4316.504	17801.399	0.407	0.074	0.393	0.344	0.471	0.288	0.129	0.266	0.712	0.129	0.734	-1710.615	-1898.041	-1526.294	0.001
500	0.134	0.073	0.119	0.067	0.210	-	-	-	-	-	9658.106	5303.684	8542.196	4975.075	14675.892	0.390	0.051	0.384	0.343	0.435	0.285	0.108	0.273	0.715	0.108	0.727	-4251.136	-4642.135	-3850.724	0.000
1000	0.141	0.062	0.131	0.081	0.205	-	-	-	-	-	7022.508	2960.886	6739.554	4074.374	9569.533	0.392	0.046	0.388	0.350	0.436	0.292	0.095	0.285	0.708	0.095	0.715	-8256.456	-8941.355	-7518.009	0.000

Table D.1: exponential-weibull mixture on ABFLN mid-price waiting time data: removal of zeros.

n	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	-	-	-	-	-	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.209	0.148	0.173	0.083	0.333	-	-	-	-	-	1298.673	1410.190	891.982	326.634	2208.301	0.728	1.254	0.580	0.470	0.812	0.512	0.205	0.514	0.488	0.205	0.486	-285.026	-351.154	-215.103	0.005
100	0.201	0.125	0.176	0.095	0.301	-	-	-	-	-	1123.224	1043.512	841.438	360.321	1822.436	0.590	0.572	0.540	0.461	0.664	0.498	0.192	0.501	0.502	0.192	0.499	-573.084	-695.043	-448.242	0.001
200	0.197	0.108	0.178	0.105	0.281	-	-	-	-	-	1019.850	826.538	811.621	389.637	1601.874	0.536	0.166	0.519	0.457	0.601	0.489	0.181	0.495	0.511	0.181	0.505	-1150.728	-1377.497	-918.583	0.000
500	0.195	0.095	0.180	0.114	0.265	-	-	-	-	-	943.604	663.599	787.736	418.323	1434.817	0.511	0.096	0.503	0.456	0.562	0.482	0.171	0.493	0.518	0.171	0.507	-2883.337	-3412.542	-2335.236	0.000
1000	0.195	0.089	0.184	0.119	0.258	-	-	-	-	-	907.826	582.564	776.897	432.010	1338.499	0.501	0.061	0.496	0.456	0.544	0.480	0.166	0.495	0.520	0.166	0.505	-5762.342	-6779.756	-4696.868	0.000

Table D.2: exponential-weibull mixture on BARC mid-price waiting time data: removal of zeros.

n	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	-	-	-	-	-	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.204	0.150	0.179	0.064	0.332	-	-	-	-	-	7773.484	9888.252	4894.843	1180.745	13861.968	0.751	2.314	0.458	0.363	0.650	0.463	0.249	0.435	0.537	0.249	0.565	-343.524	-449.035	-213.403	0.005
100	0.194	0.122	0.182	0.073	0.300	-	-	-	-	-	6530.684	6827.871	4671.445	1404.737	11242.475	0.516	0.814	0.425	0.357	0.546	0.454	0.238	0.428	0.546	0.238	0.572	-688.677	-890.678	-453.437	0.001
200	0.189	0.103	0.186	0.082	0.283	-	-	-	-	-	5707.412	4865.076	4513.247	1622.166	9573.735	0.437	0.284	0.406	0.351	0.489	0.449	0.228	0.423	0.551	0.228	0.577	-1375.605	-1767.670	-948.118	0.000
500	0.189	0.084	0.194	0.097	0.268	-	-	-	-	-	4975.422	3519.621	4165.934	1760.386	8081.481	0.399	0.057	0.391	0.349	0.446	0.456	0.215	0.449	0.544	0.215	0.551	-3390.474	-4325.058	-2433.605	0.000
1000	0.193	0.075	0.203	0.105	0.263	-	-	-	-	-	4433.660	2700.215	3818.430	1822.100	7058.994	0.389	0.046	0.379	0.352	0.427	0.471	0.207	0.498	0.529	0.207	0.502	-6624.313	-8501.038	-4873.330	0.000

Table D.3: exponential-weibull mixture on RRLN mid-price waiting time data: removal of zeros.

n	$\hat{\lambda}_{ave}$	$\pm\hat{\lambda}_{ave}$	$\hat{\lambda}_{median}$	$\hat{\lambda}_{lower}$	$\hat{\lambda}_{upper}$	-	-	-	-	-	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.146	0.134	0.106	0.047	0.238	-	-	-	-	-	3555.854	3611.132	2510.844	926.579	6097.825	0.632	0.955	0.533	0.425	0.728	0.400	0.181	0.388	0.600	0.181	0.612	-352.739	-413.697	-290.120	0.006
100	0.136	0.107	0.108	0.054	0.208	-	-	-	-	-	3110.712	2665.266	2378.661	999.947	5139.998	0.561	0.860	0.499	0.420	0.623	0.388	0.167	0.377	0.612	0.167	0.623	-708.353	-819.352	-600.129	0.002
200	0.130	0.089	0.109	0.060	0.190	-	-	-	-	-	2854.502	2132.582	2302.543	1081.980	4587.525	0.506	0.259	0.481	0.417	0.570	0.381	0.155	0.370	0.619	0.155	0.630	-1421.031	-1622.992	-1226.145	0.002
500	0.125	0.069	0.114	0.067	0.172	-	-	-	-	-	2651.290	1705.555	2244.702	1166.710	4139.529	0.477	0.099	0.465	0.417	0.532	0.376	0.143	0.372	0.624	0.143	0.628	-3557.401	-4027.851	-3130.824	0.000
1000	0.123	0.059	0.113	0.072	0.166	-	-	-	-	-	2538.676	1476.795	2196.952	1228.086	3906.481	0.466	0.055	0.459	0.416	0.514	0.375	0.132	0.371	0.625	0.132	0.629	-7110.349	-7969.153	-6312.260	0.000

Table D.4: exponential-weibull mixture on VOD mid-price waiting time data: removal of zeros.

n	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.236	0.161	0.197	0.094	0.382	-	-	-	-	-	980.434	1051.354	658.325	252.607	1657.000	0.692	1.014	0.586	0.481	0.786	0.482	0.191	0.479	0.518	0.191	0.521	-282.849	-342.380	-225.109	0.003
100	0.229	0.136	0.201	0.110	0.347	-	-	-	-	-	848.342	763.685	625.069	269.783	1401.239	0.591	0.654	0.549	0.472	0.661	0.465	0.173	0.462	0.535	0.173	0.538	-569.038	-677.511	-468.891	0.000
200	0.226	0.118	0.205	0.122	0.323	-	-	-	-	-	772.985	606.148	607.629	283.308	1244.156	0.543	0.195	0.529	0.469	0.606	0.454	0.161	0.452	0.546	0.161	0.548	-1142.591	-1342.676	-962.044	0.000
500	0.225	0.103	0.209	0.134	0.305	-	-	-	-	-	719.348	486.918	596.010	298.510	1131.078	0.521	0.117	0.515	0.467	0.571	0.446	0.150	0.445	0.554	0.150	0.555	-2863.193	-3329.742	-2454.539	0.000
1000	0.224	0.095	0.212	0.141	0.297	-	-	-	-	-	694.778	428.806	595.960	307.068	1071.742	0.511	0.050	0.508	0.466	0.555	0.444	0.144	0.443	0.556	0.144	0.557	-5727.705	-6621.191	-4962.476	0.000

Table D.5: exponential-weibull mixture on RIOTINTO mid-price waiting time data: removal of zeros.

n	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.152	0.146	0.107	0.040	0.261	-	-	-	-	-	12918.173	12514.061	9197.586	3350.563	22047.724	0.483	0.367	0.435	0.354	0.568	0.311	0.162	0.292	0.689	0.162	0.708	-421.242	-479.766	-364.061	0.002
100	0.142	0.119	0.108	0.049	0.234	-	-	-	-	-	11032.541	8823.855	8759.322	3669.391	18313.916	0.442	0.572	0.413	0.350	0.503	0.296	0.141	0.281	0.704	0.141	0.719	-846.374	-946.958	-750.156	0.001
200	0.135	0.096	0.110	0.058	0.208	-	-	-	-	-	9781.822	6596.655	8391.148	3999.482	15625.054	0.410	0.074	0.398	0.349	0.471	0.286	0.124	0.271	0.714	0.124	0.729	-1695.862	-1866.213	-1528.856	0.001
500	0.125	0.067	0.111	0.069	0.178	-	-	-	-	-	8698.879	4547.329	7945.662	4413.892	12991.156	0.392	0.046	0.385	0.348	0.435	0.282	0.099	0.273	0.718	0.099	0.727	-4232.148	-4560.937	-3914.242	0.000
1000	0.126	0.049	0.119	0.084	0.176	-	-	-	-	-	7526.535	3078.595	7442.846	4440.625	10529.185	0.386	0.040	0.379	0.349	0.420	0.277	0.091	0.270	0.723	0.091	0.730	-8402.156	-8966.728	-7797.586	0.000

Table D.6: exponential-weibull mixture on SSELN mid-price waiting time data: removal of zeros.

n	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.761	3.835	3.072	2.150	8.918	0.977	0.158	1.000	0.820	1.143	7816.868	7887.118	5561.792	1837.618	15325.570	0.399	0.570	0.359	0.304	0.464	0.178	0.132	0.146	0.822	0.132	0.854	-425.691	-488.615	-356.706	0.033
100	4.419	3.457	2.922	2.169	7.580	1.011	0.131	1.037	0.884	1.133	6970.915	5883.286	5354.081	2113.294	12005.322	0.365	0.102	0.348	0.302	0.418	0.166	0.117	0.140	0.834	0.117	0.860	-858.136	-968.064	-753.975	0.003
200	4.214	3.225	2.874	2.206	6.546	1.039	0.106	1.057	0.936	1.137	6300.153	4481.026	5215.834	2347.668	10302.022	0.351	0.058	0.341	0.302	0.402	0.163	0.104	0.141	0.837	0.104	0.859	-1716.546	-1906.068	-1534.091	0.000
500	3.750	2.388	2.830	2.314	5.407	1.065	0.080	1.079	0.984	1.146	5586.530	3221.905	5013.258	2706.295	8655.234	0.342	0.044	0.333	0.303	0.390	0.164	0.087	0.146	0.836	0.087	0.854	-4267.912	-4661.452	-3872.207	0.000
1000	3.402	1.824	2.708	2.325	4.888	1.068	0.071	1.079	0.977	1.139	4189.250	2005.587	3848.667	2259.226	5953.092	0.346	0.042	0.342	0.307	0.399	0.174	0.071	0.154	0.826	0.071	0.846	-8286.542	-8963.602	-7570.597	0.000

Table D.7: gamma-weibull mixture on ABFLN mid-price waiting time data: removal of zeros.

n	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.644	2.628	2.655	2.064	5.692	0.987	0.150	1.011	0.842	1.133	601.927	663.614	409.254	148.495	1048.880	0.517	0.697	0.462	0.391	0.570	0.347	0.206	0.309	0.653	0.206	0.691	-284.463	-351.050	-211.540	0.013
100	3.357	2.220	2.596	2.076	4.155	1.009	0.132	1.033	0.886	1.131	546.287	499.983	407.207	174.978	888.232	0.465	0.199	0.446	0.390	0.519	0.338	0.191	0.311	0.662	0.191	0.689	-574.617	-697.217	-447.950	0.002
200	3.154	1.906	2.570	2.108	3.484	1.024	0.116	1.046	0.921	1.130	518.753	416.792	407.355	196.548	816.776	0.447	0.104	0.437	0.390	0.495	0.335	0.180	0.318	0.665	0.180	0.682	-1154.659	-1383.010	-920.773	0.000
500	2.943	1.498	2.552	2.145	3.187	1.036	0.101	1.054	0.948	1.127	499.333	356.338	407.406	218.741	759.442	0.437	0.070	0.431	0.392	0.479	0.334	0.169	0.328	0.666	0.169	0.672	-2893.071	-3426.624	-2339.859	0.000
1000	2.844	1.251	2.543	2.175	3.069	1.041	0.093	1.057	0.963	1.125	488.085	321.208	406.118	230.109	722.751	0.432	0.055	0.427	0.393	0.471	0.336	0.163	0.334	0.664	0.163	0.666	-5781.282	-6808.659	-4710.950	0.000

Table D.8: gamma-weibull mixture on BARC mid-price waiting time data: removal of zeros.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.752	2.906	2.716	2.135	5.838	0.995	0.150	1.014	0.853	1.149	3688.202	4384.394	2307.946	467.293	7255.197	0.449	1.028	0.376	0.310	0.493	0.343	0.248	0.258	0.657	0.248	0.742	-340.756	-446.202	-200.397	0.025
100	3.436	2.454	2.657	2.156	4.060	1.020	0.124	1.039	0.904	1.139	3301.452	3403.010	2318.772	584.827	5966.945	0.386	0.289	0.359	0.308	0.442	0.333	0.237	0.254	0.667	0.237	0.746	-688.969	-892.035	-447.298	0.005
200	3.195	2.086	2.618	2.204	3.395	1.036	0.105	1.053	0.943	1.130	3042.033	2767.678	2243.852	726.458	5272.244	0.364	0.118	0.348	0.308	0.415	0.333	0.228	0.257	0.667	0.228	0.743	-1378.737	-1771.659	-945.293	0.000
500	2.932	1.481	2.599	2.273	3.106	1.052	0.079	1.061	0.982	1.127	2728.690	2168.590	2151.561	847.036	4731.759	0.350	0.050	0.339	0.308	0.389	0.346	0.216	0.294	0.654	0.216	0.706	-3397.141	-4337.032	-2421.918	0.000
1000	2.819	1.966	2.584	2.323	2.959	1.057	0.067	1.064	1.002	1.118	2451.005	1751.265	2000.902	869.244	4122.049	0.345	0.044	0.335	0.310	0.375	0.365	0.208	0.389	0.635	0.208	0.611	-6634.612	-8519.285	-4865.064	0.000

Table D.9: gamma-weibull mixture on RRLN mid-price waiting time data: removal of zeros.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.679	3.790	2.923	2.083	8.788	0.976	0.157	1.004	0.820	1.133	1551.140	1502.681	1126.557	387.603	2857.004	0.455	0.327	0.414	0.349	0.533	0.212	0.140	0.179	0.788	0.140	0.821	-352.754	-413.378	-285.333	0.024
100	4.520	3.744	2.823	2.079	8.024	1.005	0.135	1.034	0.869	1.129	1411.566	1202.334	1097.571	434.504	2351.906	0.425	0.226	0.399	0.346	0.486	0.198	0.124	0.170	0.802	0.124	0.830	-712.255	-822.166	-604.911	0.003
200	4.456	3.852	2.772	2.115	7.655	1.031	0.115	1.057	0.914	1.133	1319.247	1015.440	1073.346	480.429	2128.405	0.407	0.091	0.391	0.343	0.463	0.191	0.112	0.167	0.809	0.112	0.833	-1430.612	-1630.990	-1238.723	0.001
500	4.462	4.013	2.742	2.173	7.538	1.053	0.095	1.076	0.955	1.138	1235.720	833.903	1045.571	509.511	1942.078	0.394	0.062	0.383	0.342	0.447	0.188	0.100	0.168	0.812	0.100	0.832	-3584.712	-4047.622	-3158.834	0.000
1000	4.358	3.976	2.753	2.214	7.191	1.066	0.085	1.088	0.978	1.142	1187.768	737.799	1014.865	551.831	1825.438	0.388	0.052	0.379	0.342	0.437	0.188	0.093	0.173	0.812	0.093	0.827	-7166.077	-8006.718	-6357.960	0.000

Table D.10: gamma-weibull mixture on VOD mid-price waiting time data: removal of zeros.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.337	2.316	2.508	2.004	4.539	0.968	0.152	0.991	0.814	1.118	514.517	530.583	355.967	144.833	885.506	0.529	0.576	0.487	0.412	0.594	0.338	0.190	0.307	0.662	0.190	0.693	-281.962	-342.213	-221.554	0.011
100	3.069	1.930	2.457	2.003	3.507	0.987	0.133	1.008	0.853	1.115	470.883	395.824	357.564	163.649	770.012	0.487	0.289	0.471	0.411	0.546	0.329	0.173	0.309	0.671	0.173	0.691	-569.013	-678.821	-467.108	0.001
200	2.891	1.613	2.436	2.015	3.207	1.000	0.117	1.019	0.881	1.112	448.729	327.070	361.583	179.832	712.869	0.469	0.106	0.462	0.411	0.523	0.326	0.159	0.312	0.674	0.159	0.688	-1142.957	-1346.609	-959.706	0.000
500	2.768	1.348	2.428	2.035	3.040	1.008	0.103	1.026	0.903	1.108	432.075	274.259	364.220	194.419	668.365	0.460	0.113	0.454	0.412	0.508	0.324	0.147	0.316	0.676	0.147	0.684	-2863.763	-3337.575	-2449.989	0.000
1000	2.718	1.225	2.424	2.047	2.980	1.011	0.097	1.028	0.913	1.106	423.402	246.850	367.300	201.366	641.415	0.455	0.047	0.450	0.412	0.500	0.325	0.141	0.318	0.675	0.141	0.682	-5728.235	-6633.696	-4950.251	0.000

Table D.11: gamma-weibull mixture on RIOTINTO mid-price waiting time data: removal of zeros.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.669	3.806	2.995	2.127	8.893	0.973	0.161	1.001	0.809	1.140	6456.970	5989.958	4760.814	1692.329	12166.974	0.387	0.209	0.360	0.304	0.462	0.174	0.120	0.147	0.826	0.120	0.853	-420.932	-477.688	-355.499	0.035
100	4.445	3.596	2.895	2.142	7.630	1.009	0.135	1.034	0.877	1.136	5784.306	4496.307	4743.383	1889.066	9695.112	0.365	0.120	0.349	0.303	0.423	0.161	0.104	0.139	0.839	0.104	0.861	-849.060	-947.876	-752.115	0.007
200	4.218	3.313	2.875	2.191	6.781	1.039	0.109	1.062	0.935	1.138	5321.784	3602.450	4586.895	2096.192	8445.572	0.352	0.060	0.342	0.302	0.404	0.154	0.090	0.138	0.846	0.090	0.862	-1702.849	-1872.138	-1536.339	0.001
500	3.867	2.735	2.813	2.268	5.863	1.060	0.084	1.078	0.976	1.139	4842.458	2581.017	4595.792	2400.114	7368.731	0.342	0.042	0.336	0.303	0.386	0.152	0.071	0.140	0.848	0.071	0.860	-4251.185	-4579.475	-3945.892	0.000
1000	3.292	1.971	2.658	2.279	3.605	1.067	0.072	1.069	0.992	1.143	4428.441	1960.270	4401.019	2443.765	6406.491	0.342	0.037	0.336	0.309	0.379	0.156	0.059	0.142	0.844	0.059	0.858	-8434.764	-8980.212	-7845.462	0.005

Table D.12: gamma-weibull mixture on SSELN mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	14.397	17.695	7.790	2.621	25.861	4.501	13.005	1.265	0.852	2.391	26770.216	40216.894	14955.549	4816.226	44862.231	0.615	0.888	0.497	0.390	0.729	0.399	0.196	0.383	0.601	0.196	0.617	-420.377	-485.276	-356.538	0.007
100	13.477	16.075	7.811	2.975	22.988	3.586	11.213	1.184	0.836	1.859	23497.849	34755.732	14591.524	5396.073	38955.841	0.546	0.829	0.471	0.387	0.618	0.392	0.175	0.380	0.608	0.175	0.620	-846.610	-957.655	-739.058	0.003
200	12.124	14.040	7.424	3.170	18.898	3.056	10.002	1.152	0.849	1.621	19859.100	20271.498	14176.008	5700.651	33932.003	0.494	0.488	0.451	0.383	0.555	0.385	0.155	0.377	0.615	0.155	0.623	-1695.120	-1888.502	-1506.435	0.001
500	9.846	9.832	7.200	3.318	14.239	2.072	6.760	1.145	0.893	1.488	15562.853	10027.426	13514.535	6728.010	24285.526	0.442	0.087	0.429	0.384	0.500	0.374	0.130	0.369	0.626	0.130	0.631	-4234.187	-4633.571	-3825.951	0.000
1000	8.110	7.303	6.469	3.141	10.886	2.701	8.872	1.131	0.958	1.500	11461.213	7009.820	10316.252	5268.453	16067.519	0.439	0.057	0.426	0.390	0.505	0.380	0.124	0.383	0.620	0.124	0.617	-8221.926	-8913.468	-7489.367	0.000

Table D.13: loglogistic-weibull mixture on ABFLN mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	7.262	9.960	4.085	1.881	11.608	7.969	21.420	1.533	1.071	2.877	1931.713	2548.161	1225.290	387.162	3428.635	0.909	1.477	0.649	0.494	1.112	0.564	0.213	0.584	0.436	0.213	0.416	-281.645	-348.237	-208.258	0.015
100	6.344	8.119	3.918	1.981	9.350	7.639	20.732	1.487	1.086	2.317	1646.598	1827.732	1145.952	419.491	2778.211	0.690	0.619	0.596	0.483	0.818	0.555	0.197	0.574	0.445	0.197	0.426	-564.980	-689.454	-435.829	0.005
200	5.736	6.825	3.804	2.085	8.245	7.223	19.762	1.471	1.103	2.070	1465.018	1503.321	1091.403	452.137	2377.733	0.611	0.417	0.569	0.480	0.703	0.547	0.184	0.569	0.453	0.184	0.431	-1133.934	-1366.886	-895.198	0.003
500	5.215	5.565	3.693	2.236	7.435	6.554	18.236	1.466	1.123	1.899	1319.608	1110.043	1053.919	496.226	2080.697	0.568	0.245	0.548	0.482	0.637	0.542	0.172	0.567	0.458	0.172	0.433	-2843.533	-3387.861	-2281.570	0.001
1000	4.914	4.796	3.611	2.331	6.982	6.007	17.046	1.470	1.143	1.821	1247.852	949.125	1027.004	520.286	1904.430	0.547	0.103	0.538	0.481	0.611	0.540	0.165	0.567	0.460	0.165	0.433	-5686.901	-6744.332	-4589.732	0.001

Table D.14: loglogistic-weibull mixture on BARC mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	9.065	13.560	3.967	2.011	15.032	4.865	13.928	1.567	0.983	2.649	13076.176	24460.573	6621.860	1436.199	21824.351	0.894	2.548	0.515	0.390	0.820	0.521	0.240	0.529	0.479	0.240	0.471	-337.944	-444.193	-206.151	0.010
100	8.010	11.481	3.802	2.102	12.884	4.209	12.517	1.499	0.973	2.234	10529.987	14114.041	6393.637	1670.088	18143.149	0.621	1.237	0.471	0.380	0.649	0.515	0.226	0.525	0.485	0.226	0.475	-678.695	-883.843	-439.462	0.003
200	7.341	10.148	3.650	2.203	11.769	3.633	11.200	1.477	0.978	2.047	9145.072	9917.540	6194.468	2015.408	15912.232	0.497	0.404	0.449	0.377	0.559	0.512	0.213	0.524	0.488	0.213	0.476	-1359.936	-1759.206	-925.468	0.001
500	5.930	7.117	3.334	2.280	9.185	3.230	10.016	1.521	1.020	1.950	7475.335	6361.515	5499.677	2221.155	13061.854	0.437	0.072	0.426	0.371	0.498	0.514	0.198	0.524	0.486	0.198	0.476	-3355.284	-4306.284	-2381.980	0.000
1000	4.949	4.469	3.134	2.321	7.987	3.286	9.950	1.598	1.047	1.900	6451.730	4637.563	4964.897	2271.375	11526.546	0.422	0.056	0.414	0.370	0.471	0.524	0.191	0.559	0.476	0.191	0.441	-6554.706	-8466.501	-4781.471	0.000

Table D.15: loglogistic-weibull mixture on RRLN mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	11.980	13.876	7.431	2.916	20.177	5.261	16.092	1.261	0.929	2.090	5754.145	8199.021	3654.165	1204.606	9793.882	0.810	1.249	0.619	0.472	0.964	0.483	0.201	0.486	0.517	0.201	0.514	-349.224	-410.526	-284.880	0.010
100	10.683	11.663	7.167	3.260	16.672	4.642	14.900	1.210	0.946	1.672	4968.733	5960.457	3521.890	1314.965	8267.274	0.667	0.763	0.575	0.465	0.771	0.474	0.182	0.478	0.526	0.182	0.522	-702.976	-814.495	-592.305	0.005
200	9.553	9.197	7.077	3.612	14.525	4.212	13.947	1.180	0.963	1.502	4435.842	3785.069	3437.202	1475.636	7251.503	0.595	0.318	0.550	0.464	0.682	0.468	0.166	0.473	0.532	0.166	0.527	-1410.335	-1616.272	-1212.846	0.001
500	8.514	6.631	6.823	3.993	12.653	3.726	12.657	1.167	0.992	1.392	4069.104	2877.591	3342.855	1594.438	6596.001	0.546	0.129	0.532	0.462	0.617	0.464	0.152	0.471	0.536	0.152	0.529	-3535.434	-4012.702	-3105.407	0.000
1000	7.911	4.967	6.777	4.114	11.705	3.306	11.505	1.158	1.012	1.337	3854.247	2432.766	3308.105	1700.789	6048.659	0.529	0.089	0.519	0.462	0.590	0.462	0.141	0.472	0.538	0.141	0.528	-7069.901	-7939.983	-6269.088	0.000

Table D.16: loglogistic-weibull mixture on VOD mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	6.123	8.967	3.398	1.343	9.390	13.408	28.554	1.685	1.127	5.042	1375.887	2263.915	804.486	253.539	2392.325	0.818	1.169	0.620	0.487	0.988	0.513	0.212	0.525	0.487	0.212	0.475	-276.795	-339.168	-213.270	0.010
100	5.256	7.432	3.230	1.000	7.569	14.797	29.504	1.641	1.157	69.087	1134.919	1386.784	739.731	249.648	1936.898	0.640	0.394	0.576	0.476	0.759	0.493	0.195	0.509	0.507	0.195	0.491	-554.433	-670.284	-442.736	0.004
200	4.638	6.207	3.107	1.000	6.534	15.809	29.789	1.628	1.196	70.909	985.823	990.208	700.969	248.734	1662.642	0.576	0.176	0.552	0.470	0.666	0.477	0.182	0.496	0.523	0.182	0.504	-1111.008	-1328.382	-906.844	0.002
500	4.071	4.873	2.994	1.000	5.707	16.970	29.927	1.630	1.243	70.606	873.178	757.510	668.876	250.969	1454.945	0.542	0.092	0.533	0.465	0.614	0.461	0.174	0.486	0.539	0.174	0.514	-2780.507	-3286.513	-2310.902	0.001
1000	3.831	4.392	2.936	1.000	5.347	17.463	29.735	1.635	1.270	69.895	823.674	655.066	664.655	252.476	1367.706	0.528	0.074	0.525	0.462	0.593	0.454	0.171	0.483	0.546	0.171	0.517	-5559.319	-6517.713	-4671.535	0.000

Table D.17: loglogistic-weibull mixture on RIOTINTO mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	14.015	17.052	7.701	2.897	24.353	4.333	12.671	1.229	0.855	2.219	23578.452	39390.969	13849.608	4581.616	37993.455	0.606	0.712	0.502	0.395	0.719	0.403	0.194	0.389	0.597	0.194	0.611	-416.191	-475.878	-356.631	0.006
100	12.650	15.050	7.584	3.062	20.001	4.054	12.304	1.170	0.862	1.764	19685.786	26377.706	13350.150	5006.691	31282.278	0.526	0.634	0.470	0.390	0.605	0.391	0.172	0.383	0.609	0.172	0.617	-838.352	-941.364	-741.295	0.002
200	11.895	13.952	7.541	3.508	18.474	3.507	10.994	1.132	0.875	1.526	17032.915	17881.338	12817.136	5421.187	27123.396	0.491	0.648	0.449	0.389	0.545	0.383	0.155	0.378	0.617	0.155	0.622	-1683.217	-1856.909	-1516.093	0.001
500	10.116	10.518	7.306	3.866	13.954	2.857	9.495	1.113	0.897	1.399	13994.510	9770.157	12259.121	5705.095	21519.805	0.439	0.060	0.433	0.388	0.492	0.372	0.129	0.375	0.628	0.129	0.625	-4215.063	-4545.930	-3897.372	0.000
1000	8.715	7.647	6.893	4.284	11.538	2.197	7.302	1.116	0.920	1.322	11643.645	5943.373	11096.027	5880.258	16326.981	0.429	0.048	0.422	0.385	0.472	0.369	0.121	0.368	0.631	0.121	0.632	-8373.393	-8948.305	-7749.836	0.000

Table D.18: loglogistic-weibull mixture on SSELN mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	28.794	107.054	11.372	4.476	37.166	1.302	1.689	0.982	0.706	1.670	20445.650	29265.834	12264.947	4260.609	34357.313	0.540	0.533	0.459	0.365	0.644	0.349	0.181	0.323	0.651	0.181	0.677	-424.726	-488.704	-358.704	0.015
100	25.878	69.567	11.328	4.842	34.731	1.053	0.574	0.915	0.678	1.356	18621.438	41194.864	12027.717	4568.850	30360.323	0.477	0.265	0.438	0.362	0.566	0.342	0.163	0.320	0.658	0.163	0.680	-853.162	-963.670	-744.301	0.008
200	22.369	57.454	11.099	4.867	29.967	0.963	0.390	0.878	0.671	1.194	15690.750	15340.176	11397.384	4852.308	26076.815	0.440	0.093	0.420	0.360	0.516	0.336	0.143	0.321	0.664	0.143	0.679	-1707.985	-1896.064	-1517.880	0.005
500	15.257	20.075	10.906	5.222	23.862	0.920	0.310	0.867	0.679	1.097	12285.932	7400.098	11104.478	5327.944	18903.066	0.413	0.057	0.405	0.360	0.464	0.326	0.118	0.316	0.674	0.118	0.684	-4246.686	-4637.979	-3844.438	0.000
1000	12.974	11.777	9.853	4.862	18.471	0.903	0.266	0.837	0.706	1.057	9047.896	4907.089	8613.074	4096.612	12835.407	0.414	0.052	0.403	0.366	0.475	0.333	0.111	0.325	0.667	0.111	0.675	-8249.093	-8936.567	-7510.629	0.000

Table D.19: weibull-weibull mixture on ABFLN mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	9.870	17.220	6.058	3.076	15.366	1.334	1.242	1.107	0.845	1.785	1433.369	1725.250	936.861	321.628	2668.402	0.715	0.856	0.587	0.468	0.910	0.520	0.201	0.528	0.480	0.201	0.472	-283.712	-349.060	-206.915	0.028
100	8.950	13.579	5.914	3.195	13.312	1.204	0.626	1.064	0.840	1.533	1264.330	5610.384	880.480	344.896	2204.104	0.593	0.307	0.545	0.456	0.721	0.508	0.182	0.516	0.492	0.182	0.484	-571.349	-691.664	-435.954	0.021
200	8.186	9.760	5.813	3.339	12.105	1.148	0.462	1.044	0.846	1.391	1110.981	1015.369	842.385	363.406	1891.019	0.543	0.144	0.522	0.450	0.638	0.499	0.167	0.508	0.501	0.167	0.492	-1147.003	-1371.046	-897.021	0.020
500	7.462	7.832	5.667	3.531	10.823	1.118	0.415	1.031	0.863	1.270	1000.243	817.699	810.880	386.282	1637.783	0.514	0.077	0.503	0.448	0.587	0.490	0.154	0.503	0.510	0.154	0.497	-2873.816	-3396.472	-2286.987	0.018
1000	7.066	5.713	5.551	3.661	10.169	1.104	0.400	1.029	0.877	1.221	945.222	687.164	789.945	399.963	1505.934	0.502	0.061	0.494	0.447	0.566	0.487	0.147	0.502	0.513	0.147	0.498	-5742.118	-6757.131	-4616.632	0.019

Table D.20: weibull-weibull mixture on BARC mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	13.226	29.900	6.014	3.061	20.684	1.386	1.590	1.105	0.796	1.843	9117.605	13563.338	5168.056	1091.807	16907.813	0.663	1.809	0.466	0.365	0.691	0.476	0.238	0.468	0.524	0.238	0.532	-340.826	-446.492	-202.798	0.021
100	12.271	27.036	5.860	3.158	18.698	1.213	1.005	1.052	0.774	1.553	7890.429	10928.556	4969.052	1212.291	14075.694	0.489	0.520	0.434	0.355	0.576	0.466	0.221	0.465	0.534	0.221	0.535	-685.473	-887.665	-440.616	0.011
200	10.908	18.184	5.679	3.243	17.446	1.106	0.428	1.020	0.755	1.427	6888.220	7747.791	4734.987	1362.019	12164.639	0.435	0.178	0.413	0.348	0.510	0.463	0.206	0.466	0.537	0.206	0.534	-1370.803	-1763.506	-934.611	0.007
500	9.329	15.148	5.183	3.411	14.983	1.062	0.342	1.021	0.760	1.314	5782.609	5326.664	4234.527	1508.553	10367.114	0.406	0.069	0.396	0.342	0.469	0.468	0.190	0.471	0.532	0.190	0.529	-3380.007	-4310.517	-2415.987	0.008
1000	7.851	6.458	4.884	3.439	13.535	1.048	0.301	1.039	0.770	1.267	4925.501	3858.282	3616.121	1527.611	9517.407	0.392	0.056	0.382	0.341	0.440	0.481	0.180	0.505	0.519	0.180	0.495	-6592.736	-8463.960	-4800.514	0.010

Table D.21: weibull-weibull mixture on RRLN mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{2\text{ave}}$	$\pm\hat{\alpha}_{2\text{ave}}$	$\hat{\alpha}_{2\text{median}}$	$\hat{\alpha}_{2\text{lower}}$	$\hat{\alpha}_{2\text{upper}}$	$\hat{\beta}_{2\text{ave}}$	$\pm\hat{\beta}_{2\text{ave}}$	$\hat{\beta}_{2\text{median}}$	$\hat{\beta}_{2\text{lower}}$	$\hat{\beta}_{2\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	19.815	43.787	11.145	5.175	28.907	1.183	1.409	0.962	0.734	1.439	4463.499	6072.003	2979.511	1086.307	7886.616	0.697	0.998	0.570	0.445	0.843	0.440	0.188	0.431	0.560	0.188	0.569	-352.118	-412.414	-286.058	0.019
100	16.855	26.692	10.839	5.543	24.794	1.037	0.951	0.919	0.731	1.218	3924.037	4548.714	2851.354	1156.434	6672.353	0.596	0.671	0.533	0.438	0.700	0.429	0.171	0.422	0.571	0.171	0.578	-707.862	-817.530	-595.157	0.012
200	14.657	16.740	10.613	5.944	22.137	0.965	0.722	0.896	0.738	1.101	3518.942	2861.655	2754.791	1239.516	5824.596	0.557	0.822	0.511	0.435	0.628	0.421	0.155	0.415	0.579	0.155	0.585	-1419.831	-1620.626	-1218.809	0.009
500	13.012	10.304	10.358	6.404	19.643	0.911	0.244	0.881	0.753	1.027	3191.069	2161.351	2651.180	1341.555	5166.660	0.504	0.108	0.491	0.433	0.573	0.414	0.140	0.411	0.586	0.140	0.589	-3555.255	-4023.301	-3115.481	0.008
1000	12.130	8.058	10.340	6.673	17.823	0.902	0.236	0.876	0.766	0.989	3016.670	1867.975	2565.831	1395.145	4681.055	0.489	0.061	0.483	0.432	0.548	0.409	0.129	0.412	0.591	0.129	0.588	-7109.074	-7959.673	-6296.716	0.004

Table D.22: weibull-weibull mixture on VOD mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{2\text{ave}}$	$\pm\hat{\alpha}_{2\text{ave}}$	$\hat{\alpha}_{2\text{median}}$	$\hat{\alpha}_{2\text{lower}}$	$\hat{\alpha}_{2\text{upper}}$	$\hat{\beta}_{2\text{ave}}$	$\pm\hat{\beta}_{2\text{ave}}$	$\hat{\beta}_{2\text{median}}$	$\hat{\beta}_{2\text{lower}}$	$\hat{\beta}_{2\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	8.069	11.866	5.144	2.607	13.697	1.443	1.288	1.185	0.888	2.137	1040.652	1315.349	662.437	250.800	2061.511	0.681	0.658	0.581	0.474	0.886	0.482	0.191	0.481	0.518	0.191	0.519	-281.546	-339.631	-211.281	0.047
100	7.377	10.229	5.035	2.654	12.047	1.325	0.822	1.136	0.886	1.912	907.077	1045.867	625.029	259.605	1740.603	0.582	0.236	0.545	0.463	0.720	0.466	0.170	0.468	0.534	0.170	0.532	-567.193	-672.386	-444.513	0.047
200	6.708	7.583	4.910	2.731	10.852	1.278	0.593	1.113	0.901	1.797	805.968	800.732	595.827	265.242	1519.881	0.539	0.102	0.523	0.456	0.641	0.453	0.153	0.455	0.547	0.153	0.545	-1138.393	-1332.894	-912.210	0.050
500	6.096	5.973	4.782	2.862	9.923	1.271	0.573	1.100	0.926	1.798	723.670	581.111	569.785	270.969	1356.235	0.513	0.069	0.505	0.451	0.596	0.441	0.138	0.445	0.559	0.138	0.555	-2848.293	-3295.010	-2313.905	0.059
1000	5.777	4.991	4.698	2.889	9.504	1.279	0.588	1.097	0.948	1.924	682.584	495.896	560.614	274.171	1269.985	0.502	0.059	0.497	0.449	0.576	0.437	0.132	0.442	0.563	0.132	0.558	-5689.567	-6525.687	-4643.829	0.064

Table D.23: weibull-weibull mixture on RIOTINTO mid-price waiting time data: removal of zeros.

n	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{2\text{ave}}$	$\pm\hat{\alpha}_{2\text{ave}}$	$\hat{\alpha}_{2\text{median}}$	$\hat{\alpha}_{2\text{lower}}$	$\hat{\alpha}_{2\text{upper}}$	$\hat{\beta}_{2\text{ave}}$	$\pm\hat{\beta}_{2\text{ave}}$	$\hat{\beta}_{2\text{median}}$	$\hat{\beta}_{2\text{lower}}$	$\hat{\beta}_{2\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	28.108	99.798	11.835	5.052	36.096	1.305	2.030	0.953	0.699	1.592	17114.214	20577.758	11070.867	4084.359	30349.036	0.530	0.471	0.462	0.371	0.650	0.352	0.177	0.330	0.648	0.177	0.670	-420.249	-478.306	-357.689	0.020
100	22.626	69.641	11.391	5.289	29.706	1.092	1.377	0.901	0.688	1.309	14511.181	14931.218	10545.764	4299.657	24031.589	0.465	0.137	0.436	0.367	0.556	0.337	0.155	0.323	0.663	0.155	0.677	-845.120	-945.481	-745.308	0.010
200	17.888	26.466	11.432	5.592	25.806	0.959	0.466	0.868	0.692	1.154	12689.948	10613.481	10269.495	4617.658	20579.925	0.436	0.087	0.420	0.365	0.509	0.328	0.136	0.316	0.672	0.136	0.684	-1693.536	-1863.023	-1517.847	0.008
500	13.877	9.886	11.160	6.163	21.481	0.905	0.300	0.854	0.700	1.039	10629.083	5982.670	9698.955	4788.632	16200.334	0.411	0.052	0.406	0.360	0.463	0.319	0.110	0.315	0.681	0.110	0.685	-4228.549	-4558.253	-3908.584	0.000
1000	12.814	8.498	10.677	6.630	17.390	0.860	0.210	0.837	0.710	0.959	9129.509	4007.024	8885.143	5156.198	12931.852	0.405	0.045	0.401	0.363	0.453	0.317	0.098	0.313	0.683	0.098	0.687	-8396.831	-8962.303	-7793.016	0.000

Table D.24: weibull-weibull mixture on SSELN mid-price waiting time data: removal of zeros.

	AIC			BIC		
exponential/weibull	0.3372	0.6070	-	0.8205	-	-
gamma/weibull	0.0233	0.1115	0.1793	0.0025	0.0316	0.1793
loglogistic/weibull	0.5384	-	-	0.1579	0.7212	-
weibull/weibull	0.1011	0.2815	0.8167	0.0191	0.2471	0.8167

Table D.25: ABFLN, N = 50: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.2091	0.5590	-	0.6888	-	-
gamma/weibull	0.0197	0.2139	0.2709	0.0084	0.0242	0.2709
loglogistic/weibull	0.7104	-	-	0.2867	0.8417	-
weibull/weibull	0.0605	0.2268	0.7266	0.0159	0.1337	0.7266

Table D.26: BARC, N = 50: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.2194	0.5463	-	0.6462	-	-
gamma/weibull	0.0174	0.2004	0.2648	0.0069	0.0228	0.2648
loglogistic/weibull	0.6992	-	-	0.3318	0.8246	-
weibull/weibull	0.0636	0.2529	0.7286	0.0146	0.1521	0.7286

Table D.27: RRLN, N = 50: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.3578	0.6388	-	0.8421	-	-
gamma/weibull	0.0256	0.1202	0.1747	0.0061	0.0341	0.1747
loglogistic/weibull	0.5347	-	-	0.1378	0.7328	-
weibull/weibull	0.0819	0.2409	0.8231	0.0138	0.2329	0.8231

Table D.28: VOD, N = 50: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.1873	0.5098	-	0.6450	-	-
gamma/weibull	0.0151	0.2585	0.3274	0.0068	0.0187	0.3274
loglogistic/weibull	0.7429	-	-	0.3346	0.8687	-
weibull/weibull	0.0544	0.2314	0.6696	0.0134	0.1124	0.6696

Table D.29: RIOTINTO, N = 50: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.3494	0.6269	-	0.8358	-	-
gamma/weibull	0.0251	0.1087	0.1761	0.0031	0.0377	0.1761
loglogistic/weibull	0.5408	-	-	0.1464	0.7255	-
weibull/weibull	0.0846	0.2644	0.8192	0.0147	0.2367	0.8192

Table D.30: SSELN, N = 50: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.1672	0.5137	-	0.7387	-	-
gamma/weibull	0.0175	0.1326	0.1647	0.0030	0.0180	0.1647
loglogistic/weibull	0.7511	-	-	0.2438	0.8516	-
weibull/weibull	0.0642	0.3537	0.8349	0.0145	0.1305	0.8349

Table D.31: ABFLN, N = 100: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0790	0.5097	-	0.5190	-	-
gamma/weibull	0.0126	0.2461	0.2778	0.0046	0.0136	0.2778
loglogistic/weibull	0.8790	-	-	0.4681	0.9323	-
weibull/weibull	0.0294	0.2441	0.7219	0.0084	0.0541	0.7219

Table D.32: BARC, N = 100: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0959	0.4591	-	0.4804	0.8304	-
gamma/weibull	0.0106	0.2482	0.2815	0.0034	0.1070	0.2815
loglogistic/weibull	0.8549	-	-	0.5062	-	-
weibull/weibull	0.0386	0.2927	0.7174	0.0101	0.0627	0.7174

Table D.33: RRLN, N = 100: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.1855	0.5839	-	0.7605	-	-
gamma/weibull	0.0157	0.1183	0.1451	0.0042	0.0184	0.1451
loglogistic/weibull	0.7462	-	-	0.2257	0.8533	-
weibull/weibull	0.0526	0.2978	0.8547	0.0096	0.1283	0.8547

Table D.34: VOD, N = 100: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0618	0.4359	-	0.4814	0.7981	-
gamma/weibull	0.0084	0.3209	0.3579	0.0034	0.1212	0.3579
loglogistic/weibull	0.9072	-	-	0.5101	-	-
weibull/weibull	0.0226	0.2432	0.6418	0.0051	0.0807	0.6418

Table D.35: RIOTINTO, N = 100: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.1772	0.5528	-	0.7540	-	-
gamma/weibull	0.0174	0.1200	0.1644	0.0030	0.0210	0.1644
loglogistic/weibull	0.7537	-	-	0.2315	0.8606	-
weibull/weibull	0.0516	0.3272	0.8349	0.0115	0.1184	0.8349

Table D.36: SSELN, N = 100: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0568	0.4126	0.7901	0.5036	-	-
gamma/weibull	0.0126	0.1423	0.2099	0.0054	0.0126	0.1595
loglogistic/weibull	0.8982	-	-	0.4811	0.9306	-
weibull/weibull	0.0324	0.4450	-	0.0099	0.0568	0.8405

Table D.37: ABFLN, N = 200: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0169	0.4630	-	0.2754	0.7986	-
gamma/weibull	0.0091	0.2635	0.2790	0.0046	0.1437	0.2790
loglogistic/weibull	0.9586	-	-	0.7131	-	-
weibull/weibull	0.0154	0.2735	0.7210	0.0070	0.0577	0.7210

Table D.38: BARC, N = 200: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0281	0.3721	-	0.2800	0.7517	-
gamma/weibull	0.0066	0.2661	0.2860	0.0010	0.1592	0.2860
loglogistic/weibull	0.9530	-	-	0.7150	-	-
weibull/weibull	0.0122	0.3618	0.7140	0.0040	0.0890	0.7140

Table D.39: RRLN, N = 200: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0680	0.5130	-	0.5326	-	-
gamma/weibull	0.0093	0.1044	0.1186	0.0037	0.0097	0.1186
loglogistic/weibull	0.8959	-	-	0.4572	0.9312	-
weibull/weibull	0.0268	0.3826	0.8813	0.0066	0.0590	0.8813

Table D.40: VOD, N = 200: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0130	0.3733	-	0.2360	0.7126	-
gamma/weibull	0.0051	0.3615	0.3817	0.0022	0.2028	0.3817
loglogistic/weibull	0.9738	-	-	0.7599	-	-
weibull/weibull	0.0081	0.2652	0.6182	0.0019	0.0846	0.6182

Table D.41: RIOTINTO, N = 200: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0515	0.4578	-	0.5580	-	-
gamma/weibull	0.0082	0.1215	0.1352	0.0014	0.0089	0.1352
loglogistic/weibull	0.9218	-	-	0.4372	0.9520	-
weibull/weibull	0.0185	0.4207	0.8648	0.0034	0.0391	0.8648

Table D.42: SSELN, N = 200: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0053	0.3048	0.8155	0.1096	0.7032	-
gamma/weibull	0.0000	0.1390	0.1845	0.0000	0.0695	0.1551
loglogistic/weibull	0.9947	-	-	0.8904	-	-
weibull/weibull	0.0000	0.5561	-	0.0000	0.2273	0.8449

Table D.43: ABFLN, N = 500: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0014	0.4035	-	0.0589	0.7297	-
gamma/weibull	0.0068	0.2610	0.2662	0.0061	0.1941	0.2662
loglogistic/weibull	0.9837	-	-	0.9287	-	-
weibull/weibull	0.0081	0.3355	0.7338	0.0062	0.0762	0.7338

Table D.44: BARC, N = 500: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0018	0.2429	0.6392	0.0550	0.6126	-
gamma/weibull	0.0009	0.3067	0.3608	0.0009	0.2296	0.3129
loglogistic/weibull	0.9911	-	-	0.9415	-	-
weibull/weibull	0.0062	0.4504	-	0.0027	0.1578	0.6871

Table D.45: RRLN, N = 500: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0130	0.3743	0.8837	0.1615	0.8113	-
gamma/weibull	0.0042	0.0905	0.1163	0.0037	0.0535	0.0950
loglogistic/weibull	0.9678	-	-	0.8297	-	-
weibull/weibull	0.0150	0.5352	-	0.0051	0.1352	0.9050

Table D.46: VOD, N = 500: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0006	0.2911	0.4473	0.0488	0.5984	-
gamma/weibull	0.0029	0.3915	-	0.0019	0.2995	0.3992
loglogistic/weibull	0.9940	-	-	0.9482	-	-
weibull/weibull	0.0024	0.3174	0.5527	0.0011	0.1021	0.6008

Table D.47: RIOTINTO, N = 500: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0059	0.3020	0.8588	0.1157	0.7824	-
gamma/weibull	0.0000	0.1176	0.1412	0.0000	0.0451	0.1255
loglogistic/weibull	0.9902	-	-	0.8824	-	-
weibull/weibull	0.0039	0.5804	-	0.0020	0.1725	0.8745

Table D.48: SSELN, N = 500: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0000	0.1981	0.7925	0.0000	0.5943	-
gamma/weibull	0.0000	0.1698	0.2075	0.0000	0.1132	0.1698
loglogistic/weibull	1.0000	-	-	1.0000	-	-
weibull/weibull	0.0000	0.6321	-	0.0000	0.2925	0.8302

Table D.49: ABFLN, N = 1000: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0003	0.3417	0.6884	0.0059	0.6857	-
gamma/weibull	0.0071	0.2581	0.3116	0.0069	0.2173	0.2603
loglogistic/weibull	0.9862	-	-	0.9814	-	-
weibull/weibull	0.0065	0.4001	-	0.0058	0.0970	0.7397

Table D.50: BARC, N = 1000: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0000	0.1733	0.6056	0.0060	0.5020	-
gamma/weibull	0.0000	0.3327	0.3944	0.0000	0.2789	0.3347
loglogistic/weibull	0.9980	-	-	0.9920	-	-
weibull/weibull	0.0020	0.4940	-	0.0020	0.2191	0.6653

Table D.51: RRLN, N = 1000: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0023	0.2485	0.8921	0.0387	0.7098	-
gamma/weibull	0.0041	0.0809	0.1079	0.0041	0.0528	0.0821
loglogistic/weibull	0.9818	-	-	0.9525	-	-
weibull/weibull	0.0117	0.6706	-	0.0047	0.2374	0.9179

Table D.52: VOD, N = 1000: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0001	0.2309	0.3822	0.0079	0.5090	-
gamma/weibull	0.0018	0.4000	-	0.0018	0.3558	0.4036
loglogistic/weibull	0.9967	-	-	0.9890	-	-
weibull/weibull	0.0014	0.3691	0.6178	0.0013	0.1352	0.5964

Table D.53: RIOTINTO, N = 1000: mid-price waiting times, removal of zeros.

	AIC			BIC		
exponential/weibull	0.0000	0.2053	0.8579	0.0105	0.6947	-
gamma/weibull	0.0000	0.0895	0.1421	0.0000	0.0526	0.0895
loglogistic/weibull	0.9947	-	-	0.9842	-	-
weibull/weibull	0.0053	0.7053	-	0.0053	0.2526	0.9105

Table D.54: SSELN, N = 1000: mid-price waiting times, removal of zeros.

n	θ	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	3.309	208.490	0.153	0.049	0.513	-	-	-	-	-	12995.223	15150.148	8309.365	2566.943	22788.306	0.614	4.158	0.405	0.323	0.558	0.288	0.165	0.262	0.044	0.083	0.017	0.667	0.177	0.697	-398.241	-472.241	-327.428	0.003
100	0.5	0.313	1.106	0.158	0.058	0.459	-	-	-	-	-	10685.447	10260.953	7599.575	2805.049	18638.624	0.523	3.824	0.380	0.316	0.489	0.275	0.147	0.248	0.042	0.077	0.020	0.683	0.157	0.708	-800.662	-930.032	-677.013	0.002
200	0.5	0.321	1.956	0.166	0.068	0.418	-	-	-	-	-	9162.363	7463.900	7068.828	3182.353	15270.968	0.397	0.433	0.364	0.318	0.444	0.263	0.130	0.240	0.041	0.073	0.022	0.695	0.137	0.717	-1605.018	-1830.791	-1398.150	0.002
500	0.5	0.225	0.168	0.170	0.090	0.384	-	-	-	-	-	7543.205	4629.891	6382.555	3492.493	11969.678	0.360	0.051	0.351	0.312	0.403	0.259	0.107	0.243	0.037	0.054	0.021	0.704	0.108	0.718	-3996.636	-4435.025	-3573.180	0.000
1000	0.5	0.245	0.154	0.208	0.106	0.403	-	-	-	-	-	5535.980	2737.229	4985.547	3053.140	8213.171	0.358	0.046	0.353	0.315	0.400	0.261	0.097	0.251	0.039	0.044	0.025	0.700	0.090	0.706	-7771.340	-8603.200	-6927.892	0.000

Table D.55: exponential-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) ABFLN data.

n	θ	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	0.668	18.301	0.237	0.104	0.667	-	-	-	-	-	1066.908	1350.972	647.370	174.730	1915.062	0.794	3.475	0.524	0.411	0.797	0.465	0.209	0.460	0.059	0.098	0.021	0.476	0.207	0.475	-256.689	-329.702	-179.275	0.007
100	0.5	5.355	1137.031	0.245	0.121	0.602	-	-	-	-	-	874.609	965.466	587.679	194.352	1501.615	0.651	3.479	0.478	0.400	0.619	0.450	0.197	0.449	0.058	0.093	0.027	0.492	0.192	0.493	-516.779	-648.945	-381.923	0.001
200	0.5	0.594	15.715	0.252	0.136	0.553	-	-	-	-	-	753.588	741.380	545.268	213.810	1253.548	0.574	2.333	0.454	0.396	0.542	0.438	0.187	0.444	0.056	0.088	0.031	0.506	0.179	0.504	-1039.209	-1281.386	-794.019	0.000
500	0.5	0.787	28.976	0.259	0.151	0.510	-	-	-	-	-	664.160	576.623	516.475	228.235	1066.455	0.574	3.743	0.437	0.394	0.495	0.430	0.178	0.442	0.054	0.085	0.034	0.516	0.167	0.512	-2607.888	-3159.829	-2046.025	0.000
1000	0.5	0.474	4.178	0.263	0.157	0.484	-	-	-	-	-	624.201	486.003	508.337	239.752	995.066	0.572	5.714	0.429	0.394	0.477	0.427	0.173	0.445	0.052	0.082	0.034	0.521	0.160	0.513	-5220.799	-6279.564	-4160.430	0.000

Table D.56: exponential-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	0.807	27.927	0.241	0.087	0.553	-	-	-	-	-	6518.044	9314.016	3715.876	761.090	11634.956	0.757	3.116	0.425	0.330	0.636	0.437	0.250	0.401	0.041	0.074	0.009	0.522	0.245	0.545	-318.069	-428.331	-187.452	0.006
100	0.5	0.444	3.564	0.247	0.098	0.483	-	-	-	-	-	5293.415	6279.670	3428.741	875.105	9342.622	0.627	5.254	0.390	0.323	0.515	0.430	0.242	0.393	0.039	0.072	0.013	0.531	0.234	0.558	-637.902	-844.250	-405.346	0.001
200	0.5	0.464	6.988	0.255	0.111	0.441	-	-	-	-	-	4403.213	4196.809	3184.176	972.722	7649.292	0.492	2.913	0.369	0.319	0.454	0.426	0.233	0.388	0.035	0.066	0.015	0.539	0.222	0.565	-1276.151	-1672.479	-867.516	0.000
500	0.5	0.304	0.577	0.266	0.132	0.397	-	-	-	-	-	3678.984	2986.767	2901.566	1101.770	6251.417	0.385	0.783	0.352	0.315	0.404	0.432	0.223	0.400	0.030	0.057	0.013	0.538	0.209	0.559	-3156.024	-4087.761	-2213.204	0.000
1000	0.5	0.287	0.149	0.282	0.143	0.381	-	-	-	-	-	3181.831	2215.419	2673.920	1118.310	5192.505	0.352	0.044	0.342	0.315	0.385	0.448	0.216	0.477	0.025	0.042	0.010	0.527	0.200	0.501	-6177.406	-8040.285	-4545.265	0.000

Table D.57: exponential-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) RRLN data.

n	θ	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	0.657	17.605	0.137	0.055	0.462	-	-	-	-	-	3029.787	3467.472	1969.458	571.181	5389.388	0.703	3.012	0.491	0.380	0.719	0.360	0.178	0.344	0.064	0.098	0.034	0.576	0.188	0.591	-322.943	-395.241	-251.420	0.006
100	0.5	1.067	52.962	0.142	0.063	0.420	-	-	-	-	-	2546.971	2554.967	1795.843	613.698	4399.265	0.614	2.601	0.452	0.372	0.602	0.345	0.164	0.329	0.063	0.095	0.039	0.592	0.171	0.606	-649.335	-779.555	-527.567	0.002
200	0.5	1.676	101.693	0.145	0.072	0.383	-	-	-	-	-	2241.931	1976.781	1687.347	664.930	3823.005	0.559	2.009	0.431	0.369	0.534	0.334	0.154	0.320	0.063	0.091	0.042	0.603	0.159	0.615	-1304.263	-1538.371	-1087.858	0.002
500	0.5	1.897	94.115	0.151	0.081	0.352	-	-	-	-	-	2009.527	1549.839	1621.295	699.598	3290.490	0.521	1.868	0.415	0.368	0.488	0.325	0.144	0.315	0.062	0.087	0.046	0.613	0.144	0.624	-3270.268	-3805.830	-2797.347	0.000
1000	0.5	1.165	25.747	0.153	0.087	0.323	-	-	-	-	-	1882.187	1307.299	1576.382	735.160	2996.276	0.468	1.265	0.407	0.367	0.463	0.321	0.134	0.314	0.061	0.085	0.046	0.617	0.131	0.623	-6546.444	-7517.483	-5685.547	0.000

Table D.58: exponential-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) VOD data.

n	θ	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	1.159	64.926	0.300	0.126	0.880	-	-	-	-	-	777.784	990.081	461.327	135.895	1394.827	0.831	4.397	0.527	0.419	0.769	0.432	0.197	0.422	0.070	0.118	0.023	0.498	0.199	0.506	-249.126	-320.885	-179.036	0.008
100	0.5	1.292	63.626	0.313	0.147	0.831	-	-	-	-	-	641.658	699.680	422.604	146.172	1116.221	0.703	3.996	0.484	0.409	0.614	0.415	0.181	0.404	0.069	0.115	0.028	0.517	0.181	0.526	-501.294	-631.834	-382.049	0.002
200	0.5	2.629	369.974	0.324	0.166	0.806	-	-	-	-	-	557.586	528.714	401.920	153.781	951.582	0.667	4.641	0.463	0.406	0.544	0.402	0.169	0.392	0.067	0.113	0.031	0.531	0.166	0.541	-1007.519	-1246.579	-795.563	0.001
500	0.5	1.017	20.189	0.336	0.186	0.780	-	-	-	-	-	497.763	413.239	386.173	160.329	830.505	0.651	4.676	0.447	0.403	0.503	0.393	0.160	0.382	0.064	0.109	0.033	0.543	0.152	0.550	-2528.101	-3072.998	-2060.904	0.000
1000	0.5	1.368	52.680	0.342	0.196	0.752	-	-	-	-	-	471.423	352.511	383.009	163.213	785.592	0.641	5.291	0.441	0.402	0.487	0.390	0.154	0.380	0.063	0.107	0.034	0.547	0.144	0.555	-5062.447	-6079.479	-4178.127	0.000

Table D.59: exponential-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) RIOTINTO data.

n	θ	$\hat{\lambda}_{\text{ave}}$	$\pm\hat{\lambda}_{\text{ave}}$	$\hat{\lambda}_{\text{median}}$	$\hat{\lambda}_{\text{lower}}$	$\hat{\lambda}_{\text{upper}}$	-	-	-	-	-	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	0.375	3.700	0.140	0.049	0.469	-	-	-	-	-	11297.262	12518.862	7556.725	2390.428	19728.549	0.525	2.205	0.408	0.328	0.556	0.289	0.156	0.267	0.048	0.079	0.024	0.663	0.169	0.684	-393.462	-462.109	-326.898	0.003
100	0.5	0.603	13.177	0.144	0.061	0.435	-	-	-	-	-	9229.210	8461.265	6872.236	2563.068	15706.515	0.502	2.505	0.383	0.323	0.482	0.272	0.138	0.255	0.047	0.078	0.027	0.681	0.150	0.699	-789.807	-909.901	-677.833	0.000
200	0.5	0.423	5.841	0.146	0.072	0.422	-	-	-	-	-	7869.063	6018.102	6417.339	2701.898	12891.807	0.442	2.063	0.368	0.319	0.437	0.260	0.120	0.245	0.045	0.070	0.030	0.695	0.130	0.714	-1585.462	-1782.905	-1395.233	0.001
500	0.5	0.251	0.615	0.151	0.084	0.371	-	-	-	-	-	6818.648	4099.433	6010.239	2920.786	10599.859	0.367	0.151	0.355	0.316	0.401	0.252	0.101	0.244	0.043	0.067	0.031	0.704	0.106	0.722	-3967.082	-4372.684	-3599.559	0.000
1000	0.5	0.211	0.156	0.163	0.102	0.350	-	-	-	-	-	5961.272	2829.834	5644.590	3263.860	8979.711	0.353	0.038	0.348	0.319	0.386	0.247	0.089	0.237	0.042	0.061	0.031	0.711	0.091	0.726	-7901.392	-8551.081	-7303.467	0.000

Table D.60: exponential-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) SSELN data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	4.151	3.691	2.743	1.398	7.753	0.987	0.131	0.997	0.877	1.112	6902.675	7567.410	4627.719	1290.615	12647.193	0.501	4.117	0.343	0.286	0.445	0.170	0.128	0.140	0.046	0.090	0.016	0.785	0.155	0.823	-398.693	-471.890	-325.150	0.012
100	0.5	3.949	3.432	2.652	1.493	6.955	1.022	0.096	1.024	0.936	1.116	6023.950	5813.871	4386.915	1487.200	10433.565	0.447	3.797	0.329	0.284	0.402	0.158	0.113	0.132	0.044	0.082	0.020	0.798	0.140	0.831	-803.358	-932.639	-681.925	0.001
200	0.5	3.742	3.210	2.561	1.603	6.315	1.045	0.076	1.048	0.975	1.121	5311.984	4570.960	4158.901	1688.814	8677.543	0.349	0.527	0.320	0.284	0.381	0.152	0.100	0.129	0.043	0.077	0.024	0.805	0.125	0.836	-1610.718	-1836.031	-1407.818	0.001
500	0.5	3.508	2.699	2.569	1.772	5.618	1.066	0.057	1.070	1.009	1.126	4481.913	2868.003	3876.909	2049.557	6881.159	0.320	0.043	0.312	0.282	0.364	0.150	0.083	0.134	0.040	0.055	0.024	0.810	0.101	0.829	-4012.401	-4451.014	-3589.356	0.000
1000	0.5	3.107	2.081	2.357	1.784	4.791	1.067	0.050	1.068	1.015	1.117	3477.157	1990.061	3044.019	1783.351	4875.693	0.322	0.039	0.316	0.287	0.370	0.157	0.072	0.145	0.043	0.046	0.030	0.799	0.085	0.804	-7799.361	-8633.195	-6990.187	0.000

Table D.61: gamma-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) ABFLN data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	3.089	2.626	2.302	1.216	4.975	1.002	0.107	1.006	0.907	1.106	487.846	629.450	296.003	83.054	855.540	0.579	2.720	0.423	0.354	0.540	0.314	0.194	0.277	0.064	0.101	0.028	0.622	0.208	0.661	-257.533	-331.158	-180.232	0.004
100	0.5	2.845	2.253	2.239	1.264	3.880	1.022	0.088	1.024	0.944	1.108	416.330	449.080	284.148	98.471	702.925	0.507	2.669	0.402	0.350	0.477	0.303	0.177	0.275	0.063	0.095	0.036	0.634	0.191	0.665	-519.709	-652.025	-384.877	0.001
200	0.5	2.654	1.951	2.203	1.322	3.301	1.033	0.077	1.036	0.966	1.109	380.046	362.717	278.450	112.451	619.166	0.469	1.700	0.391	0.349	0.448	0.296	0.163	0.278	0.063	0.090	0.042	0.641	0.177	0.663	-1045.216	-1287.610	-799.654	0.000
500	0.5	2.456	1.556	2.172	1.373	2.984	1.040	0.069	1.044	0.981	1.107	355.911	302.123	274.312	127.066	559.266	0.441	1.231	0.382	0.349	0.430	0.292	0.150	0.282	0.064	0.085	0.047	0.644	0.165	0.660	-2622.137	-3172.454	-2060.003	0.000
1000	0.5	2.365	1.351	2.163	1.402	2.852	1.043	0.066	1.048	0.986	1.105	344.809	272.392	272.352	133.155	533.782	0.449	1.685	0.379	0.350	0.423	0.291	0.143	0.286	0.064	0.082	0.050	0.644	0.158	0.654	-5248.344	-6308.594	-4193.548	0.000

Table D.62: gamma-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	3.333	2.988	2.480	1.460	5.064	1.009	0.120	1.014	0.907	1.125	3183.309	4387.384	1737.443	262.323	6064.532	0.499	2.507	0.351	0.286	0.465	0.318	0.235	0.240	0.048	0.080	0.022	0.634	0.240	0.701	-317.807	-428.361	-185.644	0.009
100	0.5	3.035	2.433	2.417	1.537	3.844	1.035	0.092	1.038	0.953	1.123	2690.410	3156.698	1666.790	328.190	4959.858	0.514	5.189	0.331	0.283	0.413	0.310	0.224	0.235	0.047	0.076	0.028	0.643	0.228	0.709	-640.643	-847.402	-408.162	0.001
200	0.5	2.883	2.230	2.362	1.653	3.364	1.050	0.072	1.051	0.985	1.120	2373.422	2498.534	1624.979	403.701	4248.083	0.397	2.422	0.320	0.282	0.386	0.308	0.213	0.235	0.046	0.068	0.034	0.646	0.217	0.705	-1281.650	-1677.694	-874.623	0.000
500	0.5	2.578	1.445	2.324	1.777	2.966	1.060	0.054	1.063	1.006	1.114	2058.926	1924.628	1454.833	492.228	3697.958	0.322	0.055	0.310	0.281	0.358	0.316	0.201	0.265	0.045	0.057	0.037	0.639	0.203	0.676	-3170.537	-4098.523	-2222.964	0.000
1000	0.5	2.450	0.987	2.329	1.891	2.817	1.064	0.045	1.068	1.021	1.107	1806.294	1518.973	1340.154	511.721	3127.275	0.315	0.042	0.305	0.282	0.347	0.333	0.193	0.349	0.043	0.041	0.037	0.624	0.194	0.601	-6205.877	-8042.639	-4562.087	0.000

Table D.63: gamma-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) RRLN data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	3.838	3.648	2.464	1.164	7.340	0.985	0.121	0.993	0.878	1.102	1316.389	1455.188	877.649	248.625	2362.616	0.521	2.304	0.388	0.323	0.507	0.196	0.136	0.167	0.058	0.100	0.025	0.746	0.165	0.783	-323.745	-396.412	-252.333	0.006
100	0.5	3.777	3.682	2.382	1.220	6.949	1.013	0.095	1.016	0.928	1.105	1147.982	1108.333	829.025	283.819	1985.321	0.499	2.174	0.370	0.320	0.458	0.182	0.119	0.155	0.056	0.096	0.029	0.762	0.148	0.796	-653.060	-783.456	-532.699	0.000
200	0.5	3.691	3.681	2.337	1.288	6.669	1.031	0.082	1.035	0.957	1.110	1040.732	914.719	789.356	314.642	1746.755	0.457	1.819	0.360	0.318	0.430	0.173	0.105	0.151	0.055	0.093	0.033	0.772	0.136	0.801	-1311.922	-1545.250	-1098.248	0.000
500	0.5	3.707	3.801	2.298	1.378	6.536	1.045	0.072	1.051	0.978	1.114	957.529	760.944	752.721	338.649	1574.855	0.451	1.870	0.351	0.316	0.411	0.166	0.092	0.150	0.055	0.087	0.037	0.779	0.122	0.799	-3291.833	-3821.594	-2821.824	0.000
1000	0.5	3.632	3.806	2.303	1.450	6.203	1.052	0.067	1.060	0.986	1.115	905.920	648.934	741.613	371.335	1453.554	0.373	0.283	0.347	0.316	0.402	0.164	0.084	0.150	0.055	0.085	0.038	0.781	0.115	0.798	-6589.271	-7546.812	-5708.093	0.000

Table D.64: gamma-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) VOD data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	2.724	2.311	2.073	1.097	4.029	0.991	0.105	0.994	0.899	1.093	413.378	513.861	256.695	83.775	723.236	0.648	3.524	0.444	0.371	0.567	0.307	0.181	0.279	0.077	0.123	0.033	0.616	0.204	0.647	-249.699	-322.083	-179.580	0.006
100	0.5	2.499	1.985	1.999	1.131	3.320	1.009	0.087	1.009	0.934	1.092	356.599	357.238	248.503	96.126	608.202	0.572	2.911	0.423	0.367	0.502	0.297	0.162	0.279	0.076	0.118	0.039	0.626	0.186	0.650	-503.211	-634.260	-383.410	0.002
200	0.5	2.340	1.719	1.966	1.164	3.001	1.018	0.077	1.017	0.953	1.092	326.541	279.588	244.996	106.995	546.338	0.549	3.076	0.412	0.366	0.472	0.292	0.147	0.281	0.076	0.115	0.043	0.632	0.171	0.648	-1011.202	-1251.483	-798.841	0.001
500	0.5	2.222	1.484	1.941	1.196	2.817	1.024	0.070	1.024	0.965	1.090	306.052	226.578	244.785	117.427	498.386	0.539	2.823	0.403	0.366	0.454	0.288	0.133	0.284	0.076	0.109	0.048	0.636	0.156	0.645	-2536.594	-3084.523	-2066.081	0.000
1000	0.5	2.173	1.347	1.929	1.226	2.739	1.025	0.066	1.025	0.970	1.088	296.508	201.528	245.774	123.816	474.809	0.547	4.336	0.398	0.366	0.445	0.287	0.125	0.285	0.076	0.106	0.050	0.637	0.148	0.644	-5078.597	-6108.118	-4195.599	0.000

Table D.65: gamma-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) RIOTINTO data.

n	θ	$\hat{k}_{\text{ave}}$	$\pm\hat{k}_{\text{ave}}$	$\hat{k}_{\text{median}}$	$\hat{k}_{\text{lower}}$	$\hat{k}_{\text{upper}}$	$\hat{\theta}_{\text{ave}}$	$\pm\hat{\theta}_{\text{ave}}$	$\hat{\theta}_{\text{median}}$	$\hat{\theta}_{\text{lower}}$	$\hat{\theta}_{\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	3.951	3.547	2.644	1.298	7.403	0.987	0.128	0.996	0.876	1.113	5818.192	9897.028	3963.496	1184.380	10298.661	0.446	2.241	0.342	0.286	0.443	0.164	0.117	0.140	0.046	0.083	0.020	0.789	0.142	0.821	-393.827	-462.027	-325.779	0.011
100	0.5	3.826	3.429	2.589	1.384	6.696	1.021	0.097	1.025	0.935	1.114	4899.377	4238.397	3785.508	1328.032	8423.257	0.410	1.359	0.330	0.284	0.400	0.152	0.102	0.131	0.044	0.079	0.024	0.804	0.127	0.834	-793.465	-912.076	-680.792	0.000
200	0.5	3.627	3.129	2.493	1.485	6.083	1.042	0.079	1.043	0.970	1.119	4362.084	3294.950	3583.597	1538.456	7098.907	0.349	0.504	0.321	0.283	0.380	0.144	0.086	0.126	0.043	0.075	0.028	0.813	0.111	0.838	-1591.521	-1789.877	-1404.441	0.000
500	0.5	3.518	2.953	2.504	1.644	5.922	1.059	0.062	1.064	0.996	1.118	3859.715	2308.568	3450.225	1682.021	5967.881	0.326	0.149	0.313	0.283	0.358	0.139	0.069	0.128	0.042	0.066	0.029	0.820	0.090	0.834	-3983.543	-4384.893	-3641.367	0.000
1000	0.5	3.152	2.537	2.420	1.732	3.640	1.066	0.053	1.070	1.008	1.126	3601.934	1856.610	3408.004	1832.678	5493.884	0.318	0.035	0.313	0.285	0.348	0.140	0.059	0.129	0.041	0.061	0.031	0.818	0.077	0.832	-7930.578	-8566.750	-7333.926	0.000

Table D.66: gamma-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) SSELN data.

n	θ	$\hat{\alpha}_{\text{log ave}}$	$\pm\hat{\alpha}_{\text{log ave}}$	$\hat{\alpha}_{\text{log median}}$	$\hat{\alpha}_{\text{log lower}}$	$\hat{\alpha}_{\text{log upper}}$	$\hat{\beta}_{\text{log ave}}$	$\pm\hat{\beta}_{\text{log ave}}$	$\hat{\beta}_{\text{log median}}$	$\hat{\beta}_{\text{log lower}}$	$\hat{\beta}_{\text{log upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	42.042	291.930	7.641	2.203	41.365	3.283	10.910	1.011	0.665	1.962	38138.545	88672.624	16403.872	4761.748	58703.551	0.783	2.575	0.523	0.382	0.879	0.426	0.204	0.412	0.043	0.075	0.017	0.530	0.205	0.546	-399.566	-470.291	-323.154	0.016
100	0.5	24.577	76.300	7.082	2.331	27.853	3.083	15.940	0.952	0.663	1.496	32386.344	112427.527	15011.615	5079.002	46137.889	0.602	0.850	0.481	0.378	0.702	0.420	0.188	0.408	0.041	0.074	0.019	0.540	0.187	0.556	-800.619	-927.323	-672.362	0.009
200	0.5	16.983	42.398	6.748	2.564	19.992	2.034	7.494	0.912	0.677	1.248	23573.558	32318.077	14483.881	5771.838	36809.093	0.580	1.580	0.453	0.373	0.597	0.417	0.170	0.402	0.037	0.066	0.018	0.546	0.165	0.562	-1601.695	-1825.747	-1391.121	0.004
500	0.5	8.372	9.246	5.960	2.586	12.149	1.496	5.744	0.903	0.720	1.126	16164.920	11783.574	13346.165	6446.132	26152.391	0.444	0.098	0.426	0.372	0.518	0.405	0.134	0.396	0.030	0.039	0.015	0.565	0.128	0.570	-3991.440	-4430.676	-3563.304	0.005
1000	0.5	6.548	6.112	4.788	2.587	9.316	1.302	4.193	0.897	0.761	1.089	12108.913	7991.204	10461.842	5987.133	17817.756	0.437	0.070	0.415	0.377	0.505	0.408	0.128	0.410	0.033	0.043	0.020	0.559	0.117	0.559	-7745.533	-8575.697	-6897.413	0.000

Table D.67: loglogistic-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) ABFLN data.

n	θ	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	12.417	954.415	3.463	1.492	11.437	7.053	22.275	1.232	0.858	2.594	2059.773	2897.753	1237.903	312.218	3988.956	1.180	3.379	0.696	0.470	1.468	0.558	0.216	0.584	0.066	0.099	0.031	0.376	0.206	0.352	-256.977	-327.911	-173.202	0.032
100	0.5	6.589	198.075	3.272	1.559	8.523	6.695	21.729	1.153	0.849	1.882	1790.704	2187.552	1199.863	356.006	3177.630	0.862	3.086	0.619	0.454	0.993	0.561	0.205	0.589	0.063	0.095	0.033	0.375	0.191	0.355	-514.457	-645.535	-374.101	0.012
200	0.5	5.047	13.764	3.123	1.636	7.256	6.203	20.728	1.116	0.852	1.614	1585.038	1621.156	1157.972	395.023	2713.181	0.719	2.301	0.580	0.450	0.807	0.562	0.196	0.592	0.060	0.092	0.033	0.378	0.178	0.359	-1032.900	-1275.480	-781.112	0.006
500	0.5	4.344	5.204	3.013	1.730	6.392	5.771	19.675	1.095	0.859	1.431	1409.416	1235.047	1096.345	446.539	2308.285	0.622	1.259	0.549	0.450	0.690	0.563	0.186	0.595	0.056	0.088	0.032	0.381	0.161	0.364	-2590.525	-3144.710	-2016.644	0.003
1000	0.5	4.027	3.938	2.915	1.799	5.929	5.305	16.801	1.092	0.868	1.355	1325.966	1074.296	1073.247	476.881	2119.168	0.618	2.106	0.534	0.453	0.642	0.563	0.180	0.600	0.054	0.087	0.032	0.383	0.153	0.365	-5182.771	-6255.957	-4104.688	0.002

Table D.68: loglogistic-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	37.725	1213.595	3.645	1.743	20.105	4.221	14.767	1.220	0.759	2.309	16319.426	36131.094	6987.392	1344.512	27901.308	1.054	3.111	0.546	0.381	1.039	0.536	0.231	0.547	0.045	0.070	0.019	0.419	0.226	0.402	-318.126	-426.628	-181.962	0.022
100	0.5	14.229	79.250	3.297	1.798	13.662	3.318	12.548	1.148	0.752	1.861	12254.516	19195.328	6567.482	1533.322	21032.287	0.682	1.516	0.485	0.363	0.738	0.535	0.220	0.548	0.039	0.062	0.019	0.426	0.211	0.409	-637.159	-841.663	-398.583	0.010
200	0.5	9.967	97.151	3.060	1.872	10.670	2.755	10.217	1.113	0.764	1.686	9937.559	12545.222	6246.556	1798.467	17215.146	0.543	0.994	0.448	0.354	0.616	0.534	0.209	0.554	0.034	0.052	0.018	0.431	0.199	0.410	-1273.514	-1667.431	-857.843	0.006
500	0.5	5.586	10.818	2.829	1.941	8.015	2.313	8.133	1.115	0.802	1.562	7834.683	7734.570	5502.471	1961.991	13648.476	0.452	0.448	0.416	0.350	0.524	0.538	0.195	0.559	0.029	0.046	0.016	0.432	0.183	0.414	-3144.699	-4076.944	-2192.761	0.004
1000	0.5	4.158	3.872	2.647	1.933	6.506	1.966	6.669	1.161	0.827	1.507	6428.991	5058.338	4719.373	2009.318	11816.418	0.410	0.071	0.400	0.345	0.476	0.549	0.185	0.586	0.026	0.042	0.012	0.425	0.173	0.394	-6142.483	-8011.770	-4505.403	0.000

Table D.69: loglogistic-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) RRLN data.

n	θ	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	83.555	7029.714	6.480	2.178	21.075	4.444	18.547	1.037	0.753	1.863	6278.182	9973.867	3688.440	1080.368	11203.075	0.979	2.413	0.657	0.456	1.140	0.487	0.205	0.495	0.058	0.086	0.031	0.455	0.204	0.447	-323.921	-393.844	-247.870	0.021
100	0.5	20.070	960.765	6.098	2.344	16.341	3.856	14.441	0.969	0.751	1.421	5411.661	7653.108	3596.858	1213.175	9317.704	0.799	3.413	0.598	0.448	0.868	0.487	0.188	0.497	0.056	0.082	0.033	0.458	0.184	0.450	-650.511	-777.584	-522.479	0.012
200	0.5	10.074	38.627	5.885	2.601	13.658	3.311	14.027	0.930	0.757	1.206	4743.611	4622.791	3491.398	1393.528	8061.891	0.667	1.400	0.565	0.446	0.748	0.489	0.170	0.497	0.053	0.074	0.034	0.458	0.163	0.452	-1305.214	-1534.408	-1080.335	0.007
500	0.5	7.773	12.667	5.539	2.909	11.536	2.713	10.901	0.906	0.771	1.101	4268.213	3229.839	3440.290	1569.431	7054.004	0.678	4.508	0.538	0.447	0.659	0.488	0.156	0.499	0.050	0.066	0.033	0.462	0.145	0.455	-3271.936	-3795.153	-2781.504	0.005
1000	0.5	6.763	5.247	5.388	3.045	10.565	2.724	13.934	0.899	0.785	1.050	4035.540	2690.297	3390.861	1632.514	6627.896	0.570	1.231	0.522	0.448	0.618	0.489	0.148	0.501	0.047	0.061	0.032	0.464	0.134	0.457	-6552.180	-7495.297	-5648.433	0.006

Table D.70: loglogistic-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) VOD data.

n	θ	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	14.216	1625.280	2.814	1.064	9.202	11.792	29.812	1.331	0.887	4.494	1440.440	2225.440	774.036	189.090	2785.991	1.079	3.140	0.639	0.455	1.296	0.504	0.217	0.521	0.084	0.120	0.041	0.412	0.208	0.398	-247.606	-318.902	-169.680	0.028
100	0.5	5.424	128.837	2.629	1.000	6.805	12.446	30.580	1.248	0.885	3.649	1210.231	1545.268	724.980	196.733	2210.061	0.816	3.313	0.574	0.441	0.905	0.496	0.208	0.519	0.083	0.119	0.043	0.421	0.191	0.410	-496.114	-628.442	-366.177	0.012
200	0.5	4.073	14.935	2.492	1.000	5.631	13.315	33.036	1.214	0.901	7.819	1040.987	1162.624	685.156	204.161	1859.862	0.691	2.662	0.540	0.435	0.740	0.486	0.201	0.516	0.081	0.115	0.044	0.432	0.175	0.422	-994.978	-1239.668	-765.616	0.007
500	0.5	3.385	4.609	2.359	1.000	4.760	14.326	30.880	1.199	0.931	68.016	904.921	895.897	641.228	210.670	1577.225	0.626	2.696	0.514	0.433	0.636	0.475	0.195	0.514	0.080	0.113	0.044	0.445	0.160	0.432	-2492.535	-3055.201	-1982.521	0.004
1000	0.5	3.131	3.568	2.291	1.000	4.363	14.343	28.946	1.191	0.953	67.566	838.403	771.224	627.368	217.281	1445.106	0.578	1.978	0.503	0.432	0.595	0.471	0.193	0.514	0.078	0.111	0.044	0.452	0.152	0.437	-4991.315	-6046.171	-4031.093	0.004

Table D.71: loglogistic-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) RIOTINTO data.

n	θ	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	33.354	102.060	7.564	2.315	40.599	3.558	12.817	0.992	0.673	1.982	31384.647	62009.932	14639.199	4346.404	52154.881	0.799	2.524	0.526	0.385	0.883	0.428	0.204	0.417	0.044	0.072	0.020	0.528	0.205	0.539	-394.119	-460.380	-322.105	0.020
100	0.5	24.424	98.745	6.961	2.494	28.221	3.045	11.583	0.927	0.672	1.434	24395.552	40824.624	13987.172	4643.644	38561.289	0.690	2.552	0.481	0.377	0.688	0.421	0.184	0.413	0.041	0.070	0.022	0.538	0.184	0.549	-790.999	-907.642	-670.316	0.013
200	0.5	16.054	46.501	6.558	2.705	19.189	2.626	9.441	0.887	0.686	1.225	19848.540	26152.503	13197.065	4935.688	30364.470	0.555	1.202	0.456	0.374	0.594	0.414	0.166	0.412	0.039	0.061	0.023	0.548	0.162	0.556	-1584.243	-1778.885	-1386.663	0.006
500	0.5	9.176	13.895	5.942	2.789	11.901	2.232	8.469	0.879	0.730	1.090	14948.981	13139.442	11821.514	5592.793	23520.952	0.456	0.324	0.428	0.374	0.514	0.403	0.139	0.402	0.033	0.053	0.022	0.565	0.134	0.568	-3967.266	-4359.154	-3581.734	0.007
1000	0.5	7.042	5.874	5.554	3.036	10.168	0.900	0.196	0.878	0.740	1.044	12180.978	7219.935	11071.854	5876.542	18114.134	0.424	0.057	0.414	0.375	0.484	0.399	0.116	0.397	0.027	0.026	0.021	0.574	0.111	0.574	-7923.987	-8531.048	-7274.818	0.013

Table D.72: loglogistic-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) SSELN data.

n	θ	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	37.456	170.883	10.710	3.491	41.586	1.189	4.255	0.822	0.578	1.500	22531.376	36234.426	12416.028	3900.702	39371.509	0.654	2.801	0.466	0.356	0.723	0.366	0.185	0.341	0.043	0.078	0.015	0.591	0.191	0.616	-398.084	-470.009	-319.120	0.023
100	0.5	31.357	132.870	10.274	3.707	33.913	0.944	1.250	0.746	0.561	1.114	19549.357	28098.375	11524.674	4229.734	32584.652	0.526	0.801	0.436	0.352	0.606	0.362	0.169	0.341	0.039	0.074	0.016	0.599	0.172	0.620	-799.328	-927.144	-667.527	0.014
200	0.5	28.042	152.694	9.527	3.812	26.761	0.810	0.545	0.714	0.562	0.959	16315.204	18746.837	10808.968	4673.673	27321.390	0.444	0.123	0.415	0.347	0.535	0.354	0.152	0.338	0.037	0.071	0.016	0.609	0.152	0.625	-1602.220	-1827.900	-1392.339	0.007
500	0.5	12.994	14.099	9.162	3.782	19.774	0.740	0.259	0.687	0.570	0.882	12263.596	8361.342	10685.457	4986.535	18554.445	0.409	0.068	0.397	0.348	0.472	0.349	0.125	0.342	0.031	0.052	0.014	0.620	0.121	0.632	-3989.960	-4432.223	-3567.295	0.000
1000	0.5	11.776	18.619	8.373	3.734	16.495	0.732	0.260	0.676	0.572	0.846	9433.567	5961.375	8316.516	4473.396	14330.992	0.407	0.061	0.390	0.357	0.467	0.357	0.123	0.346	0.031	0.043	0.017	0.612	0.112	0.617	-7759.177	-8596.510	-6918.240	0.000

Table D.73: weibull-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) ABFLN data.

n	θ	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	10.458	25.734	5.229	2.364	17.348	1.190	3.360	0.897	0.669	1.592	1553.749	2399.127	958.907	270.353	3166.958	0.904	2.131	0.610	0.448	1.176	0.530	0.203	0.541	0.057	0.094	0.019	0.413	0.199	0.401	-257.645	-327.626	-168.541	0.045
100	0.5	8.825	18.780	4.960	2.456	13.705	1.057	2.685	0.845	0.659	1.265	1350.772	3205.217	893.890	294.853	2556.384	0.691	1.617	0.548	0.432	0.839	0.525	0.189	0.537	0.053	0.089	0.019	0.422	0.182	0.414	-517.973	-645.537	-365.184	0.034
200	0.5	7.619	13.199	4.764	2.573	11.794	0.994	2.459	0.821	0.662	1.105	1204.569	7210.955	829.578	312.868	2145.219	0.594	1.230	0.514	0.425	0.699	0.520	0.177	0.534	0.049	0.086	0.019	0.431	0.166	0.423	-1039.932	-1274.968	-761.290	0.032
500	0.5	6.534	8.331	4.579	2.710	10.148	0.968	3.071	0.809	0.670	0.989	1010.956	1332.997	776.067	333.934	1768.964	0.569	2.350	0.489	0.423	0.609	0.516	0.165	0.534	0.044	0.083	0.017	0.440	0.150	0.433	-2604.041	-3141.768	-1963.244	0.032
1000	0.5	6.023	5.800	4.474	2.822	9.376	0.991	5.866	0.806	0.677	0.940	931.232	739.182	752.943	352.712	1608.174	0.532	1.066	0.478	0.423	0.570	0.514	0.159	0.536	0.042	0.081	0.016	0.444	0.142	0.436	-5208.420	-6250.628	-3999.503	0.032

Table D.74: weibull-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) BARC data.

n	θ	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	17.323	82.499	5.247	2.521	22.152	1.238	2.962	0.903	0.642	1.599	10032.876	17866.131	5137.643	970.391	18843.029	0.757	2.191	0.472	0.354	0.795	0.492	0.235	0.489	0.039	0.070	0.007	0.469	0.229	0.466	-316.268	-425.263	-177.092	0.028
100	0.5	13.550	51.012	4.901	2.616	18.054	1.087	3.939	0.844	0.625	1.313	8442.815	12932.056	4843.883	1104.950	15462.194	0.521	0.673	0.432	0.342	0.621	0.488	0.225	0.490	0.034	0.067	0.006	0.478	0.214	0.471	-636.539	-840.981	-395.044	0.017
200	0.5	11.452	51.821	4.640	2.699	15.756	0.973	2.827	0.822	0.610	1.133	6950.901	7822.916	4579.970	1229.764	12314.709	0.480	1.329	0.407	0.335	0.541	0.488	0.212	0.495	0.029	0.064	0.005	0.483	0.200	0.476	-1274.045	-1669.542	-852.078	0.010
500	0.5	8.214	16.024	4.324	2.849	12.995	0.867	0.819	0.805	0.620	1.039	5736.265	5607.326	3934.938	1364.971	10904.191	0.400	0.081	0.386	0.330	0.472	0.497	0.197	0.505	0.023	0.053	0.001	0.480	0.183	0.471	-3143.562	-4077.829	-2179.314	0.014
1000	0.5	6.549	6.029	4.021	2.924	10.374	0.829	0.220	0.811	0.627	1.001	4779.439	3793.197	3599.873	1454.008	9201.994	0.382	0.060	0.372	0.329	0.434	0.510	0.188	0.546	0.018	0.040	0.000	0.471	0.173	0.447	-6150.884	-8009.741	-4503.035	0.009

Table D.75: weibull-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) RRLN data.

n	θ	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	22.361	63.362	10.106	3.928	31.428	1.098	3.003	0.788	0.589	1.284	4857.208	7101.419	3068.487	969.278	8926.872	0.860	2.473	0.595	0.433	0.985	0.453	0.189	0.448	0.057	0.092	0.028	0.490	0.194	0.494	-323.315	-393.717	-245.749	0.026
100	0.5	18.171	61.802	9.528	4.176	25.093	0.955	3.152	0.740	0.588	1.043	4282.702	10731.083	2870.360	1040.928	7423.674	0.712	2.316	0.542	0.424	0.778	0.444	0.174	0.441	0.055	0.089	0.029	0.501	0.176	0.505	-650.544	-777.464	-519.105	0.020
200	0.5	14.982	49.212	9.063	4.433	20.630	0.940	4.608	0.717	0.591	0.909	3671.425	3791.873	2730.486	1108.722	6241.141	0.603	1.472	0.511	0.418	0.670	0.438	0.160	0.437	0.052	0.086	0.030	0.510	0.159	0.512	-1304.241	-1534.648	-1076.978	0.015
500	0.5	11.323	13.510	8.493	4.852	17.627	0.797	1.176	0.700	0.603	0.832	3201.332	2376.859	2616.722	1209.316	5349.897	0.535	0.880	0.487	0.415	0.593	0.430	0.145	0.432	0.049	0.082	0.030	0.521	0.141	0.521	-3269.552	-3793.652	-2768.074	0.014
1000	0.5	10.140	7.110	8.331	4.860	15.646	0.801	1.900	0.699	0.611	0.801	2973.455	1984.576	2520.068	1228.738	4800.605	0.503	0.472	0.473	0.415	0.553	0.425	0.137	0.429	0.049	0.085	0.028	0.527	0.129	0.527	-6540.154	-7502.485	-5656.477	0.008

Table D.76: weibull-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) VOD data.

n	θ	$\hat{\alpha}_{1\text{ave}}$	$\pm\hat{\alpha}_{1\text{ave}}$	$\hat{\alpha}_{1\text{median}}$	$\hat{\alpha}_{1\text{lower}}$	$\hat{\alpha}_{1\text{upper}}$	$\hat{\beta}_{1\text{ave}}$	$\pm\hat{\beta}_{1\text{ave}}$	$\hat{\beta}_{1\text{median}}$	$\hat{\beta}_{1\text{lower}}$	$\hat{\beta}_{1\text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	8.353	26.132	4.348	1.944	16.659	1.326	3.982	0.945	0.692	2.047	1115.604	2198.439	656.429	205.590	2585.272	0.884	2.182	0.591	0.449	1.232	0.489	0.198	0.491	0.071	0.114	0.026	0.440	0.198	0.436	-251.089	-318.493	-157.935	0.071
100	0.5	7.246	23.013	4.093	1.966	13.297	1.201	3.748	0.893	0.688	1.675	950.877	1431.071	601.099	211.721	2102.937	0.690	1.727	0.534	0.434	0.870	0.478	0.182	0.484	0.068	0.112	0.027	0.455	0.179	0.453	-505.138	-628.347	-344.138	0.068
200	0.5	6.132	10.470	3.894	2.019	11.601	1.154	4.227	0.869	0.696	1.542	823.740	2305.827	559.560	216.722	1815.659	0.618	1.993	0.504	0.428	0.720	0.468	0.168	0.477	0.064	0.112	0.026	0.468	0.161	0.468	-1014.146	-1239.285	-715.190	0.076
500	0.5	5.256	7.138	3.738	2.144	11.099	1.170	4.367	0.856	0.717	1.639	706.848	793.152	527.762	224.492	1634.782	0.577	1.965	0.482	0.423	0.635	0.461	0.156	0.470	0.058	0.109	0.023	0.481	0.143	0.480	-2543.430	-3054.195	-1801.450	0.094
1000	0.5	4.864	5.587	3.647	2.196	10.981	1.222	6.104	0.855	0.735	2.384	651.093	549.178	511.383	229.123	1548.253	0.571	1.796	0.472	0.420	0.599	0.458	0.151	0.467	0.055	0.107	0.022	0.487	0.135	0.487	-5089.339	-6039.186	-3604.386	0.107

Table D.77: weibull-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) RIOTINTO data.

n	θ	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.5	37.548	140.922	10.983	3.921	39.568	1.183	3.068	0.790	0.573	1.358	19125.130	28175.463	11367.484	3584.914	32832.545	0.602	1.108	0.470	0.361	0.714	0.371	0.180	0.353	0.044	0.073	0.019	0.585	0.187	0.604	-393.053	-460.461	-320.317	0.021
100	0.5	26.456	77.163	10.241	4.047	31.347	1.029	4.424	0.726	0.564	1.088	16178.175	23559.545	10803.662	3929.354	27046.812	0.636	4.167	0.440	0.357	0.598	0.363	0.163	0.349	0.041	0.074	0.020	0.596	0.168	0.614	-789.293	-907.080	-667.805	0.016
200	0.5	23.373	89.933	9.811	4.279	24.866	0.854	2.127	0.695	0.566	0.934	13263.258	13385.194	10016.436	4076.815	21891.653	0.459	0.479	0.418	0.352	0.531	0.352	0.143	0.340	0.038	0.066	0.021	0.610	0.145	0.622	-1583.318	-1779.239	-1381.509	0.012
500	0.5	13.579	26.371	9.400	4.779	17.916	0.737	0.293	0.683	0.582	0.850	10849.001	9197.502	9083.338	4313.537	16955.697	0.411	0.110	0.396	0.348	0.464	0.339	0.120	0.337	0.035	0.065	0.020	0.626	0.120	0.634	-3962.557	-4365.087	-3586.569	0.005
1000	0.5	10.735	8.233	8.756	5.295	14.445	0.691	0.176	0.670	0.585	0.768	9177.675	4632.665	8388.619	4619.869	13627.438	0.396	0.050	0.388	0.350	0.442	0.338	0.102	0.331	0.033	0.062	0.019	0.629	0.097	0.639	-7889.491	-8535.907	-7282.199	0.000

Table D.78: weibull-uniform-weibull mixture on mid-price waiting time zero inflated (uniformly distributed) SSELN data.

	AIC			BIC		
exponential/uniform/weibull	0.3988	0.6273	-	0.8420	-	-
gamma/uniform/weibull	0.0148	0.0358	0.1346	0.0011	0.0392	0.1346
loglogistic/uniform/weibull	0.4228	-	-	0.1188	0.5962	-
weibull/uniform/weibull	0.1630	0.3364	0.8637	0.0375	0.3640	0.8637

Table D.79: ABFLN, N = 50: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.3451	0.6437	-	0.8158	-	-
gamma/uniform/weibull	0.0143	0.0597	0.1778	0.0042	0.0396	0.1778
loglogistic/uniform/weibull	0.5120	-	-	0.1544	0.6655	-
weibull/uniform/weibull	0.1270	0.2946	0.8199	0.0240	0.2931	0.8199

Table D.80: BARC, N = 50: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.3097	0.5944	-	0.7611	-	-
gamma/uniform/weibull	0.0152	0.0782	0.1670	0.0049	0.0334	0.1670
loglogistic/uniform/weibull	0.5449	-	-	0.1995	0.6793	-
weibull/uniform/weibull	0.1293	0.3263	0.8304	0.0336	0.2865	0.8304

Table D.81: RRLN, N = 50: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4093	-	-	0.8627	-	-
gamma/uniform/weibull	0.0145	0.0369	0.1302	0.0029	0.0369	0.1302
loglogistic/uniform/weibull	0.4067	0.5763	-	0.1042	0.5763	-
weibull/uniform/weibull	0.1688	0.3861	0.8687	0.0296	0.3861	0.8687

Table D.82: VOD, N = 50: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.3478	0.6689	-	0.7896	-	-
gamma/uniform/weibull	0.0110	0.0577	0.2112	0.0030	0.0360	0.2112
loglogistic/uniform/weibull	0.5228	-	-	0.1842	0.6914	-
weibull/uniform/weibull	0.1153	0.2697	0.7844	0.0202	0.2691	0.7844

Table D.83: RIOTINTO, N = 50: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.4061	0.6266	-	0.8465	-	-
gamma/uniform/weibull	0.0124	0.0356	0.1397	0.0022	0.0381	0.1397
loglogistic/uniform/weibull	0.4213	-	-	0.1169	0.5901	-
weibull/uniform/weibull	0.1598	0.3374	0.8593	0.0340	0.3714	0.8593

Table D.84: SSELN, N = 50: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.2452	0.5072	-	0.7642	-	-
gamma/uniform/weibull	0.0070	0.0289	0.0980	0.0008	0.0199	0.0980
loglogistic/uniform/weibull	0.5884	-	-	0.1964	0.6915	-
weibull/uniform/weibull	0.1589	0.4635	0.9016	0.0383	0.2882	0.9016

Table D.85: ABFLN, N = 100: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.1915	0.5559	-	0.7347	-	-
gamma/uniform/weibull	0.0074	0.0585	0.1549	0.0015	0.0217	0.1549
loglogistic/uniform/weibull	0.6900	-	-	0.2430	0.7744	-
weibull/uniform/weibull	0.1108	0.3852	0.8447	0.0206	0.2036	0.8447

Table D.86: BARC, N = 100: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.1741	0.4903	-	0.6494	-	-
gamma/uniform/weibull	0.0054	0.0802	0.1453	0.0013	0.0177	0.1453
loglogistic/uniform/weibull	0.7090	-	-	0.3212	0.7838	-
weibull/uniform/weibull	0.1113	0.4293	0.8544	0.0280	0.1983	0.8544

Table D.87: RRLN, N = 100: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.2493	0.4838	0.9447	0.7888	-	-
gamma/uniform/weibull	0.0070	0.0285	0.0553	0.0013	0.0212	0.0989
loglogistic/uniform/weibull	0.5656	-	-	0.1765	0.6618	-
weibull/uniform/weibull	0.1781	0.4877	-	0.0334	0.3170	0.9009

Table D.88: VOD, N = 100: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.2018	0.6119	-	0.7266	-	-
gamma/uniform/weibull	0.0065	0.0625	0.2099	0.0012	0.0216	0.2099
loglogistic/uniform/weibull	0.6986	-	-	0.2569	0.7962	-
weibull/uniform/weibull	0.0923	0.3245	0.7886	0.0146	0.1811	0.7886

Table D.89: RIOTINTO, N = 100: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.2485	0.5036	-	0.7754	-	-
gamma/uniform/weibull	0.0084	0.0257	0.1066	0.0006	0.0249	0.1066
loglogistic/uniform/weibull	0.5862	-	-	0.1883	0.6847	-
weibull/uniform/weibull	0.1569	0.4707	0.8934	0.0356	0.2904	0.8934

Table D.90: SSELN, N = 100: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.1115	0.3770	0.9402	0.5926	-	-
gamma/uniform/weibull	0.0025	0.0189	0.0590	0.0008	0.0139	0.0730
loglogistic/uniform/weibull	0.7402	-	-	0.3500	0.7795	-
weibull/uniform/weibull	0.1451	0.6033	-	0.0557	0.2057	0.9262

Table D.91: ABFLN, N = 200: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0776	0.4311	0.9034	0.5237	-	-
gamma/uniform/weibull	0.0051	0.0579	0.0964	0.0013	0.0138	0.1342
loglogistic/uniform/weibull	0.8260	-	-	0.4519	0.8569	-
weibull/uniform/weibull	0.0912	0.5109	-	0.0230	0.1293	0.8656

Table D.92: BARC, N = 200: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0740	0.3874	0.8868	0.4135	0.8162	-
gamma/uniform/weibull	0.0027	0.0772	0.1132	0.0006	0.0302	0.1219
loglogistic/uniform/weibull	0.8344	-	-	0.5563	-	-
weibull/uniform/weibull	0.0889	0.5353	-	0.0296	0.1536	0.8781

Table D.93: RRLN, N = 200: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.1164	0.3165	0.9483	0.5959	-	-
gamma/uniform/weibull	0.0043	0.0225	0.0517	0.0008	0.0124	0.0717
loglogistic/uniform/weibull	0.7027	-	-	0.3502	0.7398	-
weibull/uniform/weibull	0.1766	0.6610	-	0.0532	0.2478	0.9283

Table D.94: VOD, N = 200: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0890	0.5199	-	0.5514	-	-
gamma/uniform/weibull	0.0034	0.0752	0.2119	0.0008	0.0131	0.2119
loglogistic/uniform/weibull	0.8438	-	-	0.4352	0.8822	-
weibull/uniform/weibull	0.0636	0.4047	0.7876	0.0124	0.1044	0.7876

Table D.95: RIOTINTO, N = 200: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.1029	0.3414	0.9504	0.5905	-	-
gamma/uniform/weibull	0.0043	0.0204	0.0496	0.0006	0.0161	0.0843
loglogistic/uniform/weibull	0.7515	-	-	0.3668	0.7912	-
weibull/uniform/weibull	0.1413	0.6382	-	0.0421	0.1927	0.9157

Table D.96: SSELN, N = 200: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0239	0.1914	0.9354	0.2416	0.5789	-
gamma/uniform/weibull	0.0048	0.0287	0.0646	0.0000	0.0024	0.0359
loglogistic/uniform/weibull	0.9019	-	-	0.7201	-	-
weibull/uniform/weibull	0.0694	0.7799	-	0.0383	0.4187	0.9641

Table D.97: ABFLN, N = 500: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0188	0.2424	0.9015	0.1702	0.7634	-
gamma/uniform/weibull	0.0027	0.0543	0.0985	0.0012	0.0068	0.1054
loglogistic/uniform/weibull	0.9248	-	-	0.8029	-	-
weibull/uniform/weibull	0.0536	0.7033	-	0.0256	0.2298	0.8946

Table D.98: BARC, N = 500: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0175	0.2432	0.9149	0.1312	0.6844	-
gamma/uniform/weibull	0.0008	0.0580	0.0851	0.0000	0.0199	0.0914
loglogistic/uniform/weibull	0.9436	-	-	0.8458	-	-
weibull/uniform/weibull	0.0382	0.6987	-	0.0231	0.2957	0.9086

Table D.99: RRLN, N = 500: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0293	0.1203	0.9401	0.2359	0.5425	-
gamma/uniform/weibull	0.0018	0.0261	0.0597	0.0003	0.0023	0.0589
loglogistic/uniform/weibull	0.8423	-	-	0.6947	-	-
weibull/uniform/weibull	0.1266	0.8533	-	0.0692	0.4550	0.9408

Table D.100: VOD, N = 500: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0225	0.3616	0.8164	0.1937	0.8642	-
gamma/uniform/weibull	0.0017	0.1109	0.1835	0.0005	0.0064	0.2132
loglogistic/uniform/weibull	0.9414	-	-	0.7933	-	-
weibull/uniform/weibull	0.0344	0.5275	-	0.0125	0.1294	0.7867

Table D.101: RIOTINTO, N = 500: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0262	0.1731	0.9336	0.2150	0.5612	-
gamma/uniform/weibull	0.0017	0.0227	0.0664	0.0000	0.0000	0.0559
loglogistic/uniform/weibull	0.8846	-	-	0.7378	-	-
weibull/uniform/weibull	0.0874	0.8042	-	0.0472	0.4388	0.9441

Table D.102: SSELN, N = 500: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0000	0.0677	0.9248	0.0526	0.4060	0.9850
gamma/uniform/weibull	0.0000	0.0526	0.0752	0.0000	0.0150	0.0150
loglogistic/uniform/weibull	0.9549	-	-	0.9173	-	-
weibull/uniform/weibull	0.0451	0.8797	-	0.0301	0.5789	-

Table D.103: ABFLN, N = 1000: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0071	0.1374	0.8960	0.0521	0.5598	-
gamma/uniform/weibull	0.0035	0.0610	0.1040	0.0007	0.0057	0.0984
loglogistic/uniform/weibull	0.9582	-	-	0.9248	-	-
weibull/uniform/weibull	0.0312	0.8016	-	0.0223	0.4345	0.9016

Table D.104: BARC, N = 1000: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0000	0.1800	0.9198	0.0285	0.5348	-
gamma/uniform/weibull	0.0000	0.0428	0.0802	0.0000	0.0178	0.0588
loglogistic/uniform/weibull	0.9875	-	-	0.9608	-	-
weibull/uniform/weibull	0.0125	0.7772	-	0.0107	0.4474	0.9412

Table D.105: RRLN, N = 1000: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0140	0.0518	0.9249	0.0792	0.2857	0.9974
gamma/uniform/weibull	0.0026	0.0290	0.0751	0.0000	0.0010	0.0026
loglogistic/uniform/weibull	0.9017	-	-	0.8608	-	-
weibull/uniform/weibull	0.0818	0.9193	-	0.0600	0.7133	-

Table D.106: VOD, N = 1000: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0080	0.2324	0.7645	0.0605	0.7600	-
gamma/uniform/weibull	0.0009	0.1489	0.2353	0.0003	0.0097	0.2181
loglogistic/uniform/weibull	0.9697	-	-	0.9286	-	-
weibull/uniform/weibull	0.0215	0.6186	-	0.0106	0.2302	0.7818

Table D.107: RIOTINTO, N = 1000: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

	AIC			BIC		
exponential/uniform/weibull	0.0045	0.0717	0.9462	0.0628	0.3229	0.9865
gamma/uniform/weibull	0.0000	0.0314	0.0538	0.0000	0.0090	0.0135
loglogistic/uniform/weibull	0.9552	-	-	0.9148	-	-
weibull/uniform/weibull	0.0404	0.8969	-	0.0224	0.6682	-

Table D.108: SSELN, N = 1000: mid-price waiting times, zero inflated (uniformly distributed, $\theta = 0.5$).

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	- - - - -	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.290	0.775	0.137	0.040	0.458	- - - - -	24.868	152.040	10.676	0.638	26.337	14011.191	17010.161	8603.004	2610.908	24542.725	0.535	1.430	0.414	0.324	0.577	0.275	0.157	0.248	0.065	0.081	0.038	0.660	0.177	0.688	-396.315	-470.625	-324.268	0.004
100	0.267	0.565	0.147	0.050	0.424	- - - - -	21.640	85.142	11.975	5.695	25.567	11148.819	11335.256	7656.399	2717.470	19373.555	0.429	0.337	0.384	0.312	0.498	0.263	0.141	0.237	0.057	0.073	0.034	0.680	0.158	0.704	-796.697	-927.588	-669.933	0.002
200	0.250	0.477	0.157	0.060	0.390	- - - - -	18.388	42.538	11.835	8.176	20.294	9219.251	7835.533	6799.166	3018.231	15407.749	0.381	0.104	0.361	0.307	0.447	0.252	0.125	0.231	0.052	0.069	0.031	0.697	0.137	0.718	-1597.560	-1827.049	-1389.571	0.002
500	0.212	0.156	0.165	0.088	0.352	- - - - -	14.050	7.653	12.253	9.585	17.732	7218.981	4646.949	5979.830	3259.294	11664.188	0.354	0.054	0.342	0.305	0.403	0.248	0.105	0.232	0.041	0.053	0.025	0.711	0.109	0.728	-3979.417	-4422.309	-3556.848	0.000
1000	0.234	0.144	0.202	0.104	0.380	- - - - -	13.308	3.994	12.517	10.255	16.169	5222.053	2732.090	4597.478	2768.898	7787.170	0.350	0.048	0.346	0.303	0.393	0.249	0.094	0.239	0.041	0.042	0.031	0.710	0.091	0.717	-7735.271	-8576.802	-6898.448	0.000

Table D.109: exponential-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) ABFLN data.

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	- - - - -	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.374	0.857	0.219	0.091	0.588	- - - - -	33.407	1018.355	12.684	5.054	31.032	1131.429	1453.955	685.247	175.769	2034.953	0.744	1.901	0.536	0.409	0.834	0.446	0.201	0.438	0.086	0.099	0.054	0.468	0.207	0.466	-253.907	-327.764	-175.552	0.008
100	0.364	0.701	0.230	0.112	0.532	- - - - -	27.337	272.177	13.516	8.794	27.032	902.136	1035.732	591.019	184.072	1557.824	0.567	1.163	0.477	0.391	0.637	0.432	0.190	0.429	0.077	0.083	0.053	0.491	0.193	0.491	-511.374	-645.066	-374.990	0.001
200	0.360	0.671	0.239	0.131	0.493	- - - - -	22.030	95.681	13.693	9.971	22.629	751.406	767.850	531.656	198.212	1262.748	0.485	0.496	0.446	0.384	0.545	0.420	0.181	0.425	0.070	0.071	0.052	0.510	0.181	0.507	-1028.853	-1273.985	-781.586	0.000
500	0.346	0.547	0.245	0.146	0.454	- - - - -	16.580	22.836	13.719	10.874	19.336	645.935	582.543	491.908	211.364	1047.395	0.448	0.336	0.427	0.382	0.490	0.410	0.171	0.424	0.066	0.061	0.053	0.523	0.170	0.518	-2584.132	-3139.807	-2018.869	0.001
1000	0.339	0.501	0.248	0.153	0.433	- - - - -	14.913	6.817	13.770	11.153	17.771	601.781	484.002	484.648	221.318	969.823	0.430	0.111	0.419	0.381	0.469	0.407	0.166	0.425	0.065	0.057	0.053	0.529	0.163	0.519	-5174.694	-6244.863	-4098.721	0.001

Table D.110: exponential-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) BARC data.

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	- - - - -	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.339	0.796	0.224	0.072	0.503	- - - - -	29.521	131.787	12.393	1.261	32.751	6969.585	10344.805	3855.346	778.763	12538.757	0.724	2.225	0.431	0.328	0.655	0.420	0.243	0.379	0.064	0.082	0.037	0.516	0.244	0.536	-315.844	-426.591	-184.587	0.007
100	0.314	0.606	0.233	0.085	0.442	- - - - -	33.293	466.460	13.466	8.096	30.294	5446.029	6762.031	3385.740	846.524	9650.995	0.497	0.972	0.390	0.316	0.525	0.414	0.235	0.375	0.055	0.068	0.035	0.531	0.233	0.554	-633.910	-841.053	-402.377	0.001
200	0.296	0.474	0.243	0.105	0.402	- - - - -	23.129	57.461	14.038	9.508	25.790	4374.030	4453.874	2997.137	927.122	7665.338	0.405	0.341	0.362	0.309	0.455	0.410	0.226	0.372	0.048	0.055	0.034	0.542	0.222	0.567	-1268.529	-1665.946	-861.479	0.000
500	0.280	0.291	0.255	0.128	0.371	- - - - -	19.275	60.975	14.250	10.414	21.754	3522.096	3049.337	2663.513	1001.681	6040.397	0.355	0.064	0.344	0.305	0.399	0.413	0.215	0.382	0.041	0.051	0.032	0.545	0.209	0.567	-3138.291	-4073.943	-2196.774	0.000
1000	0.271	0.135	0.270	0.140	0.347	- - - - -	16.598	9.037	14.677	11.102	20.404	2984.215	2172.220	2453.727	1017.524	5005.299	0.342	0.046	0.333	0.306	0.377	0.427	0.208	0.449	0.038	0.040	0.030	0.535	0.202	0.510	-6143.294	-8003.028	-4515.569	0.000

Table D.111: exponential-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) RRLN data.

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	- - - - -	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.292	1.014	0.125	0.046	0.411	- - - - -	32.552	1360.540	11.071	1.338	25.995	3239.252	3798.197	2077.835	579.987	5835.104	0.659	1.964	0.506	0.381	0.745	0.346	0.171	0.327	0.086	0.095	0.057	0.568	0.188	0.582	-320.404	-393.677	-247.657	0.008
100	0.279	0.768	0.133	0.055	0.382	- - - - -	19.799	61.440	11.819	7.075	22.857	2649.919	2752.087	1814.636	598.756	4606.246	0.548	1.493	0.456	0.366	0.616	0.332	0.159	0.315	0.078	0.085	0.055	0.589	0.171	0.603	-644.391	-776.568	-520.642	0.003
200	0.266	0.668	0.140	0.067	0.353	- - - - -	16.836	35.850	11.990	8.550	19.484	2265.264	2119.671	1656.089	614.073	3881.789	0.483	0.713	0.426	0.359	0.542	0.323	0.150	0.308	0.071	0.075	0.051	0.606	0.158	0.618	-1295.010	-1532.253	-1075.159	0.002
500	0.250	0.537	0.148	0.079	0.327	- - - - -	14.102	15.144	11.938	9.627	16.289	1944.275	1566.224	1533.763	649.737	3201.026	0.437	0.606	0.405	0.356	0.485	0.315	0.140	0.303	0.064	0.065	0.049	0.622	0.142	0.633	-3250.143	-3789.826	-2763.802	0.001
1000	0.231	0.407	0.151	0.086	0.310	- - - - -	13.403	13.887	11.896	10.062	15.261	1791.905	1297.153	1473.109	668.215	2865.947	0.412	0.152	0.397	0.354	0.457	0.310	0.131	0.300	0.061	0.062	0.048	0.629	0.129	0.635	-6516.263	-7482.733	-5622.996	0.002

Table D.112: exponential-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) VOD data.

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	- - - - -	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.477	0.971	0.275	0.112	0.775	- - - - -	29.733	317.186	12.780	6.461	30.489	821.569	1067.204	480.537	135.317	1482.393	0.723	2.130	0.535	0.415	0.801	0.414	0.188	0.401	0.094	0.109	0.058	0.492	0.199	0.498	-245.950	-319.079	-174.401	0.008
100	0.470	0.865	0.290	0.137	0.732	- - - - -	30.070	1129.588	13.541	9.056	26.531	656.497	747.439	420.776	138.057	1148.198	0.582	2.151	0.481	0.400	0.627	0.396	0.174	0.383	0.087	0.098	0.057	0.517	0.182	0.525	-495.183	-628.405	-373.384	0.002
200	0.466	0.770	0.302	0.159	0.711	- - - - -	21.479	74.266	13.720	10.211	22.425	553.448	554.140	389.653	143.120	945.134	0.503	0.885	0.454	0.393	0.546	0.383	0.162	0.370	0.081	0.089	0.056	0.536	0.167	0.546	-995.613	-1240.014	-779.743	0.001
500	0.461	0.699	0.312	0.179	0.684	- - - - -	16.763	21.565	13.760	11.024	19.272	481.763	408.365	369.333	148.132	814.202	0.466	0.853	0.436	0.390	0.498	0.372	0.152	0.360	0.078	0.080	0.056	0.551	0.152	0.557	-2500.067	-3054.550	-2021.185	0.001
1000	0.449	0.639	0.316	0.189	0.656	- - - - -	15.071	7.259	13.787	11.426	17.952	452.672	346.607	364.495	151.913	757.080	0.442	0.163	0.429	0.389	0.480	0.368	0.146	0.355	0.076	0.076	0.056	0.556	0.144	0.562	-5012.025	-6049.634	-4103.256	0.002

Table D.113: exponential-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) RIOTINTO data.

n	$\hat\lambda_{1\text{ ave}}$	$\pm\hat\lambda_{1\text{ ave}}$	$\hat\lambda_{1\text{ median}}$	$\hat\lambda_{1\text{ lower}}$	$\hat\lambda_{1\text{ upper}}$	-	-	-	-	-	$\hat\lambda_{2\text{ ave}}$	$\pm\hat\lambda_{2\text{ ave}}$	$\hat\lambda_{2\text{ median}}$	$\hat\lambda_{2\text{ lower}}$	$\hat\lambda_{2\text{ upper}}$	$\hat\alpha_{\text{ave}}$	$\pm\hat\alpha_{\text{ave}}$	$\hat\alpha_{\text{median}}$	$\hat\alpha_{\text{lower}}$	$\hat\alpha_{\text{upper}}$	$\hat\beta_{\text{ave}}$	$\pm\hat\beta_{\text{ave}}$	$\hat\beta_{\text{median}}$	$\hat\beta_{\text{lower}}$	$\hat\beta_{\text{upper}}$	$\hat\pi_{1\text{ ave}}$	$\pm\hat\pi_{1\text{ ave}}$	$\hat\pi_{1\text{ median}}$	$\hat\pi_{2\text{ ave}}$	$\pm\hat\pi_{2\text{ ave}}$	$\hat\pi_{2\text{ median}}$	$\hat\pi_{3\text{ ave}}$	$\pm\hat\pi_{3\text{ ave}}$	$\hat\pi_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.265	0.674	0.126	0.040	0.417	-	-	-	-	-	37.541	1199.067	10.683	0.885	27.123	12034.638	13767.202	7943.435	2380.499	20865.105	0.487	0.668	0.416	0.328	0.574	0.276	0.149	0.255	0.067	0.079	0.043	0.656	0.170	0.677	-391.065	-460.615	-323.018	0.004
100	0.267	0.917	0.131	0.051	0.387	-	-	-	-	-	23.994	284.956	11.618	5.595	23.160	9656.511	9586.500	6933.334	2473.442	16703.990	0.428	0.398	0.386	0.318	0.494	0.260	0.133	0.241	0.062	0.074	0.040	0.678	0.150	0.694	-785.658	-907.103	-671.285	0.000
200	0.251	0.538	0.139	0.064	0.391	-	-	-	-	-	16.158	21.785	12.052	8.345	19.338	7944.146	6532.338	6256.190	2544.298	13007.778	0.386	0.152	0.364	0.314	0.443	0.250	0.117	0.235	0.053	0.063	0.038	0.697	0.130	0.714	-1577.781	-1777.968	-1386.439	0.001
500	0.232	0.450	0.150	0.084	0.351	-	-	-	-	-	13.272	6.132	11.827	9.507	15.992	6545.601	4109.175	5710.652	2650.989	10480.985	0.356	0.061	0.347	0.308	0.401	0.242	0.099	0.235	0.045	0.056	0.033	0.713	0.105	0.729	-3948.868	-4358.606	-3581.140	0.000
1000	0.204	0.159	0.157	0.101	0.319	-	-	-	-	-	12.946	4.065	12.135	9.923	15.363	5581.280	2778.356	5255.566	2880.759	8569.175	0.345	0.040	0.340	0.306	0.380	0.236	0.087	0.227	0.042	0.059	0.033	0.722	0.091	0.736	-7864.446	-8508.393	-7240.238	0.000

Table D.114: exponential-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) SSELN data.

n	$\hat k_{\text{ave}}$	$\pm\hat k_{\text{ave}}$	$\hat k_{\text{median}}$	$\hat k_{\text{lower}}$	$\hat k_{\text{upper}}$	$\hat\theta_{\text{ave}}$	$\pm\hat\theta_{\text{ave}}$	$\hat\theta_{\text{median}}$	$\hat\theta_{\text{lower}}$	$\hat\theta_{\text{upper}}$	$\hat\lambda_{2\text{ ave}}$	$\pm\hat\lambda_{2\text{ ave}}$	$\hat\lambda_{2\text{ median}}$	$\hat\lambda_{2\text{ lower}}$	$\hat\lambda_{2\text{ upper}}$	$\hat\alpha_{\text{ave}}$	$\pm\hat\alpha_{\text{ave}}$	$\hat\alpha_{\text{median}}$	$\hat\alpha_{\text{lower}}$	$\hat\alpha_{\text{upper}}$	$\hat\beta_{\text{ave}}$	$\pm\hat\beta_{\text{ave}}$	$\hat\beta_{\text{median}}$	$\hat\beta_{\text{lower}}$	$\hat\beta_{\text{upper}}$	$\hat\pi_{1\text{ ave}}$	$\pm\hat\pi_{1\text{ ave}}$	$\hat\pi_{1\text{ median}}$	$\hat\pi_{2\text{ ave}}$	$\pm\hat\pi_{2\text{ ave}}$	$\hat\pi_{2\text{ median}}$	$\hat\pi_{3\text{ ave}}$	$\pm\hat\pi_{3\text{ ave}}$	$\hat\pi_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.623	4.513	2.832	1.562	8.644	0.968	0.122	0.974	0.868	1.088	21.958	151.197	9.785	0.414	24.142	8555.731	11127.410	5099.909	1353.424	15569.717	0.438	1.212	0.352	0.289	0.482	0.152	0.120	0.124	0.087	0.118	0.044	0.761	0.166	0.804	-395.682	-469.611	-319.075	0.017
100	4.402	4.290	2.721	1.657	7.658	1.006	0.089	1.004	0.927	1.094	16.906	77.141	10.604	1.046	20.803	6844.878	7301.968	4613.172	1466.490	11890.959	0.367	0.258	0.333	0.283	0.418	0.142	0.107	0.118	0.075	0.106	0.037	0.783	0.148	0.821	-798.305	-929.807	-673.403	0.001
200	4.254	4.407	2.637	1.720	6.973	1.030	0.071	1.032	0.963	1.102	13.782	27.415	10.629	6.564	16.479	5604.197	5150.482	4181.272	1681.760	9133.751	0.335	0.080	0.318	0.278	0.387	0.138	0.097	0.116	0.062	0.092	0.030	0.800	0.126	0.829	-1601.501	-1830.138	-1392.778	0.002
500	4.372	5.319	2.633	1.889	6.441	1.052	0.057	1.051	0.992	1.111	11.281	5.693	10.824	8.261	14.597	4452.820	3071.251	3730.915	1990.934	6889.808	0.317	0.045	0.305	0.277	0.365	0.135	0.079	0.124	0.053	0.080	0.027	0.812	0.105	0.834	-3991.504	-4431.533	-3566.941	0.000
1000	3.844	4.719	2.452	1.873	5.056	1.052	0.052	1.050	0.996	1.109	10.834	3.215	10.906	9.136	13.187	3343.272	2033.081	2833.959	1710.522	4769.295	0.316	0.041	0.310	0.278	0.367	0.145	0.071	0.134	0.049	0.063	0.030	0.805	0.087	0.817	-7756.677	-8604.501	-6956.082	0.000

Table D.115: gamma-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) ABFLN data.

n	$\hat k_{\text{ave}}$	$\pm\hat k_{\text{ave}}$	$\hat k_{\text{median}}$	$\hat k_{\text{lower}}$	$\hat k_{\text{upper}}$	$\hat\theta_{\text{ave}}$	$\pm\hat\theta_{\text{ave}}$	$\hat\theta_{\text{median}}$	$\hat\theta_{\text{lower}}$	$\hat\theta_{\text{upper}}$	$\hat\lambda_{2\text{ ave}}$	$\pm\hat\lambda_{2\text{ ave}}$	$\hat\lambda_{2\text{ median}}$	$\hat\lambda_{2\text{ lower}}$	$\hat\lambda_{2\text{ upper}}$	$\hat\alpha_{\text{ave}}$	$\pm\hat\alpha_{\text{ave}}$	$\hat\alpha_{\text{median}}$	$\hat\alpha_{\text{lower}}$	$\hat\alpha_{\text{upper}}$	$\hat\beta_{\text{ave}}$	$\pm\hat\beta_{\text{ave}}$	$\hat\beta_{\text{median}}$	$\hat\beta_{\text{lower}}$	$\hat\beta_{\text{upper}}$	$\hat\pi_{1\text{ ave}}$	$\pm\hat\pi_{1\text{ ave}}$	$\hat\pi_{1\text{ median}}$	$\hat\pi_{2\text{ ave}}$	$\pm\hat\pi_{2\text{ ave}}$	$\hat\pi_{2\text{ median}}$	$\hat\pi_{3\text{ ave}}$	$\pm\hat\pi_{3\text{ ave}}$	$\hat\pi_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.591	3.482	2.413	1.396	6.299	0.989	0.096	0.985	0.908	1.084	21.208	230.455	11.336	1.607	23.989	615.582	965.189	318.908	81.651	1055.254	0.547	1.101	0.430	0.352	0.586	0.284	0.186	0.244	0.114	0.140	0.065	0.602	0.215	0.636	-253.992	-328.683	-175.592	0.005
100	3.329	3.308	2.343	1.464	4.954	1.011	0.076	1.006	0.943	1.089	16.879	65.775	11.723	7.525	20.203	468.237	632.470	280.804	91.379	774.924	0.449	0.815	0.397	0.343	0.486	0.274	0.171	0.244	0.097	0.122	0.058	0.629	0.194	0.657	-513.277	-647.371	-376.850	0.001
200	3.084	3.090	2.306	1.523	3.569	1.024	0.066	1.019	0.961	1.093	14.034	28.934	11.691	8.682	16.749	387.806	442.892	261.249	99.940	629.164	0.406	0.293	0.381	0.339	0.444	0.270	0.158	0.250	0.083	0.105	0.055	0.647	0.177	0.668	-1033.137	-1278.385	-785.337	0.000
500	2.728	2.381	2.275	1.582	3.064	1.032	0.058	1.030	0.974	1.094	12.227	5.299	11.590	9.633	14.441	336.006	312.566	247.711	111.041	529.576	0.384	0.114	0.370	0.337	0.419	0.270	0.146	0.260	0.071	0.080	0.053	0.659	0.162	0.675	-2594.002	-3149.342	-2025.888	0.000
1000	2.586	2.071	2.270	1.603	2.936	1.035	0.054	1.034	0.978	1.093	11.768	2.454	11.533	10.032	13.436	317.986	270.289	242.536	116.803	496.287	0.376	0.073	0.366	0.336	0.409	0.270	0.139	0.265	0.068	0.072	0.054	0.663	0.154	0.673	-5193.187	-6266.868	-4121.083	0.000

Table D.116: gamma-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) BARC data.

n	$\hat k_{\text{ave}}$	$\pm\hat k_{\text{ave}}$	$\hat k_{\text{median}}$	$\hat k_{\text{lower}}$	$\hat k_{\text{upper}}$	$\hat\theta_{\text{ave}}$	$\pm\hat\theta_{\text{ave}}$	$\hat\theta_{\text{median}}$	$\hat\theta_{\text{lower}}$	$\hat\theta_{\text{upper}}$	$\hat\lambda_{2\text{ ave}}$	$\pm\hat\lambda_{2\text{ ave}}$	$\hat\lambda_{2\text{ median}}$	$\hat\lambda_{2\text{ lower}}$	$\hat\lambda_{2\text{ upper}}$	$\hat\alpha_{\text{ave}}$	$\pm\hat\alpha_{\text{ave}}$	$\hat\alpha_{\text{median}}$	$\hat\alpha_{\text{lower}}$	$\hat\alpha_{\text{upper}}$	$\hat\beta_{\text{ave}}$	$\pm\hat\beta_{\text{ave}}$	$\hat\beta_{\text{median}}$	$\hat\beta_{\text{lower}}$	$\hat\beta_{\text{upper}}$	$\hat\pi_{1\text{ ave}}$	$\pm\hat\pi_{1\text{ ave}}$	$\hat\pi_{1\text{ median}}$	$\hat\pi_{2\text{ ave}}$	$\pm\hat\pi_{2\text{ ave}}$	$\hat\pi_{2\text{ median}}$	$\hat\pi_{3\text{ ave}}$	$\pm\hat\pi_{3\text{ ave}}$	$\hat\pi_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.771	3.833	2.540	1.634	6.296	0.988	0.113	0.988	0.895	1.101	20.926	118.152	10.800	0.876	24.906	3931.270	6364.107	1954.520	287.766	7223.779	0.535	1.586	0.358	0.286	0.502	0.287	0.225	0.208	0.096	0.129	0.052	0.616	0.246	0.675	-314.863	-425.693	-180.286	0.013
100	3.596	3.718	2.477	1.704	4.858	1.018	0.085	1.015	0.942	1.103	16.003	67.002	11.202	6.005	20.016	3004.013	3986.398	1740.082	319.743	5419.152	0.389	0.589	0.332	0.279	0.422	0.283	0.218	0.202	0.082	0.116	0.046	0.636	0.230	0.688	-635.301	-843.702	-402.037	0.001
200	3.339	3.585	2.435	1.822	3.509	1.036	0.067	1.035	0.969	1.105	13.465	16.432	11.421	8.053	16.585	2433.693	2882.076	1535.855	369.388	4372.421	0.342	0.219	0.314	0.275	0.383	0.288	0.209	0.212	0.065	0.089	0.044	0.648	0.214	0.694	-1272.227	-1671.283	-866.200	0.000
500	3.005	3.282	2.405	1.922	3.040	1.046	0.052	1.046	0.991	1.100	11.490	4.104	11.106	8.943	14.189	1993.480	2073.363	1358.018	422.214	3656.613	0.315	0.057	0.302	0.273	0.353	0.297	0.198	0.245	0.055	0.072	0.042	0.647	0.202	0.681	-3147.417	-4079.841	-2200.995	0.001
1000	2.647	2.225	2.401	2.006	2.869	1.052	0.044	1.053	1.006	1.094	11.374	2.231	11.206	9.591	13.159	1660.784	1489.666	1194.180	432.424	2897.776	0.307	0.044	0.296	0.273	0.339	0.317	0.188	0.335	0.046	0.045	0.040	0.637	0.194	0.618	-6163.825	-8011.997	-4530.575	0.000

Table D.117: gamma-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) RRLN data.

n	$\hat k_{\text{ave}}$	$\pm\hat k_{\text{ave}}$	$\hat k_{\text{median}}$	$\hat k_{\text{lower}}$	$\hat k_{\text{upper}}$	$\hat\theta_{\text{ave}}$	$\pm\hat\theta_{\text{ave}}$	$\hat\theta_{\text{median}}$	$\hat\theta_{\text{lower}}$	$\hat\theta_{\text{upper}}$	$\hat\lambda_{2\text{ ave}}$	$\pm\hat\lambda_{2\text{ ave}}$	$\hat\lambda_{2\text{ median}}$	$\hat\lambda_{2\text{ lower}}$	$\hat\lambda_{2\text{ upper}}$	$\hat\alpha_{\text{ave}}$	$\pm\hat\alpha_{\text{ave}}$	$\hat\alpha_{\text{median}}$	$\hat\alpha_{\text{lower}}$	$\hat\alpha_{\text{upper}}$	$\hat\beta_{\text{ave}}$	$\pm\hat\beta_{\text{ave}}$	$\hat\beta_{\text{median}}$	$\hat\beta_{\text{lower}}$	$\hat\beta_{\text{upper}}$	$\hat\pi_{1\text{ ave}}$	$\pm\hat\pi_{1\text{ ave}}$	$\hat\pi_{1\text{ median}}$	$\hat\pi_{2\text{ ave}}$	$\pm\hat\pi_{2\text{ ave}}$	$\hat\pi_{2\text{ median}}$	$\hat\pi_{3\text{ ave}}$	$\pm\hat\pi_{3\text{ ave}}$	$\hat\pi_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.283	4.259	2.579	1.308	8.136	0.975	0.109	0.975	0.886	1.081	29.337	1360.108	10.597	0.970	23.712	1598.734	2138.243	951.573	253.442	2835.888	0.483	1.096	0.395	0.322	0.544	0.177	0.127	0.149	0.097	0.129	0.052	0.726	0.177	0.768	-320.467	-394.117	-247.143	0.008
100	4.267	4.456	2.479	1.392	7.790	1.004	0.083	0.999	0.933	1.086	16.372	52.534	11.162	5.738	20.209	1298.204	1534.805	846.091	273.495	2200.121	0.439	2.732	0.369	0.315	0.470	0.162	0.111	0.138	0.085	0.120	0.045	0.752	0.157	0.792	-647.036	-779.684	-524.829	0.001
200	4.263	4.846	2.418	1.469	7.510	1.022	0.070	1.019	0.959	1.094	13.987	35.798	11.247	7.818	16.991	1115.771	1194.606	765.643	291.066	1856.122	0.387	0.430	0.354	0.311	0.434	0.154	0.099	0.134	0.075	0.109	0.041	0.771	0.139	0.802	-1300.551	-1537.971	-1082.315	0.000
500	4.287	5.228	2.391	1.556	7.127	1.036	0.061	1.035	0.975	1.101	11.873	9.309	11.154	8.957	14.416	965.920	918.229	708.910	306.037	1586.323	0.362	0.106	0.342	0.308	0.407	0.148	0.087	0.134	0.065	0.094	0.039	0.787	0.119	0.808	-3265.459	-3801.934	-2790.664	0.001
1000	4.134	5.077	2.393	1.621	6.761	1.043	0.058	1.046	0.981	1.102	11.357	5.112	11.091	9.303	13.299	875.689	707.276	682.494	327.148	1438.542	0.352	0.072	0.335	0.306	0.395	0.147	0.080	0.135	0.059	0.087	0.036	0.794	0.110	0.812	-6536.114	-7512.919	-5658.114	0.001

Table D.118: gamma-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) VOD data.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	3.079	2.961	2.193	1.220	4.897	0.982	0.092	0.978	0.907	1.072	22.132	296.540	11.575	4.645	24.317	486.035	716.107	266.790	80.632	835.955	0.565	1.633	0.448	0.367	0.601	0.278	0.173	0.249	0.117	0.138	0.070	0.604	0.206	0.633	-245.913	-319.727	-174.353	0.006
100	2.822	2.720	2.123	1.279	3.552	1.002	0.074	0.994	0.940	1.076	17.804	170.832	11.929	8.015	20.167	381.554	467.321	242.117	88.821	637.137	0.480	1.754	0.417	0.359	0.507	0.270	0.156	0.249	0.103	0.122	0.065	0.627	0.185	0.649	-496.145	-630.242	-374.130	0.001
200	2.602	2.374	2.089	1.330	3.086	1.011	0.064	1.003	0.957	1.078	14.404	22.020	11.916	9.075	16.885	324.265	323.158	230.119	96.618	538.240	0.434	0.493	0.401	0.356	0.466	0.265	0.141	0.252	0.093	0.108	0.063	0.642	0.167	0.656	-997.836	-1243.460	-780.928	0.001
500	2.420	2.024	2.069	1.385	2.888	1.017	0.057	1.009	0.966	1.078	12.503	6.022	11.845	9.890	14.624	288.654	232.835	223.360	104.803	470.130	0.408	0.201	0.390	0.354	0.442	0.263	0.127	0.254	0.085	0.091	0.063	0.652	0.150	0.660	-2504.372	-3062.794	-2022.521	0.000
1000	2.341	1.682	2.061	1.424	2.810	1.019	0.055	1.012	0.969	1.077	12.047	2.556	11.750	10.313	13.661	274.789	197.254	222.467	110.741	440.196	0.398	0.105	0.385	0.353	0.432	0.263	0.120	0.255	0.083	0.084	0.064	0.655	0.141	0.661	-5018.263	-6068.223	-4108.746	0.001

Table D.119: gamma-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) RIOTINTO data.

n	$\hat{k}_{ave}$	$\pm\hat{k}_{ave}$	$\hat{k}_{median}$	$\hat{k}_{lower}$	$\hat{k}_{upper}$	$\hat{\theta}_{ave}$	$\pm\hat{\theta}_{ave}$	$\hat{\theta}_{median}$	$\hat{\theta}_{lower}$	$\hat{\theta}_{upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	4.458	4.358	2.755	1.477	8.402	0.970	0.117	0.975	0.872	1.088	18.007	55.904	10.023	0.605	24.641	6998.847	11577.616	4327.788	1238.245	12348.876	0.401	0.561	0.350	0.288	0.470	0.148	0.109	0.125	0.083	0.109	0.044	0.769	0.152	0.805	-390.741	-460.193	-320.035	0.015
100	4.494	4.602	2.679	1.552	7.858	1.005	0.087	1.002	0.930	1.092	20.214	284.167	10.671	1.492	20.003	5581.464	6029.093	3975.173	1308.447	9413.245	0.373	0.505	0.332	0.282	0.414	0.135	0.096	0.115	0.073	0.103	0.038	0.792	0.135	0.826	-788.408	-908.724	-674.240	0.000
200	4.305	4.635	2.580	1.669	7.185	1.028	0.073	1.024	0.960	1.103	13.044	13.185	11.095	7.117	16.685	4660.428	4225.097	3618.118	1463.903	7603.918	0.336	0.118	0.320	0.278	0.380	0.130	0.084	0.114	0.063	0.093	0.035	0.808	0.116	0.831	-1582.189	-1784.746	-1394.897	0.000
500	4.366	5.674	2.568	1.803	6.601	1.046	0.059	1.047	0.987	1.106	11.186	5.007	10.776	8.590	13.886	3823.734	2603.567	3289.236	1608.492	5999.953	0.317	0.054	0.308	0.278	0.353	0.125	0.067	0.117	0.051	0.078	0.029	0.823	0.091	0.840	-3961.756	-4367.387	-3607.338	0.000
1000	3.693	4.463	2.492	1.882	3.797	1.054	0.052	1.054	0.998	1.111	11.114	2.979	10.808	9.022	13.151	3382.580	1885.569	3130.655	1610.163	5370.082	0.310	0.036	0.303	0.278	0.340	0.128	0.056	0.118	0.041	0.071	0.026	0.830	0.082	0.846	-7888.022	-8533.797	-7276.807	0.000

Table D.120: gamma-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) SSELN data.

n	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	59.570	422.027	9.223	2.706	70.203	3.964	12.525	1.126	0.692	2.752	29.230	169.125	10.560	0.643	32.183	40190.161	93842.752	16528.630	4852.219	66380.689	0.750	2.086	0.528	0.384	0.945	0.403	0.203	0.383	0.067	0.077	0.042	0.530	0.206	0.549	-399.337	-468.836	-317.892	0.031
100	35.054	205.478	7.792	2.747	37.558	2.981	12.556	0.979	0.656	1.714	29.822	124.282	12.389	6.799	31.181	33071.072	105411.959	15135.874	5104.352	49594.729	0.582	0.496	0.485	0.376	0.727	0.406	0.186	0.393	0.054	0.066	0.034	0.540	0.188	0.556	-800.832	-924.860	-666.584	0.019
200	19.154	48.475	7.223	2.863	24.090	2.365	8.613	0.911	0.642	1.382	24.692	76.389	12.491	8.622	24.032	24469.451	36073.311	14461.485	5668.310	38724.237	0.529	0.551	0.456	0.362	0.612	0.407	0.168	0.395	0.045	0.055	0.029	0.548	0.166	0.563	-1598.983	-1822.106	-1379.288	0.011
500	9.301	11.087	6.566	2.882	13.267	1.527	5.812	0.881	0.661	1.218	19.353	62.074	13.726	10.716	20.966	16616.847	13063.451	13131.016	6395.748	27047.574	0.445	0.108	0.425	0.362	0.522	0.399	0.134	0.399	0.037	0.036	0.026	0.564	0.132	0.567	-3979.307	-4417.827	-3541.547	0.007
1000	7.091	6.576	5.335	3.004	10.377	2.095	7.658	0.871	0.720	1.178	15.479	6.764	14.472	11.041	18.025	12230.194	8659.712	10328.765	5788.150	18026.710	0.435	0.077	0.416	0.365	0.512	0.400	0.127	0.401	0.041	0.039	0.033	0.558	0.121	0.554	-7710.045	-8559.613	-6870.885	0.000

Table D.121: loglogistic-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) ABFLN data.

n	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	16.679	1517.534	3.693	1.609	12.985	9.385	25.431	1.387	0.911	3.749	30.671	792.873	11.653	6.018	30.501	2012.603	2893.017	1203.321	286.730	3958.136	1.103	2.529	0.680	0.461	1.422	0.518	0.214	0.542	0.095	0.101	0.065	0.388	0.208	0.365	-253.595	-325.728	-167.883	0.034
100	23.162	2612.882	3.497	1.689	9.020	8.678	23.910	1.276	0.864	2.452	23.901	244.704	12.606	8.572	24.274	1738.435	2180.157	1144.371	303.633	3128.261	0.807	2.062	0.605	0.439	0.971	0.523	0.201	0.551	0.086	0.089	0.063	0.390	0.195	0.369	-507.722	-641.615	-364.884	0.012
200	5.788	84.496	3.369	1.790	7.510	7.968	21.857	1.234	0.848	1.970	19.682	156.879	12.996	9.741	20.142	1516.755	1619.710	1078.929	332.029	2630.856	0.658	1.562	0.564	0.430	0.793	0.522	0.188	0.553	0.083	0.082	0.063	0.395	0.182	0.376	-1019.666	-1267.975	-763.392	0.006
500	4.763	14.446	3.297	1.939	6.719	7.175	20.184	1.224	0.844	1.701	15.052	17.145	13.268	10.692	17.513	1325.946	1223.694	1011.806	367.932	2226.557	0.577	0.582	0.533	0.429	0.678	0.519	0.172	0.552	0.082	0.077	0.065	0.399	0.163	0.380	-2559.804	-3123.601	-1979.192	0.003
1000	4.274	4.128	3.220	2.020	6.281	6.684	18.833	1.226	0.848	1.612	14.207	13.571	13.451	11.177	16.472	1240.451	1079.265	972.460	395.594	2039.178	0.538	0.162	0.516	0.432	0.631	0.517	0.163	0.553	0.081	0.069	0.066	0.402	0.153	0.383	-5129.873	-6212.405	-4026.521	0.003

Table D.122: loglogistic-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) BARC data.

n	$\hat{\alpha}_{log\ ave}$	$\pm\hat{\alpha}_{log\ ave}$	$\hat{\alpha}_{log\ median}$	$\hat{\alpha}_{log\ lower}$	$\hat{\alpha}_{log\ upper}$	$\hat{\beta}_{log\ ave}$	$\pm\hat{\beta}_{log\ ave}$	$\hat{\beta}_{log\ median}$	$\hat{\beta}_{log\ lower}$	$\hat{\beta}_{log\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	20.874	146.089	3.795	1.852	21.624	5.417	16.475	1.400	0.813	2.878	26.120	127.542	11.583	2.602	29.637	15623.336	33891.356	6702.660	1264.931	26538.018	1.004	2.684	0.534	0.375	1.003	0.498	0.231	0.509	0.072	0.080	0.051	0.430	0.230	0.415	-314.498	-424.478	-178.917	0.019
100	91.255	5492.846	3.504	1.955	14.468	4.242	13.507	1.275	0.766	2.158	23.706	95.237	12.707	8.167	26.492	12219.337	20501.952	6300.179	1421.339	20518.263	0.655	1.203	0.480	0.355	0.728	0.503	0.216	0.520	0.062	0.067	0.047	0.435	0.215	0.419	-631.782	-838.037	-393.977	0.009
200	8.169	19.701	3.308	2.045	10.893	3.425	12.072	1.219	0.747	1.878	20.898	57.436	13.377	9.707	21.465	9721.320	12514.407	5916.592	1600.995	16690.957	0.521	0.663	0.441	0.347	0.609	0.506	0.198	0.528	0.056	0.060	0.045	0.438	0.200	0.417	-1262.176	-1662.162	-849.289	0.004
500	6.037	12.946	3.062	2.138	8.372	2.420	8.200	1.226	0.761	1.754	18.424	56.678	13.599	10.925	18.305	7691.090	8038.238	5147.077	1728.331	13608.500	0.432	0.120	0.410	0.340	0.522	0.509	0.176	0.528	0.052	0.041	0.047	0.439	0.180	0.422	-3127.308	-4062.704	-2173.520	0.003
1000	4.488	3.963	2.889	2.169	6.903	2.550	8.364	1.322	0.785	1.727	14.693	5.453	13.916	11.718	17.210	6225.988	5236.780	4349.704	1783.799	11574.762	0.406	0.078	0.390	0.337	0.477	0.515	0.164	0.538	0.052	0.037	0.047	0.433	0.169	0.408	-6109.167	-7983.323	-4469.200	0.002

Table D.123: loglogistic-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) RRLN data.

n	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	88.898	7046.979	7.243	2.543	26.380	5.141	17.562	1.137	0.780	2.553	38.281	1385.998	11.222	1.800	30.678	6338.620	10620.385	3709.861	1082.635	11479.575	0.967	2.452	0.656	0.456	1.163	0.459	0.202	0.465	0.083	0.088	0.057	0.458	0.203	0.453	-322.127	-392.303	-243.449	0.027
100	51.991	4255.004	6.587	2.685	18.270	4.362	15.281	1.011	0.748	1.638	25.254	103.090	12.320	7.864	26.144	5415.516	7868.237	3592.049	1183.605	9355.067	0.738	1.264	0.597	0.443	0.877	0.467	0.186	0.474	0.072	0.080	0.051	0.461	0.185	0.454	-645.977	-774.777	-515.340	0.015
200	41.603	2264.446	6.288	2.927	14.284	3.755	15.507	0.950	0.736	1.317	21.370	84.591	12.925	9.286	21.898	4724.202	4691.188	3472.801	1329.921	8056.299	0.664	1.969	0.565	0.440	0.753	0.473	0.169	0.484	0.066	0.075	0.048	0.461	0.167	0.453	-1293.541	-1528.007	-1066.219	0.007
500	8.086	11.707	5.954	3.241	11.988	3.028	11.741	0.913	0.742	1.174	15.837	16.791	13.225	10.509	18.300	4246.473	3288.802	3418.377	1512.000	6983.087	0.628	2.961	0.539	0.441	0.667	0.475	0.151	0.485	0.063	0.074	0.048	0.463	0.148	0.457	-3239.619	-3779.198	-2748.045	0.004
1000	7.151	5.121	5.836	3.376	10.979	2.460	9.987	0.903	0.746	1.118	14.407	4.944	13.402	11.182	17.227	4007.668	2698.893	3374.057	1609.599	6589.812	0.545	0.292	0.522	0.442	0.625	0.477	0.139	0.490	0.059	0.062	0.047	0.464	0.134	0.456	-6496.808	-7469.539	-5587.839	0.005

Table D.124: loglogistic-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) VOD data.

n	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	19.310	1948.069	2.912	1.000	11.205	21.633	55.237	1.935	1.070	83.112	16.354	57.073	10.315	1.437	22.682	1305.906	3205.890	643.628	137.837	2573.633	0.870	1.753	0.581	0.417	1.137	0.411	0.220	0.397	0.131	0.126	0.092	0.458	0.214	0.454	-242.544	-316.349	-159.921	0.029
100	19.274	2084.190	2.715	1.000	7.831	21.441	40.052	1.634	0.982	79.161	15.380	40.636	11.131	7.294	19.380	1112.384	2545.286	603.226	138.139	2080.877	0.657	1.039	0.532	0.406	0.848	0.418	0.212	0.427	0.116	0.113	0.082	0.465	0.201	0.457	-486.220	-623.963	-347.331	0.014
200	10.023	991.793	2.618	1.000	6.074	21.695	35.768	1.520	0.961	77.000	14.497	49.647	11.716	8.588	17.527	948.581	1159.056	569.916	139.848	1752.205	0.569	0.518	0.506	0.400	0.713	0.418	0.204	0.445	0.107	0.105	0.077	0.475	0.187	0.464	-975.317	-1231.849	-730.824	0.009
500	18.641	1981.765	2.538	1.000	5.032	22.227	34.655	1.474	0.976	74.774	13.266	15.377	12.138	9.336	16.233	811.397	870.928	538.531	139.886	1473.675	0.512	0.198	0.487	0.398	0.619	0.411	0.194	0.455	0.101	0.095	0.075	0.488	0.172	0.469	-2444.661	-3036.763	-1897.785	0.006
1000	10.836	768.158	2.506	1.000	4.635	22.147	35.590	1.461	0.997	73.076	12.813	4.479	12.442	9.441	15.629	746.064	747.230	521.065	140.098	1345.905	0.491	0.116	0.478	0.396	0.579	0.406	0.188	0.458	0.100	0.090	0.076	0.495	0.163	0.471	-4895.754	-6010.959	-3863.690	0.005

Table D.125: loglogistic-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) RIOTINTO data.

n	$\hat{\alpha}_{\log\ ave}$	$\pm\hat{\alpha}_{\log\ ave}$	$\hat{\alpha}_{\log\ median}$	$\hat{\alpha}_{\log\ lower}$	$\hat{\alpha}_{\log\ upper}$	$\hat{\beta}_{\log\ ave}$	$\pm\hat{\beta}_{\log\ ave}$	$\hat{\beta}_{\log\ median}$	$\hat{\beta}_{\log\ lower}$	$\hat{\beta}_{\log\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{ave}$	$\pm\hat{\alpha}_{ave}$	$\hat{\alpha}_{median}$	$\hat{\alpha}_{lower}$	$\hat{\alpha}_{upper}$	$\hat{\beta}_{ave}$	$\pm\hat{\beta}_{ave}$	$\hat{\beta}_{median}$	$\hat{\beta}_{lower}$	$\hat{\beta}_{upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	43.136	131.418	8.614	2.723	56.395	4.203	12.672	1.093	0.697	2.693	40.510	1214.429	10.732	0.885	32.830	31569.866	62260.631	14725.684	4356.974	54815.971	0.729	1.334	0.524	0.384	0.900	0.402	0.200	0.390	0.067	0.075	0.045	0.531	0.204	0.544	-393.209	-458.927	-317.834	0.028
100	30.032	97.948	7.783	2.935	35.630	3.317	10.900	0.961	0.666	1.697	28.841	291.018	12.192	7.043	28.147	24936.267	41810.726	13919.645	4621.436	39567.030	0.595	0.914	0.484	0.375	0.708	0.405	0.184	0.396	0.056	0.063	0.038	0.539	0.184	0.550	-788.845	-904.184	-665.175	0.016
200	23.440	126.633	7.101	3.122	22.173	2.890	9.982	0.893	0.652	1.325	20.476	43.705	12.839	8.975	23.532	21122.982	33229.439	13121.514	4804.744	32171.373	0.534	1.074	0.457	0.364	0.603	0.405	0.165	0.397	0.046	0.045	0.035	0.549	0.164	0.559	-1583.138	-1774.596	-1375.520	0.012
500	9.777	14.042	6.379	3.180	12.402	2.479	9.148	0.880	0.688	1.158	16.370	20.106	13.197	10.487	18.874	14977.243	13833.180	11870.033	5146.549	24090.059	0.452	0.318	0.425	0.362	0.514	0.393	0.139	0.394	0.039	0.033	0.033	0.568	0.135	0.570	-3958.501	-4346.570	-3560.591	0.009
1000	7.730	6.298	5.923	3.373	11.272	1.374	5.069	0.873	0.704	1.085	15.234	5.553	13.885	11.470	18.693	12272.567	8008.686	10934.052	5672.400	19102.090	0.423	0.063	0.413	0.367	0.490	0.392	0.117	0.394	0.036	0.023	0.031	0.573	0.114	0.571	-7891.226	-8491.716	-7220.009	0.013

Table D.126: loglogistic-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) SSELN data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	45.094	159.935	12.812	4.036	60.864	1.413	2.935	0.899	0.583	1.861	26.768	156.257	10.476	0.545	31.132	23516.550	38915.641	12865.873	4017.635	40863.356	0.614	0.921	0.475	0.358	0.745	0.342	0.184	0.313	0.073	0.092	0.041	0.585	0.190	0.608	-395.677	-468.319	-314.396	0.024
100	39.230	149.342	11.313	3.975	45.488	0.999	1.019	0.756	0.533	1.280	26.941	118.329	12.069	5.602	30.309	20429.337	30237.737	11708.106	4264.863	34069.360	0.508	0.384	0.441	0.348	0.625	0.350	0.172	0.330	0.056	0.079	0.029	0.595	0.174	0.617	-795.889	-924.975	-663.523	0.015
200	32.418	136.930	10.452	4.007	32.286	0.813	0.504	0.680	0.520	1.012	24.563	85.631	12.184	8.214	25.147	17280.344	20515.108	10978.936	4656.574	29299.655	0.452	0.143	0.418	0.343	0.553	0.353	0.157	0.335	0.045	0.069	0.022	0.602	0.155	0.619	-1594.805	-1822.614	-1382.884	0.006
500	14.919	24.869	9.453	4.314	20.166	0.704	0.285	0.639	0.520	0.862	19.126	26.440	13.539	10.078	21.637	12736.022	9497.713	10614.125	4994.825	19315.643	0.413	0.079	0.398	0.342	0.479	0.357	0.129	0.350	0.030	0.050	0.015	0.613	0.127	0.626	-3972.596	-4419.654	-3551.704	0.000
1000	12.285	20.239	8.561	4.172	16.426	0.674	0.262	0.619	0.516	0.798	19.071	42.298	14.091	10.638	20.078	9800.844	6444.896	8642.055	4633.699	14540.713	0.410	0.066	0.390	0.352	0.476	0.369	0.123	0.362	0.028	0.039	0.017	0.603	0.115	0.607	-7722.388	-8567.702	-6888.585	0.000

Table D.127: weibull-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) ABFLN data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	12.242	30.057	5.678	2.580	21.892	1.268	1.873	0.945	0.662	1.875	36.300	1049.228	12.441	5.319	40.379	1608.040	2477.019	999.278	280.008	3298.617	0.906	1.760	0.623	0.448	1.227	0.504	0.204	0.514	0.088	0.108	0.052	0.408	0.199	0.395	-255.035	-325.565	-164.413	0.047
100	10.174	22.155	5.244	2.658	15.690	1.034	1.619	0.845	0.623	1.368	30.154	244.998	13.681	8.873	33.797	1409.066	3203.267	924.010	291.124	2662.765	0.681	1.062	0.556	0.425	0.873	0.513	0.188	0.526	0.070	0.085	0.043	0.417	0.183	0.407	-512.639	-641.838	-358.932	0.032
200	8.425	15.930	4.992	2.768	12.605	0.913	0.617	0.799	0.612	1.151	27.388	192.228	14.309	10.234	27.957	1262.539	7906.792	857.451	307.419	2240.207	0.579	0.575	0.517	0.415	0.725	0.516	0.172	0.530	0.058	0.069	0.039	0.426	0.168	0.418	-1029.384	-1267.894	-751.095	0.029
500	7.046	9.228	4.814	2.924	10.756	0.841	0.379	0.778	0.608	1.000	19.892	41.945	14.950	11.353	23.649	1056.311	1563.211	796.764	327.735	1858.706	0.518	0.241	0.489	0.412	0.628	0.514	0.153	0.530	0.051	0.055	0.037	0.435	0.150	0.428	-2582.110	-3123.164	-1939.894	0.030
1000	6.490	6.401	4.700	3.040	9.890	0.811	0.319	0.770	0.611	0.939	17.601	20.799	15.255	11.779	21.732	971.370	798.009	769.677	346.945	1669.095	0.495	0.108	0.477	0.413	0.585	0.513	0.145	0.531	0.049	0.050	0.037	0.438	0.140	0.431	-5168.000	-6210.824	-3964.365	0.029

Table D.128: weibull-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) BARC data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	20.370	74.695	5.788	2.697	29.067	1.310	1.731	0.959	0.644	1.886	29.000	128.786	12.218	1.391	37.608	10422.110	18709.562	5299.735	1040.258	19701.981	0.807	2.268	0.482	0.353	0.838	0.465	0.233	0.458	0.070	0.094	0.039	0.465	0.229	0.460	-314.429	-423.432	-174.592	0.030
100	16.562	57.065	5.248	2.790	21.328	1.076	1.333	0.845	0.595	1.440	34.732	465.661	13.640	8.170	33.620	8864.221	14185.729	4962.318	1099.552	15989.734	0.532	0.651	0.436	0.336	0.642	0.474	0.221	0.480	0.052	0.076	0.030	0.474	0.215	0.468	-632.429	-837.899	-390.011	0.016
200	12.585	39.559	4.847	2.863	17.162	0.904	0.480	0.799	0.571	1.185	30.028	258.630	14.764	9.841	28.794	7155.871	8245.553	4666.298	1140.504	12982.955	0.478	1.507	0.405	0.326	0.552	0.479	0.205	0.487	0.040	0.056	0.025	0.482	0.198	0.473	-1265.672	-1661.330	-847.465	0.010
500	9.342	22.764	4.513	3.043	13.289	0.817	0.294	0.767	0.569	1.045	20.134	33.851	15.573	11.098	24.520	5888.643	6143.848	3958.504	1294.156	10932.539	0.399	0.094	0.381	0.318	0.483	0.491	0.186	0.499	0.031	0.049	0.020	0.479	0.180	0.470	-3126.333	-4059.207	-2169.459	0.012
1000	8.015	19.118	4.230	3.081	11.269	0.797	0.244	0.769	0.572	1.000	28.007	222.546	16.557	12.050	22.883	4958.868	4660.971	3484.349	1316.005	9419.655	0.380	0.073	0.366	0.317	0.444	0.505	0.173	0.522	0.026	0.036	0.019	0.469	0.168	0.452	-6118.415	-7961.086	-4485.144	0.009

Table D.129: weibull-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) RRLN data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	26.538	71.461	11.524	4.353	40.092	1.221	2.206	0.835	0.583	1.565	36.933	1377.335	11.254	1.276	31.707	5063.137	10662.564	3175.834	1010.957	9232.420	0.845	1.875	0.608	0.437	1.014	0.431	0.193	0.424	0.085	0.100	0.053	0.484	0.194	0.490	-320.899	-391.924	-242.109	0.027
100	21.061	61.575	10.289	4.419	29.216	0.941	2.764	0.737	0.559	1.131	25.675	117.894	12.348	7.509	27.985	4425.900	10945.109	2945.126	1047.999	7635.664	0.678	1.533	0.550	0.421	0.795	0.436	0.176	0.433	0.068	0.083	0.042	0.496	0.177	0.500	-645.501	-774.452	-513.934	0.019
200	17.778	142.439	9.464	4.605	22.182	0.778	0.567	0.689	0.553	0.928	21.758	54.587	12.974	9.146	24.132	3777.537	3956.117	2792.429	1105.299	6461.852	0.584	0.789	0.517	0.412	0.684	0.440	0.159	0.440	0.055	0.069	0.035	0.505	0.160	0.507	-1294.799	-1527.726	-1064.992	0.014
500	11.772	14.721	8.716	4.876	17.957	0.709	0.357	0.660	0.556	0.828	18.640	48.201	13.361	10.381	20.643	3291.977	2824.070	2656.039	1210.357	5467.042	0.527	0.819	0.491	0.411	0.605	0.438	0.141	0.439	0.047	0.058	0.032	0.516	0.142	0.516	-3243.688	-3778.260	-2741.073	0.012
1000	10.341	7.127	8.565	5.030	16.002	0.681	0.191	0.654	0.560	0.786	16.068	13.740	13.456	10.958	19.125	3037.979	2037.923	2558.215	1276.676	4919.346	0.489	0.123	0.476	0.409	0.565	0.434	0.130	0.436	0.043	0.054	0.030	0.523	0.127	0.523	-6508.942	-7469.455	-5587.647	0.010

Table D.130: weibull-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) VOD data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	9.735	23.674	4.710	2.149	20.735	1.374	2.598	0.999	0.679	2.357	30.579	331.156	12.219	6.354	45.001	1157.634	2107.600	683.657	208.986	2672.651	0.886	1.868	0.603	0.449	1.293	0.466	0.197	0.468	0.098	0.116	0.059	0.435	0.199	0.430	-247.921	-316.644	-153.553	0.071
100	8.203	19.257	4.347	2.187	15.305	1.137	1.921	0.895	0.643	1.820	31.339	1170.153	13.237	8.954	38.275	996.735	1631.899	619.689	210.478	2189.616	0.680	1.631	0.540	0.427	0.912	0.468	0.181	0.474	0.082	0.099	0.050	0.450	0.182	0.446	-499.108	-624.785	-337.123	0.066
200	6.827	12.346	4.115	2.265	12.474	1.009	0.804	0.848	0.636	1.580	23.240	71.849	13.815	10.209	32.273	906.751	10457.877	571.413	213.543	1895.171	0.575	0.909	0.505	0.418	0.754	0.465	0.163	0.474	0.071	0.087	0.045	0.463	0.163	0.461	-1002.889	-1233.122	-702.381	0.073
500	5.735	10.270	3.937	2.424	11.385	0.945	0.543	0.825	0.646	1.426	18.727	33.726	14.374	11.275	27.625	735.046	1076.816	529.296	221.367	1681.894	0.508	0.239	0.479	0.413	0.651	0.460	0.144	0.471	0.063	0.074	0.042	0.477	0.142	0.475	-2519.955	-3039.434	-1788.586	0.089
1000	5.262	6.751	3.855	2.509	11.133	0.908	0.429	0.818	0.655	1.324	17.079	20.099	14.787	11.845	24.683	681.962	1813.366	503.974	226.652	1573.519	0.484	0.133	0.467	0.412	0.610	0.458	0.132	0.466	0.059	0.067	0.041	0.483	0.130	0.481	-5041.043	-6009.425	-3597.777	0.101

Table D.131: weibull-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) RIOTINTO data.

n	$\hat{\alpha}_{1ave}$	$\pm\hat{\alpha}_{1ave}$	$\hat{\alpha}_{1median}$	$\hat{\alpha}_{1lower}$	$\hat{\alpha}_{1upper}$	$\hat{\beta}_{1ave}$	$\pm\hat{\beta}_{1ave}$	$\hat{\beta}_{1median}$	$\hat{\beta}_{1lower}$	$\hat{\beta}_{1upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\alpha}_{2ave}$	$\pm\hat{\alpha}_{2ave}$	$\hat{\alpha}_{2median}$	$\hat{\alpha}_{2lower}$	$\hat{\alpha}_{2upper}$	$\hat{\beta}_{2ave}$	$\pm\hat{\beta}_{2ave}$	$\hat{\beta}_{2median}$	$\hat{\beta}_{2lower}$	$\hat{\beta}_{2upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	50.837	218.176	12.939	4.385	56.843	1.320	2.112	0.850	0.570	1.718	39.858	1210.901	10.700	0.791	32.385	20267.704	31955.424	11859.604	3660.972	34448.683	0.637	2.289	0.477	0.361	0.734	0.352	0.182	0.330	0.070	0.085	0.041	0.579	0.188	0.600	-390.811	-459.079	-317.053	0.023
100	35.024	103.203	11.621	4.474	41.771	1.017	1.507	0.731	0.538	1.242	27.009	289.643	11.937	6.304	28.022	17068.964	24954.579	11107.286	3882.748	28629.540	0.514	0.718	0.444	0.352	0.622	0.353	0.167	0.337	0.056	0.074	0.033	0.591	0.169	0.607	-785.567	-903.638	-661.471	0.017
200	30.557	118.973	10.401	4.540	27.404	0.804	0.755	0.666	0.523	0.963	21.245	48.314	12.695	8.753	23.537	13950.926	14975.520	10132.886	4085.050	22714.698	0.450	0.145	0.421	0.346	0.543	0.353	0.147	0.340	0.042	0.054	0.027	0.605	0.148	0.617	-1577.794	-1774.926	-1375.205	0.011
500	14.572	29.019	9.599	4.819	18.389	0.699	0.275	0.643	0.539	0.835	15.861	10.715	13.109	10.213	19.991	11133.769	9879.185	9127.006	4202.075	17401.941	0.408	0.076	0.396	0.339	0.472	0.345	0.122	0.340	0.032	0.053	0.019	0.623	0.124	0.631	-3944.337	-4350.261	-3572.091	0.003
1000	11.179	8.689	8.943	5.293	15.495	0.654	0.210	0.617	0.536	0.742	15.280	6.236	13.637	10.990	19.323	9315.406	4966.478	8580.974	4225.694	14135.626	0.395	0.055	0.387	0.345	0.445	0.344	0.101	0.341	0.030	0.059	0.018	0.627	0.098	0.633	-7852.198	-8498.739	-7227.755	0.000

Table D.132: weibull-exponential-weibull mixture on mid-price waiting time zero inflated (exponentially distributed) SSELN data.

	AIC			BIC		
exponential/exponential/weibull	0.4171	-	-	0.8245	-	-
gamma/exponential/weibull	0.0419	0.0860	0.1888	0.0049	0.0860	0.1888
loglogistic/exponential/weibull	0.3686	0.5492	-	0.1192	0.5492	-
weibull/exponential/weibull	0.1711	0.3634	0.8085	0.0501	0.3634	0.8085

Table D.133: ABFLN, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.3477	0.6011	-	0.7878	-	-
gamma/exponential/weibull	0.0405	0.1030	0.2388	0.0121	0.0740	0.2388
loglogistic/exponential/weibull	0.4811	-	-	0.1669	0.6450	-
weibull/exponential/weibull	0.1286	0.2937	0.7583	0.0311	0.2786	0.7583

Table D.134: BARC, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.3178	0.5588	-	0.7451	-	-
gamma/exponential/weibull	0.0420	0.1230	0.2286	0.0173	0.0694	0.2286
loglogistic/exponential/weibull	0.5087	-	-	0.1939	0.6547	-
weibull/exponential/weibull	0.1298	0.3165	0.7666	0.0422	0.2736	0.7666

Table D.135: RRLN, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.4413	-	-	0.8593	-	-
gamma/exponential/weibull	0.0352	0.0714	0.1690	0.0062	0.0714	0.1690
loglogistic/exponential/weibull	0.3482	0.5360	-	0.0966	0.5360	-
weibull/exponential/weibull	0.1741	0.3912	0.8292	0.0366	0.3912	0.8292

Table D.136: VOD, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.3105	0.6218	-	0.6845	-	-
gamma/exponential/weibull	0.0199	0.0923	0.2603	0.0045	0.0407	0.2603
loglogistic/exponential/weibull	0.5582	-	-	0.2840	0.7223	-
weibull/exponential/weibull	0.1089	0.2821	0.7352	0.0245	0.2342	0.7352

Table D.137: RIOTINTO, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.4285	-	-	0.8366	-	-
gamma/exponential/weibull	0.0364	0.0786	0.1843	0.0049	0.0786	0.1843
loglogistic/exponential/weibull	0.3664	0.5490	-	0.1142	0.5490	-
weibull/exponential/weibull	0.1675	0.3709	0.8132	0.0430	0.3709	0.8132

Table D.138: SSELN, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2780	0.4920	-	0.7888	-	-
gamma/exponential/weibull	0.0297	0.0601	0.1394	0.0055	0.0496	0.1394
loglogistic/exponential/weibull	0.5264	-	-	0.1519	0.6540	-
weibull/exponential/weibull	0.1660	0.4479	0.8606	0.0539	0.2964	0.8606

Table D.139: ABFLN, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2016	0.5249	-	0.7294	-	-
gamma/exponential/weibull	0.0223	0.1011	0.2022	0.0058	0.0382	0.2022
loglogistic/exponential/weibull	0.6674	-	-	0.2393	0.7672	-
weibull/exponential/weibull	0.1083	0.3737	0.7973	0.0252	0.1942	0.7973

Table D.140: BARC, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1869	0.4647	-	0.6510	-	-
gamma/exponential/weibull	0.0205	0.1220	0.1988	0.0091	0.0336	0.1988
loglogistic/exponential/weibull	0.6769	-	-	0.3045	0.7700	-
weibull/exponential/weibull	0.1155	0.4132	0.8008	0.0353	0.1963	0.8008

Table D.141: RRLN, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.2978	0.4915	-	0.8208	-	-
gamma/exponential/weibull	0.0218	0.0502	0.1288	0.0027	0.0429	0.1288
loglogistic/exponential/weibull	0.5020	-	-	0.1394	0.6281	-
weibull/exponential/weibull	0.1782	0.4581	0.8710	0.0368	0.3288	0.8710

Table D.142: VOD, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1842	0.5681	-	0.6370	-	-
gamma/exponential/weibull	0.0103	0.1022	0.2538	0.0023	0.0195	0.2538
loglogistic/exponential/weibull	0.7237	-	-	0.3434	0.8288	-
weibull/exponential/weibull	0.0812	0.3288	0.7450	0.0168	0.1510	0.7450

Table D.143: RIOTINTO, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.3021	0.4973	-	0.8096	-	-
gamma/exponential/weibull	0.0281	0.0545	0.1389	0.0033	0.0515	0.1389
loglogistic/exponential/weibull	0.5051	-	-	0.1488	0.6500	-
weibull/exponential/weibull	0.1647	0.4482	0.8611	0.0383	0.2985	0.8611

Table D.144: SSELN, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1533	0.3770	0.9016	0.6451	-	-
gamma/exponential/weibull	0.0123	0.0361	0.0975	0.0025	0.0197	0.0902
loglogistic/exponential/weibull	0.7107	-	-	0.2975	0.7820	-
weibull/exponential/weibull	0.1230	0.5861	-	0.0541	0.1975	0.9090

Table D.145: ABFLN, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0828	0.4088	0.8475	0.5457	-	-
gamma/exponential/weibull	0.0097	0.0970	0.1524	0.0030	0.0163	0.1699
loglogistic/exponential/weibull	0.8189	-	-	0.4264	0.8592	-
weibull/exponential/weibull	0.0886	0.4943	-	0.0248	0.1244	0.8301

Table D.146: BARC, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0847	0.3689	0.8320	0.4356	0.7955	-
gamma/exponential/weibull	0.0069	0.1156	0.1680	0.0018	0.0437	0.1599
loglogistic/exponential/weibull	0.8269	-	-	0.5246	-	-
weibull/exponential/weibull	0.0814	0.5156	-	0.0380	0.1608	0.8398

Table D.147: RRLN, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1500	0.3317	0.9250	0.6673	-	-
gamma/exponential/weibull	0.0115	0.0384	0.0750	0.0025	0.0204	0.0876
loglogistic/exponential/weibull	0.6651	-	-	0.2765	0.7254	-
weibull/exponential/weibull	0.1734	0.6299	-	0.0537	0.2542	0.9124

Table D.148: VOD, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0753	0.4791	-	0.4998	-	-
gamma/exponential/weibull	0.0048	0.1231	0.2526	0.0014	0.0090	0.2526
loglogistic/exponential/weibull	0.8657	-	-	0.4868	0.9073	-
weibull/exponential/weibull	0.0540	0.3976	0.7472	0.0119	0.0835	0.7472

Table D.149: RIOTINTO, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.1289	0.3333	0.9108	0.6636	-	-
gamma/exponential/weibull	0.0112	0.0421	0.0892	0.0012	0.0211	0.0979
loglogistic/exponential/weibull	0.7187	-	-	0.2856	0.7770	-
weibull/exponential/weibull	0.1413	0.6245	-	0.0496	0.2020	0.9021

Table D.150: SSELN, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0311	0.1722	0.8971	0.2990	0.5789	-
gamma/exponential/weibull	0.0048	0.0407	0.1029	0.0024	0.0096	0.0598
loglogistic/exponential/weibull	0.8995	-	-	0.6603	-	-
weibull/exponential/weibull	0.0646	0.7871	-	0.0383	0.4115	0.9402

Table D.151: ABFLN, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0188	0.2470	0.8535	0.2010	0.7554	-
gamma/exponential/weibull	0.0042	0.0891	0.1464	0.0015	0.0180	0.1317
loglogistic/exponential/weibull	0.9158	-	-	0.7672	-	-
weibull/exponential/weibull	0.0613	0.6639	-	0.0303	0.2265	0.8683

Table D.152: BARC, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0191	0.2377	0.8521	0.1606	0.6455	-
gamma/exponential/weibull	0.0024	0.1033	0.1479	0.0024	0.0548	0.1312
loglogistic/exponential/weibull	0.9404	-	-	0.8100	-	-
weibull/exponential/weibull	0.0382	0.6590	-	0.0270	0.2997	0.8688

Table D.153: RRLN, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0336	0.1326	0.9213	0.2908	0.5385	-
gamma/exponential/weibull	0.0045	0.0266	0.0787	0.0020	0.0050	0.0519
loglogistic/exponential/weibull	0.8200	-	-	0.6225	-	-
weibull/exponential/weibull	0.1419	0.8408	-	0.0847	0.4565	0.9481

Table D.154: VOD, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0198	0.3385	0.7558	0.1929	0.8411	-
gamma/exponential/weibull	0.0016	0.1578	0.2442	0.0009	0.0195	0.2462
loglogistic/exponential/weibull	0.9470	-	-	0.7939	-	-
weibull/exponential/weibull	0.0316	0.5036	-	0.0122	0.1393	0.7538

Table D.155: RIOTINTO, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0262	0.1521	0.9108	0.2797	0.5717	-
gamma/exponential/weibull	0.0052	0.0455	0.0892	0.0017	0.0052	0.0682
loglogistic/exponential/weibull	0.9021	-	-	0.6836	-	-
weibull/exponential/weibull	0.0664	0.8024	-	0.0350	0.4231	0.9318

Table D.156: SSELN, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0000	0.0827	0.8797	0.0376	0.3308	0.9699
gamma/exponential/weibull	0.0075	0.0602	0.1203	0.0075	0.0301	0.0301
loglogistic/exponential/weibull	0.9850	-	-	0.9549	-	-
weibull/exponential/weibull	0.0075	0.8571	-	0.0000	0.6391	-

Table D.157: ABFLN, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0094	0.1500	0.8582	0.0707	0.5786	-
gamma/exponential/weibull	0.0036	0.0818	0.1417	0.0007	0.0212	0.1130
loglogistic/exponential/weibull	0.9472	-	-	0.8983	-	-
weibull/exponential/weibull	0.0398	0.7682	-	0.0302	0.4002	0.8869

Table D.158: BARC, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0000	0.1658	0.8663	0.0339	0.5152	-
gamma/exponential/weibull	0.0000	0.0820	0.1337	0.0000	0.0392	0.0927
loglogistic/exponential/weibull	0.9875	-	-	0.9554	-	-
weibull/exponential/weibull	0.0125	0.7522	-	0.0107	0.4456	0.9073

Table D.159: RRLN, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0155	0.0502	0.9151	0.1097	0.3002	0.9917
gamma/exponential/weibull	0.0041	0.0280	0.0849	0.0016	0.0047	0.0083
loglogistic/exponential/weibull	0.8737	-	-	0.8075	-	-
weibull/exponential/weibull	0.1066	0.9218	-	0.0813	0.6951	-

Table D.160: VOD, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0092	0.2466	0.7182	0.0648	0.7417	-
gamma/exponential/weibull	0.0012	0.1845	0.2814	0.0010	0.0374	0.2456
loglogistic/exponential/weibull	0.9706	-	-	0.9226	-	-
weibull/exponential/weibull	0.0189	0.5687	-	0.0116	0.2208	0.7542

Table D.161: RIOTINTO, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
exponential/exponential/weibull	0.0045	0.0538	0.9327	0.0762	0.2960	0.9776
gamma/exponential/weibull	0.0000	0.0045	0.0673	0.0000	0.0045	0.0224
loglogistic/exponential/weibull	0.9596	-	-	0.8969	-	-
weibull/exponential/weibull	0.0359	0.9417	-	0.0269	0.6996	-

Table D.162: SSELN, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

n	$\hat{\lambda}_{1\ \mathrm{ave}}$	$\pm\hat{\lambda}_{1\ \mathrm{ave}}$	$\hat{\lambda}_{1\ \mathrm{median}}$	$\hat{\lambda}_{2\ \mathrm{ave}}$	$\pm\hat{\lambda}_{2\ \mathrm{ave}}$	$\hat{\lambda}_{2\ \mathrm{median}}$	$\hat{\pi}_{1\ \mathrm{ave}}$	$\pm\hat{\pi}_{1\ \mathrm{ave}}$	$\hat{\pi}_{1\ \mathrm{median}}$	$\hat{\pi}_{2\ \mathrm{ave}}$	$\pm\hat{\pi}_{2\ \mathrm{ave}}$	$\hat{\pi}_{2\ \mathrm{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04046	0.55420	0.00003	0.30712	1.17831	0.07615	0.53220	0.17457	0.54163	0.46780	0.17457	0.45837	-416.94017	-494.96845	-344.30490	0.00000
100	0.02683	0.45520	0.00003	0.19125	0.68868	0.05807	0.51818	0.15097	0.52868	0.48182	0.15097	0.47132	-841.37427	-976.16774	-718.09870	0.00000
200	0.01742	0.35043	0.00002	0.14880	0.61225	0.04768	0.51087	0.12561	0.52478	0.48913	0.12561	0.47522	-1692.52684	-1930.76432	-1483.08280	0.00000
500	0.00004	0.00006	0.00002	0.09744	0.31412	0.04208	0.50498	0.09700	0.51426	0.49502	0.09700	0.48574	-4239.66361	-4704.69493	-3799.64694	0.00000
1000	0.00004	0.00004	0.00003	0.08623	0.11394	0.05205	0.49699	0.09185	0.49681	0.50301	0.09185	0.50319	-8258.10053	-9131.12582	-7443.06236	0.00000

Table D.163: 2-component exponential mixture on mid-price waiting time ABFLN data.

n	$\hat{\lambda}_{1\ \mathrm{ave}}$	$\pm\hat{\lambda}_{1\ \mathrm{ave}}$	$\hat{\lambda}_{1\ \mathrm{median}}$	$\hat{\lambda}_{2\ \mathrm{ave}}$	$\pm\hat{\lambda}_{2\ \mathrm{ave}}$	$\hat{\lambda}_{2\ \mathrm{median}}$	$\hat{\pi}_{1\ \mathrm{ave}}$	$\pm\hat{\pi}_{1\ \mathrm{ave}}$	$\hat{\pi}_{1\ \mathrm{median}}$	$\hat{\pi}_{2\ \mathrm{ave}}$	$\pm\hat{\pi}_{2\ \mathrm{ave}}$	$\hat{\pi}_{2\ \mathrm{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04393	0.54143	0.00063	0.50607	7.20561	0.19966	0.37808	0.17903	0.36499	0.62192	0.17903	0.63501	-262.02611	-338.10272	-183.96923	0.00000
100	0.03982	0.52761	0.00057	0.38837	1.24377	0.18557	0.37125	0.16065	0.36066	0.62875	0.16065	0.63934	-531.54253	-667.91673	-393.21979	0.00000
200	0.03863	0.52504	0.00054	0.33587	0.77718	0.17961	0.36823	0.14812	0.35783	0.63177	0.14812	0.64217	-1072.58913	-1323.07537	-819.41032	0.00000
500	0.03523	0.49708	0.00052	0.30555	0.68761	0.17692	0.36656	0.13707	0.35481	0.63344	0.13707	0.64519	-2698.88315	-3270.18187	-2117.82716	0.00000
1000	0.03038	0.45308	0.00051	0.29735	0.76597	0.17658	0.36571	0.13122	0.35265	0.63429	0.13122	0.64735	-5409.99732	-6520.32446	-4295.76057	0.00000

Table D.164: 2-component exponential mixture on mid-price waiting time BARC data.

n	$\hat{\lambda}_{1\ \mathrm{ave}}$	$\pm\hat{\lambda}_{1\ \mathrm{ave}}$	$\hat{\lambda}_{1\ \mathrm{median}}$	$\hat{\lambda}_{2\ \mathrm{ave}}$	$\pm\hat{\lambda}_{2\ \mathrm{ave}}$	$\hat{\lambda}_{2\ \mathrm{median}}$	$\hat{\pi}_{1\ \mathrm{ave}}$	$\pm\hat{\pi}_{1\ \mathrm{ave}}$	$\hat{\pi}_{1\ \mathrm{median}}$	$\hat{\pi}_{2\ \mathrm{ave}}$	$\pm\hat{\pi}_{2\ \mathrm{ave}}$	$\hat{\pi}_{2\ \mathrm{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.03084	0.46374	0.00007	0.33794	1.40930	0.16565	0.40985	0.21860	0.40178	0.59015	0.21860	0.59822	-330.39553	-446.75693	-192.73833	0.00000
100	0.02309	0.38540	0.00006	0.25250	0.92783	0.14524	0.40049	0.19729	0.39881	0.59951	0.19729	0.60119	-667.76458	-884.75168	-424.62245	0.00000
200	0.01887	0.36233	0.00005	0.20249	0.57337	0.13605	0.39274	0.18078	0.39028	0.60726	0.18078	0.60972	-1341.69505	-1753.22288	-909.91049	0.00000
500	0.00575	0.19951	0.00005	0.17371	0.39739	0.13978	0.38246	0.16373	0.37850	0.61754	0.16373	0.62150	-3334.34341	-4325.22612	-2324.28633	0.00000
1000	0.00007	0.00008	0.00005	0.16030	0.21972	0.15548	0.36928	0.15315	0.34630	0.63072	0.15315	0.65370	-6544.89620	-8490.78890	-4797.73037	0.00000

Table D.165: 2-component exponential mixture on mid-price waiting time RRLN data.

n	$\hat{\lambda}_{1\ \mathrm{ave}}$	$\pm\hat{\lambda}_{1\ \mathrm{ave}}$	$\hat{\lambda}_{1\ \mathrm{median}}$	$\hat{\lambda}_{2\ \mathrm{ave}}$	$\pm\hat{\lambda}_{2\ \mathrm{ave}}$	$\hat{\lambda}_{2\ \mathrm{median}}$	$\hat{\pi}_{1\ \mathrm{ave}}$	$\pm\hat{\pi}_{1\ \mathrm{ave}}$	$\hat{\pi}_{1\ \mathrm{median}}$	$\hat{\pi}_{2\ \mathrm{ave}}$	$\pm\hat{\pi}_{2\ \mathrm{ave}}$	$\hat{\pi}_{2\ \mathrm{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.05272	0.61592	0.00019	0.41729	3.85620	0.10891	0.46910	0.18077	0.46462	0.53090	0.18077	0.53538	-332.42407	-406.69253	-261.45369	0.00000
100	0.04936	0.59782	0.00017	0.28022	0.93521	0.09558	0.45781	0.15852	0.45376	0.54219	0.15852	0.54624	-672.17544	-803.61611	-550.10565	0.00000
200	0.04624	0.57783	0.00016	0.25166	1.82867	0.09038	0.45124	0.14191	0.44966	0.54876	0.14191	0.55034	-1354.19012	-1592.58265	-1139.57054	0.00000
500	0.04114	0.54123	0.00015	0.19456	0.68584	0.08757	0.44706	0.12570	0.44355	0.55294	0.12570	0.55644	-3406.83920	-3948.20106	-2930.21646	0.00000
1000	0.03256	0.47991	0.00015	0.18212	0.66093	0.08527	0.44516	0.11340	0.43955	0.55484	0.11340	0.56045	-6833.50618	-7821.01047	-5963.42578	0.00000

Table D.166: 2-component exponential mixture on mid-price waiting time VOD data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.06818	0.67795	0.00088	0.64096	2.13800	0.24438	0.39395	0.16983	0.38751	0.60605	0.16983	0.61249	-253.86504	-328.73764	-184.29498	0.00000
100	0.06221	0.65516	0.00081	0.53087	1.60629	0.22768	0.39009	0.14894	0.38650	0.60991	0.14894	0.61350	-515.21272	-649.53013	-395.33164	0.00000
200	0.05979	0.64332	0.00077	0.48927	2.36358	0.22240	0.38947	0.13411	0.38774	0.61053	0.13411	0.61226	-1039.92110	-1285.59214	-826.66359	0.00000
500	0.05738	0.63144	0.00073	0.44441	1.06459	0.22107	0.38892	0.12089	0.38797	0.61108	0.12089	0.61203	-2617.75420	-3178.05390	-2140.72773	0.00000
1000	0.05365	0.60919	0.00071	0.42446	0.90914	0.22202	0.38818	0.11338	0.38749	0.61182	0.11338	0.61251	-5250.13050	-6293.13095	-4363.56308	0.00000

Table D.167: 2-component exponential mixture on mid-price waiting time RIOTINTO data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.03590	0.51778	0.00003	0.29170	1.04163	0.07820	0.53056	0.17098	0.53623	0.46944	0.17098	0.46377	-411.49531	-483.28558	-344.38131	0.00000
100	0.03070	0.47634	0.00003	0.20013	0.74776	0.06037	0.51800	0.14429	0.52076	0.48200	0.14429	0.47924	-831.28758	-952.80307	-716.08466	0.00000
200	0.02240	0.39903	0.00003	0.14114	0.59358	0.05218	0.51062	0.12251	0.51341	0.48938	0.12251	0.48659	-1673.85941	-1875.18413	-1487.03076	0.00000
500	0.01384	0.33022	0.00003	0.11821	0.55037	0.05009	0.50772	0.09949	0.51344	0.49228	0.09949	0.48656	-4211.42914	-4598.42603	-3814.88797	0.00000
1000	0.00003	0.00003	0.00002	0.11479	0.47799	0.04895	0.50447	0.08595	0.50635	0.49553	0.08595	0.49365	-8407.95605	-9014.43417	-7804.38927	0.00000

Table D.168: 2-component exponential mixture on mid-price waiting time SSELN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.03594	0.49864	0.00002	0.05455	0.51823	0.00631	0.42244	0.99547	0.22394	22.34616	112.73140	9.58171	0.38346	0.17131	0.38065	0.29399	0.12600	0.28199	0.24500	0.13992	0.22496	0.07755	0.09187	0.05242	-397.70162	-475.07864	-325.67130	0.00076
100	0.02353	0.39408	0.00002	0.03514	0.40813	0.00242	0.28781	0.52462	0.15696	19.40869	124.01568	10.71081	0.35065	0.14855	0.34598	0.29334	0.10701	0.28779	0.26806	0.12661	0.24873	0.08795	0.09646	0.05936	-800.55448	-933.79203	-675.02430	0.00117
200	0.01799	0.36266	0.00001	0.02783	0.38977	0.00102	0.21648	0.50053	0.11763	15.17598	47.71351	10.97483	0.31820	0.12458	0.30865	0.29477	0.09056	0.28874	0.29360	0.11214	0.27875	0.09343	0.10158	0.05960	-1605.99903	-1836.38003	-1396.34669	0.00164
500	0.00002	0.00004	0.00001	0.00277	0.00641	0.00068	0.13117	0.12412	0.09835	11.03269	6.68088	10.92805	0.28962	0.09657	0.27712	0.29671	0.06866	0.28879	0.32442	0.09534	0.31868	0.08926	0.08994	0.05769	-4011.05846	-4464.07446	-3579.51868	0.00239
1000	0.00002	0.00003	0.00002	0.00222	0.00335	0.00078	0.13439	0.09894	0.10502	11.10351	3.65026	10.98959	0.28097	0.08723	0.28403	0.29833	0.06110	0.29802	0.34410	0.08888	0.33455	0.07661	0.07313	0.05776	-7808.56100	-8657.08122	-6960.72502	0.00752

Table D.169: 4-component exponential mixture on mid-price waiting time ABFLN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04194	0.52922	0.00044	0.08258	0.52905	0.02448	0.51238	0.65050	0.35749	47.40284	4245.93011	10.66680	0.25187	0.14828	0.23896	0.28195	0.12706	0.27206	0.36792	0.17968	0.35227	0.09826	0.10743	0.06733	-251.58843	-327.09943	-174.00351	0.00015
100	0.03834	0.51625	0.00039	0.07008	0.51553	0.01947	0.46222	0.59850	0.33011	30.48914	2195.16076	11.99219	0.24371	0.12976	0.23306	0.26950	0.11057	0.26048	0.38331	0.16619	0.37117	0.10348	0.11015	0.07114	-509.63260	-644.71783	-372.96931	0.00015
200	0.03710	0.51247	0.00036	0.06275	0.51178	0.01625	0.41737	0.57669	0.29993	15.73491	83.30883	12.68733	0.23700	0.11621	0.22649	0.25825	0.09705	0.25055	0.39470	0.15706	0.38678	0.11005	0.11829	0.07383	-1027.59484	-1274.86461	-778.25865	0.00005
500	0.03372	0.48266	0.00034	0.05394	0.48213	0.01366	0.37032	0.53881	0.26907	13.53844	12.35207	13.19343	0.23006	0.10293	0.22076	0.24712	0.08534	0.24021	0.41068	0.15154	0.40711	0.11214	0.12262	0.07554	-2583.86425	-3144.09582	-2017.00639	0.00012
1000	0.03113	0.46364	0.00033	0.04888	0.46306	0.01287	0.34902	0.52093	0.25717	13.50361	5.38433	13.55845	0.22760	0.09469	0.21867	0.24102	0.07856	0.23348	0.42485	0.14745	0.42932	0.10653	0.11614	0.07485	-5177.61116	-6254.19577	-4101.32274	0.00025

Table D.170: 4-component exponential mixture on mid-price waiting time BARC data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.03057	0.46362	0.00005	0.06701	0.47057	0.01446	0.44477	0.67905	0.31522	39.91494	975.00567	8.92839	0.29508	0.18265	0.26978	0.26436	0.12517	0.25041	0.35597	0.20585	0.32279	0.08459	0.10088	0.05411	-315.39473	-428.90398	-184.77967	0.00036
100	0.02564	0.42747	0.00004	0.05052	0.42843	0.00860	0.36070	0.56598	0.26791	16.61706	64.60236	10.52611	0.27251	0.16055	0.25044	0.25488	0.11239	0.24449	0.37532	0.19667	0.33738	0.09729	0.12108	0.05891	-636.14990	-846.90548	-404.40644	0.00029
200	0.01905	0.36522	0.00003	0.03665	0.36578	0.00399	0.28789	0.43583	0.21457	13.41835	27.39086	11.18112	0.25230	0.14205	0.23269	0.24643	0.10070	0.23782	0.38464	0.18837	0.34688	0.11664	0.15265	0.06215	-1275.76547	-1676.61237	-863.54089	0.00060
500	0.00597	0.20949	0.00003	0.01738	0.21073	0.00160	0.21075	0.26396	0.16557	11.20720	9.09326	11.30237	0.22188	0.12031	0.20177	0.23934	0.09110	0.23142	0.40281	0.18308	0.37370	0.13597	0.18180	0.06445	-3162.95270	-4109.33365	-2216.41781	0.00159
1000	0.00004	0.00005	0.00003	0.00802	0.01783	0.00118	0.17917	0.13926	0.13692	10.92509	7.01914	11.46184	0.19958	0.10585	0.18401	0.23268	0.08564	0.23011	0.41410	0.18189	0.40003	0.15365	0.20519	0.06385	-6194.56220	-8065.82968	-4538.23835	0.00178

Table D.171: 4-component exponential mixture on mid-price waiting time RRLN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.05104	0.59999	0.00014	0.07601	0.59904	0.01470	0.49225	0.89290	0.27851	31.74108	787.94088	10.45155	0.33411	0.16611	0.32847	0.30803	0.14325	0.29130	0.26839	0.15153	0.25070	0.08947	0.10106	0.06161	-318.99875	-394.21915	-246.60465	0.00015
100	0.04872	0.59214	0.00012	0.06694	0.59134	0.01029	0.41236	0.67483	0.22486	21.00631	325.92784	11.56593	0.31563	0.14815	0.30946	0.29877	0.12832	0.28255	0.29147	0.14041	0.27999	0.09412	0.10169	0.06623	-644.45609	-778.64755	-520.56055	0.00015
200	0.04739	0.58721	0.00011	0.06212	0.59075	0.00757	0.34893	0.65265	0.18212	18.49692	323.16003	11.83457	0.29932	0.13273	0.29459	0.29096	0.11483	0.27676	0.31394	0.13207	0.30618	0.09578	0.09924	0.06849	-1297.07136	-1537.85849	-1076.44568	0.00030
500	0.04227	0.55180	0.00010	0.05221	0.55144	0.00458	0.28151	0.60171	0.15014	12.86941	20.90350	11.88692	0.28086	0.11392	0.27296	0.28387	0.09911	0.26906	0.34452	0.12472	0.34208	0.09076	0.09074	0.06836	-3259.90130	-3803.23752	-2777.08539	0.00050
1000	0.03290	0.48171	0.00009	0.04060	0.48182	0.00321	0.23498	0.50887	0.13325	12.11452	5.55038	11.59836	0.26781	0.10017	0.25976	0.27780	0.08421	0.26718	0.36653	0.12084	0.36996	0.08787	0.09001	0.06735	-6534.88154	-7531.09813	-5642.28783	0.00104

Table D.172: 4-component exponential mixture on mid-price waiting time VOD data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.06469	0.65737	0.00058	0.10311	0.65615	0.02330	0.56912	0.77430	0.39458	42.72721	4416.23332	10.93129	0.24987	0.14257	0.23907	0.27456	0.12314	0.26800	0.36137	0.16721	0.34748	0.11420	0.12637	0.07538	-243.27055	-318.29561	-173.12175	0.00010
100	0.06004	0.64694	0.00052	0.08974	0.64551	0.01908	0.52280	0.70731	0.36975	21.21338	389.37837	12.09962	0.24163	0.12267	0.23350	0.26618	0.10468	0.26211	0.37427	0.15311	0.36301	0.11791	0.12472	0.07969	-492.95718	-627.50088	-371.58571	0.00008
200	0.05802	0.63760	0.00049	0.08159	0.63633	0.01632	0.48054	0.68655	0.34418	15.53483	90.66762	12.65287	0.23476	0.10758	0.22820	0.25824	0.09065	0.25519	0.38227	0.14399	0.37296	0.12472	0.12957	0.08407	-994.00659	-1239.91379	-777.67677	0.00004
500	0.05513	0.61889	0.00046	0.07370	0.61788	0.01420	0.44146	0.66738	0.31968	14.07940	86.42750	13.16815	0.22831	0.09340	0.22440	0.25092	0.07852	0.24914	0.39254	0.13848	0.38438	0.12823	0.13340	0.08550	-2499.83687	-3059.10009	-2018.14273	0.00010
1000	0.05300	0.60827	0.00044	0.06935	0.60757	0.01337	0.42474	0.65481	0.31034	13.56927	14.23105	13.32610	0.22569	0.08507	0.22287	0.24815	0.07296	0.24681	0.40259	0.13561	0.39338	0.12358	0.12946	0.08459	-5010.72904	-6062.02681	-4105.54902	0.00019

Table D.173: 4-component exponential mixture on mid-price waiting time RIOTINTO data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{4\text{ ave}}$	$\pm\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{4\text{ median}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	$\hat{\pi}_{4\text{ ave}}$	$\pm\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{4\text{ median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.03459	0.50139	0.00002	0.05329	0.51145	0.00631	0.42452	0.97354	0.22330	38.76244	583.41888	9.97770	0.38248	0.16955	0.38046	0.29658	0.12745	0.28498	0.24113	0.13247	0.22502	0.07981	0.09171	0.05448	-392.42336	-464.48887	-325.29013	0.00103
100	0.02864	0.44007	0.00002	0.03842	0.43986	0.00214	0.30677	0.55887	0.15495	28.53940	376.46109	10.79083	0.34567	0.14635	0.33793	0.29691	0.10878	0.28942	0.26917	0.11971	0.25614	0.08825	0.09364	0.06217	-790.53948	-912.99446	-677.17022	0.00060
200	0.01947	0.36194	0.00002	0.02558	0.36185	0.00110	0.22631	0.52414	0.11342	14.31856	38.95085	11.09735	0.31957	0.12596	0.30642	0.29827	0.09192	0.29488	0.29091	0.10870	0.28419	0.09125	0.09383	0.06265	-1587.98987	-1795.16801	-1392.54524	0.00124
500	0.01317	0.31397	0.00001	0.01553	0.31394	0.00069	0.14373	0.46297	0.08635	10.61527	5.73911	10.86533	0.28311	0.09164	0.27667	0.30116	0.07240	0.29790	0.32422	0.09358	0.32387	0.09151	0.09348	0.06165	-3983.57066	-4380.73581	-3604.78791	0.00350
1000	0.00002	0.00002	0.00001	0.00144	0.00358	0.00061	0.10178	0.08434	0.08558	10.79800	3.25781	10.63179	0.26661	0.07532	0.26020	0.30982	0.05686	0.31385	0.34661	0.08522	0.34047	0.07697	0.08104	0.05668	-7943.47448	-8576.87018	-7353.46841	0.00448

Table D.174: 4-component exponential mixture on mid-price waiting time SSELN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.03854	0.04208	0.09623	0.52698	11.19836	110.71736	0.30042	0.25679	0.20486	0.16823	0.03775	0.03195	-393.62400	-469.84297	-321.89440	0.00133
100	0.02418	0.02639	0.06576	0.44560	5.66746	73.54032	0.27322	0.26025	0.20673	0.18619	0.04134	0.03226	-792.69034	-925.82284	-664.97067	0.00469
200	0.01734	0.01883	0.05108	0.37567	5.17627	48.55737	0.24573	0.26383	0.20378	0.19733	0.05616	0.03318	-1590.91987	-1821.39319	-1380.51780	0.00246
500	0.00002	0.00073	0.01611	0.23220	4.77938	25.93731	0.21147	0.26197	0.19187	0.20611	0.09344	0.03514	-3969.26469	-4409.60466	-3539.70140	0.00239
1000	0.00002	0.00037	0.01357	0.19559	4.53723	17.12730	0.19258	0.25194	0.18599	0.20412	0.12681	0.03855	-7715.10824	-8566.24506	-6868.03160	0.00752

Table D.175: 6-component exponential mixture on mid-price waiting time ABFLN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.04208	0.04758	0.12453	0.56811	4.73889	80.17121	0.17071	0.18290	0.24623	0.29444	0.05544	0.05027	-250.52166	-325.67644	-173.01448	0.00122
100	0.03844	0.04315	0.11114	0.55482	4.34241	69.33903	0.16354	0.18483	0.24381	0.30134	0.05468	0.05179	-507.36231	-641.97684	-370.40144	0.00166
200	0.03705	0.04115	0.10149	0.53741	3.97277	62.31217	0.15580	0.18580	0.24030	0.30546	0.05827	0.05438	-1022.59061	-1268.75510	-772.38535	0.00148
500	0.03643	0.03966	0.09135	0.49759	3.38973	28.84779	0.14428	0.18526	0.23179	0.30297	0.07821	0.05749	-2570.48587	-3129.15968	-2005.04641	0.00043
1000	0.03293	0.03567	0.08193	0.46002	3.12864	18.34616	0.13603	0.18373	0.22193	0.29438	0.10478	0.05916	-5149.11728	-6223.15426	-4079.04646	0.00025

Table D.176: 6-component exponential mixture on mid-price waiting time BARC data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.02781	0.03293	0.10610	0.53369	5.83685	89.39159	0.21857	0.19890	0.21267	0.29142	0.04283	0.03561	-312.33986	-424.64957	-181.91129	0.00203
100	0.02658	0.02898	0.08640	0.48646	4.29360	71.25388	0.19597	0.20342	0.20860	0.30931	0.04692	0.03578	-629.87636	-838.52563	-395.65353	0.00293
200	0.01913	0.02059	0.06444	0.42197	3.79729	55.46183	0.17333	0.20707	0.20097	0.31976	0.06159	0.03729	-1262.73951	-1663.41602	-846.43678	0.00419
500	0.00652	0.00752	0.04069	0.33084	3.33523	67.68074	0.15035	0.20377	0.18733	0.32213	0.09750	0.03892	-3127.13637	-4064.65059	-2175.74079	0.00238
1000	0.00003	0.00086	0.03092	0.27913	2.72190	21.42151	0.13301	0.19601	0.17438	0.32451	0.13295	0.03914	-6121.48720	-7981.23255	-4484.16384	0.00178

Table D.177: 6-component exponential mixture on mid-price waiting time RRLN data.

n	$\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{4\text{ ave}}$	$\hat{\lambda}_{5\text{ ave}}$	$\hat{\lambda}_{6\text{ ave}}$	$\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{4\text{ ave}}$	$\hat{\pi}_{5\text{ ave}}$	$\hat{\pi}_{6\text{ ave}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.05140	0.05582	0.11359	0.58335	6.90880	76.68340	0.24478	0.22634	0.25099	0.18724	0.04864	0.04201	-316.93243	-391.74930	-245.18289	0.00085
100	0.04859	0.05189	0.10161	0.55933	5.62261	85.82038	0.22682	0.23124	0.25379	0.19597	0.04929	0.04289	-640.30471	-774.21302	-515.42233	0.00132
200	0.04633	0.04887	0.09156	0.52743	5.52402	37.25462	0.20990	0.23318	0.25295	0.20373	0.05524	0.04500	-1288.49864	-1529.38004	-1066.32355	0.00168
500	0.03875	0.04040	0.07315	0.45404	5.21193	21.46954	0.18709	0.23391	0.24273	0.21366	0.07500	0.04761	-3236.30738	-3780.81407	-2751.92018	0.00125
1000	0.03465	0.03586	0.06289	0.38884	4.87465	16.03532	0.17158	0.23306	0.22931	0.22265	0.09409	0.04930	-6483.40361	-7472.16870	-5600.10416	0.00104

Table D.178: 6-component exponential mixture on mid-price waiting time VOD data.

n	$\hat\lambda_{1\ ave}$	$\hat\lambda_{2\ ave}$	$\hat\lambda_{3\ ave}$	$\hat\lambda_{4\ ave}$	$\hat\lambda_{5\ ave}$	$\hat\lambda_{6\ ave}$	$\hat\pi_{1\ ave}$	$\hat\pi_{2\ ave}$	$\hat\pi_{3\ ave}$	$\hat\pi_{4\ ave}$	$\hat\pi_{5\ ave}$	$\hat\pi_{6\ ave}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.06553	0.07052	0.14693	0.62309	4.84243	94.10461	0.16055	0.19267	0.23083	0.29140	0.06576	0.05879	-242.30795	-316.99695	-172.09383	0.00104
100	0.06084	0.06519	0.13255	0.61227	4.50522	68.03103	0.15567	0.19397	0.22804	0.29706	0.06464	0.06063	-490.85687	-625.05063	-368.78970	0.00162
200	0.05812	0.06207	0.12142	0.59543	4.21244	65.23347	0.14946	0.19473	0.22466	0.30025	0.06750	0.06339	-989.47170	-1234.63434	-773.07992	0.00142
500	0.05584	0.05920	0.11044	0.55803	3.73083	20.43121	0.14010	0.19404	0.21774	0.29751	0.08387	0.06673	-2487.70389	-3045.53970	-2007.22149	0.00024
1000	0.05236	0.05534	0.10356	0.52733	3.45110	16.77525	0.13244	0.19312	0.21140	0.28822	0.10652	0.06830	-4984.81537	-6034.40872	-4083.00726	0.00019

Table D.179: 6-component exponential mixture on mid-price waiting time RIOTINTO data.

n	$\hat\lambda_{1\ ave}$	$\hat\lambda_{2\ ave}$	$\hat\lambda_{3\ ave}$	$\hat\lambda_{4\ ave}$	$\hat\lambda_{5\ ave}$	$\hat\lambda_{6\ ave}$	$\hat\pi_{1\ ave}$	$\hat\pi_{2\ ave}$	$\hat\pi_{3\ ave}$	$\hat\pi_{4\ ave}$	$\hat\pi_{5\ ave}$	$\hat\pi_{6\ ave}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.03218	0.03795	0.09101	0.53555	6.20290	296.24522	0.29072	0.26287	0.20990	0.16493	0.03886	0.03273	-388.02194	-459.30356	-319.62877	0.00206
100	0.03118	0.03276	0.07191	0.48165	5.93202	82.07191	0.26410	0.26858	0.21209	0.17960	0.04258	0.03306	-782.55865	-903.73987	-664.62381	0.00299
200	0.02343	0.02438	0.05211	0.40288	5.76361	39.89206	0.24259	0.26715	0.20949	0.19178	0.05489	0.03410	-1571.76504	-1777.10310	-1376.81265	0.00434
500	0.00468	0.00513	0.02267	0.29266	5.45484	19.21176	0.21838	0.26350	0.19691	0.19989	0.08478	0.03655	-3944.27461	-4342.00943	-3555.09245	0.00350
1000	0.00001	0.00035	0.01349	0.20445	4.66612	16.31273	0.20243	0.25719	0.18957	0.20202	0.10825	0.04054	-7856.86693	-8501.31862	-7230.11256	0.00448

Table D.180: 6-component exponential mixture on mid-price waiting time SSELN data.

n	$\hat\lambda_{1\ ave}$	$\hat\lambda_{2\ ave}$	$\hat\lambda_{3\ ave}$	$\hat\lambda_{4\ ave}$	$\hat\lambda_{5\ ave}$	$\hat\lambda_{6\ ave}$	$\hat\lambda_{7\ ave}$	$\hat\lambda_{8\ ave}$	$\hat\lambda_{9\ ave}$	$\hat\lambda_{10\ ave}$	$\hat\pi_{1\ ave}$	$\hat\pi_{2\ ave}$	$\hat\pi_{3\ ave}$	$\hat\pi_{4\ ave}$	$\hat\pi_{5\ ave}$	$\hat\pi_{6\ ave}$	$\hat\pi_{7\ ave}$	$\hat\pi_{8\ ave}$	$\hat\pi_{9\ ave}$	$\hat\pi_{10\ ave}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.03987	0.04026	0.05862	0.20283	0.50994	0.87955	1.81898	13.81421	23.52048	62.08893	0.22120	0.26047	0.18670	0.13340	0.08817	0.03525	0.02468	0.00994	0.01296	0.02723	-391.60256	-466.60885	-317.11918	0.01028
100	0.02751	0.02782	0.04094	0.16833	0.47284	0.68697	1.18585	10.54637	21.08140	61.27356	0.21604	0.25765	0.18325	0.13955	0.09259	0.03707	0.02486	0.00958	0.01314	0.02627	-790.39538	-923.06760	-661.30311	0.00703
200	0.01717	0.01745	0.02794	0.13629	0.45747	0.63090	0.84041	11.66907	17.50157	58.06596	0.20613	0.25705	0.17575	0.14667	0.09974	0.04006	0.02605	0.00932	0.01317	0.02606	-1587.29084	-1819.11528	-1373.94375	0.00574
500	0.00001	0.00023	0.00737	0.07374	0.38320	0.56490	0.59655	13.04006	19.80568	32.74017	0.18662	0.24962	0.16858	0.15665	0.11219	0.04849	0.03121	0.00829	0.01261	0.02574	-3963.95307	-4412.19545	-3524.27516	0.00478
1000	0.00001	0.00019	0.00550	0.05737	0.35089	0.52599	0.60769	13.40767	15.58704	20.07809	0.16134	0.23558	0.17364	0.15931	0.12006	0.06160	0.03883	0.00864	0.01326	0.02774	-7706.83920	-8549.30445	-6865.69647	0.00000

Table D.181: 10-component exponential mixture on mid-price waiting time ABFLN data.

n	$\hat\lambda_{1\ ave}$	$\hat\lambda_{2\ ave}$	$\hat\lambda_{3\ ave}$	$\hat\lambda_{4\ ave}$	$\hat\lambda_{5\ ave}$	$\hat\lambda_{6\ ave}$	$\hat\lambda_{7\ ave}$	$\hat\lambda_{8\ ave}$	$\hat\lambda_{9\ ave}$	$\hat\lambda_{10\ ave}$	$\hat\pi_{1\ ave}$	$\hat\pi_{2\ ave}$	$\hat\pi_{3\ ave}$	$\hat\pi_{4\ ave}$	$\hat\pi_{5\ ave}$	$\hat\pi_{6\ ave}$	$\hat\pi_{7\ ave}$	$\hat\pi_{8\ ave}$	$\hat\pi_{9\ ave}$	$\hat\pi_{10\ ave}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.04255	0.04713	0.11433	0.31252	0.59605	0.77577	1.21738	10.48246	28.49003	95.24428	0.16865	0.17379	0.20799	0.14064	0.13522	0.05919	0.04110	0.01402	0.01925	0.04016	-250.45201	-325.44005	-170.88594	0.00668
100	0.03884	0.04279	0.10028	0.29273	0.58858	0.70859	0.91601	11.16070	22.82700	124.68247	0.15896	0.17687	0.20168	0.14802	0.13879	0.06137	0.04196	0.01341	0.01945	0.03948	-507.03864	-641.01691	-364.54120	0.01038
200	0.03839	0.04172	0.08964	0.26963	0.58651	0.67943	0.75340	12.05328	20.35473	72.75583	0.14706	0.17797	0.19398	0.15903	0.14347	0.06388	0.04327	0.01279	0.01940	0.03916	-1021.84425	-1267.12374	-764.98070	0.00713
500	0.03535	0.03777	0.07317	0.22650	0.58242	0.64906	0.66987	13.34231	17.74193	31.84452	0.12868	0.17633	0.17887	0.17858	0.15320	0.06815	0.04558	0.01210	0.01916	0.03935	-2568.52485	-3124.81569	-1997.14730	0.00235
1000	0.03102	0.03285	0.06200	0.19021	0.57420	0.63171	0.64223	13.92086	17.16302	21.81031	0.11356	0.17468	0.16538	0.19361	0.16234	0.07230	0.04790	0.01175	0.01888	0.03958	-5145.57982	-6215.90880	-4068.28048	0.00187

Table D.182: 10-component exponential mixture on mid-price waiting time BARC data.

n	$\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{4\ ave}$	$\hat{\lambda}_{5\ ave}$	$\hat{\lambda}_{6\ ave}$	$\hat{\lambda}_{7\ ave}$	$\hat{\lambda}_{8\ ave}$	$\hat{\lambda}_{9\ ave}$	$\hat{\lambda}_{10\ ave}$	$\hat{\pi}_{1\ ave}$	$\hat{\pi}_{2\ ave}$	$\hat{\pi}_{3\ ave}$	$\hat{\pi}_{4\ ave}$	$\hat{\pi}_{5\ ave}$	$\hat{\pi}_{6\ ave}$	$\hat{\pi}_{7\ ave}$	$\hat{\pi}_{8\ ave}$	$\hat{\pi}_{9\ ave}$	$\hat{\pi}_{10\ ave}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.03055	0.03177	0.07632	0.32130	0.55404	0.75856	1.55218	10.13908	25.68859	73.51778	0.17051	0.20205	0.19137	0.15675	0.13671	0.05543	0.03487	0.00991	0.01345	0.02895	-311.38640	-422.66799	-178.55513	0.00987
100	0.02375	0.02443	0.05939	0.27597	0.51458	0.64634	0.97849	9.98248	24.47556	114.49887	0.16240	0.20349	0.18151	0.16490	0.14300	0.05840	0.03607	0.00930	0.01329	0.02765	-629.91679	-836.91920	-392.23202	0.00909
200	0.01827	0.01885	0.04732	0.22819	0.49430	0.59955	0.72799	11.02996	24.36050	94.17902	0.15077	0.20222	0.17051	0.17339	0.15347	0.06263	0.03797	0.00859	0.01325	0.02720	-1261.76901	-1660.43615	-841.65530	0.00778
500	0.00596	0.00643	0.02798	0.16698	0.46263	0.54915	0.56466	11.29124	22.09528	66.69871	0.13213	0.19468	0.15536	0.18175	0.17417	0.07189	0.04307	0.00766	0.01284	0.02645	-3126.23825	-4061.85906	-2174.84729	0.00318
1000	0.00002	0.00044	0.01862	0.13715	0.42199	0.50434	0.53559	11.90501	19.46736	23.51882	0.11458	0.18190	0.14740	0.18451	0.19495	0.08296	0.04879	0.00669	0.01214	0.02608	-6117.35806	-7974.26747	-4491.33053	0.00000

Table D.183: 10-component exponential mixture on mid-price waiting time RRLN data.

n	$\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{4\ ave}$	$\hat{\lambda}_{5\ ave}$	$\hat{\lambda}_{6\ ave}$	$\hat{\lambda}_{7\ ave}$	$\hat{\lambda}_{8\ ave}$	$\hat{\lambda}_{9\ ave}$	$\hat{\lambda}_{10\ ave}$	$\hat{\pi}_{1\ ave}$	$\hat{\pi}_{2\ ave}$	$\hat{\pi}_{3\ ave}$	$\hat{\pi}_{4\ ave}$	$\hat{\pi}_{5\ ave}$	$\hat{\pi}_{6\ ave}$	$\hat{\pi}_{7\ ave}$	$\hat{\pi}_{8\ ave}$	$\hat{\pi}_{9\ ave}$	$\hat{\pi}_{10\ ave}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.05156	0.05288	0.08598	0.21063	0.60859	0.90950	1.69653	11.07343	30.69149	97.05568	0.19508	0.22086	0.21242	0.13987	0.09482	0.04113	0.03045	0.01298	0.01711	0.03528	-316.34563	-390.72595	-242.72358	0.00689
100	0.04915	0.05029	0.07706	0.18679	0.57386	0.78475	1.17861	11.18056	19.38563	260.51110	0.18410	0.22418	0.20257	0.15466	0.09627	0.04263	0.03073	0.01260	0.01736	0.03490	-639.45691	-772.56805	-513.15478	0.00578
200	0.04553	0.04651	0.06765	0.16318	0.57623	0.74703	0.89779	12.13300	17.43973	72.95172	0.17219	0.22376	0.18982	0.17468	0.09873	0.04463	0.03181	0.01234	0.01756	0.03448	-1287.30471	-1525.06216	-1062.17059	0.00602
500	0.04133	0.04204	0.05526	0.13107	0.56651	0.70822	0.73956	13.15554	15.95627	62.72006	0.15151	0.22210	0.17434	0.20254	0.10426	0.04788	0.03356	0.01177	0.01727	0.03477	-3233.00583	-3775.40789	-2749.21426	0.00201
1000	0.03359	0.03413	0.04353	0.10637	0.54511	0.67653	0.69908	13.84914	15.56971	22.20024	0.13732	0.21953	0.16389	0.21851	0.11086	0.05101	0.03545	0.01147	0.01699	0.03497	-6479.64315	-7465.66549	-5594.19459	0.00311

Table D.184: 10-component exponential mixture on mid-price waiting time VOD data.

n	$\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{4\ ave}$	$\hat{\lambda}_{5\ ave}$	$\hat{\lambda}_{6\ ave}$	$\hat{\lambda}_{7\ ave}$	$\hat{\lambda}_{8\ ave}$	$\hat{\lambda}_{9\ ave}$	$\hat{\lambda}_{10\ ave}$	$\hat{\pi}_{1\ ave}$	$\hat{\pi}_{2\ ave}$	$\hat{\pi}_{3\ ave}$	$\hat{\pi}_{4\ ave}$	$\hat{\pi}_{5\ ave}$	$\hat{\pi}_{6\ ave}$	$\hat{\pi}_{7\ ave}$	$\hat{\pi}_{8\ ave}$	$\hat{\pi}_{9\ ave}$	$\hat{\pi}_{10\ ave}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.06596	0.07017	0.13602	0.35416	0.65485	0.81983	1.29969	10.65149	24.39477	166.21271	0.16017	0.18188	0.19619	0.13254	0.13426	0.06161	0.04522	0.01721	0.02323	0.04768	-242.18512	-316.78621	-170.41143	0.00565
100	0.06096	0.06468	0.12011	0.33179	0.65129	0.76502	0.98900	11.20131	25.27899	158.20474	0.15197	0.18434	0.19037	0.13873	0.13727	0.06373	0.04613	0.01669	0.02353	0.04723	-490.38903	-624.19791	-364.36536	0.00922
200	0.05924	0.06251	0.10807	0.30301	0.64959	0.73302	0.83582	12.11998	20.68639	154.66397	0.14172	0.18494	0.18340	0.14838	0.14147	0.06601	0.04734	0.01609	0.02352	0.04714	-988.60026	-1232.97703	-767.55571	0.00613
500	0.05801	0.06048	0.09442	0.25672	0.64872	0.70424	0.74360	13.20454	17.57715	36.16786	0.12419	0.18221	0.17358	0.16458	0.15036	0.06973	0.04944	0.01540	0.02315	0.04738	-2486.38807	-3042.84277	-2004.48283	0.00159
1000	0.05402	0.05591	0.08201	0.21687	0.64295	0.68904	0.71392	13.83634	16.42849	19.80067	0.10928	0.17902	0.16523	0.17654	0.15931	0.07337	0.05142	0.01513	0.02298	0.04772	-4982.70059	-6029.63174	-4080.02271	0.00136

Table D.185: 10-component exponential mixture on mid-price waiting time RIOTINTO data.

n	$\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{4\ ave}$	$\hat{\lambda}_{5\ ave}$	$\hat{\lambda}_{6\ ave}$	$\hat{\lambda}_{7\ ave}$	$\hat{\lambda}_{8\ ave}$	$\hat{\lambda}_{9\ ave}$	$\hat{\lambda}_{10\ ave}$	$\hat{\pi}_{1\ ave}$	$\hat{\pi}_{2\ ave}$	$\hat{\pi}_{3\ ave}$	$\hat{\pi}_{4\ ave}$	$\hat{\pi}_{5\ ave}$	$\hat{\pi}_{6\ ave}$	$\hat{\pi}_{7\ ave}$	$\hat{\pi}_{8\ ave}$	$\hat{\pi}_{9\ ave}$	$\hat{\pi}_{10\ ave}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.03459	0.03509	0.05537	0.18144	0.55386	0.80678	1.68660	10.74811	35.69001	86.80974	0.22576	0.25637	0.18345	0.13843	0.08512	0.03476	0.02428	0.01023	0.01339	0.02821	-386.19372	-456.89675	-316.72013	0.00824
100	0.02985	0.03023	0.04421	0.15561	0.52234	0.72230	1.18689	10.74377	23.57807	79.96886	0.21914	0.25589	0.17842	0.14826	0.08756	0.03592	0.02440	0.00983	0.01347	0.02711	-780.60008	-902.07213	-662.77674	0.00419
200	0.02303	0.02337	0.03416	0.12265	0.49605	0.66257	0.82771	11.87712	17.30280	68.73973	0.20995	0.25143	0.17355	0.15617	0.09383	0.03889	0.02575	0.00963	0.01365	0.02715	-1569.08864	-1773.58313	-1369.34872	0.00558
500	0.01442	0.01466	0.02195	0.08238	0.45097	0.59897	0.68019	12.71225	15.38109	21.78183	0.18784	0.24417	0.17064	0.16453	0.10930	0.04503	0.02899	0.00914	0.01346	0.02689	-3934.72021	-4342.00073	-3560.30374	0.00000
1000	0.00001	0.00022	0.00582	0.05472	0.39110	0.57420	0.65792	14.20057	15.34743	21.92165	0.17001	0.23652	0.17509	0.16821	0.11262	0.05402	0.03449	0.00876	0.01322	0.02706	-7840.97866	-8491.66723	-7217.52983	0.00000

Table D.186: 10-component exponential mixture on mid-price waiting time SSELN data.

	AIC			BIC		
2-comp-exponential	0.0581	0.1024	0.2796	0.4605	0.7588	-
4-comp-exponential	0.5759	-	-	0.5094	-	-
6-comp-exponential	0.3362	0.8376	-	0.0295	0.2404	0.9981
10-comp-exponential	0.0297	0.0598	0.7202	0.0004	0.0006	0.0017

Table D.187: ABFLN, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.1940	0.3807	0.7592	0.7860	-	-
4-comp-exponential	0.6782	-	-	0.2136	0.9985	-
6-comp-exponential	0.1269	0.6127	-	0.0004	0.0015	0.9988
10-comp-exponential	0.0009	0.0066	0.2407	0.0000	0.0000	0.0011

Table D.188: BARC, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.1302	0.2194	0.4651	0.5938	-	-
4-comp-exponential	0.5778	-	-	0.3933	0.9681	-
6-comp-exponential	0.2825	0.7552	-	0.0127	0.0316	0.9980
10-comp-exponential	0.0094	0.0250	0.5346	0.0001	0.0001	0.0015

Table D.189: RRLN, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.1136	0.2210	0.5733	0.6551	-	-
4-comp-exponential	0.6638	-	-	0.3422	0.9913	-
6-comp-exponential	0.2203	0.7677	-	0.0027	0.0087	0.9991
10-comp-exponential	0.0023	0.0112	0.4266	0.0000	0.0000	0.0007

Table D.190: VOD, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.1780	0.3688	0.7582	0.7813	-	-
4-comp-exponential	0.6948	-	-	0.2183	0.9983	-
6-comp-exponential	0.1260	0.6231	-	0.0004	0.0017	0.9989
10-comp-exponential	0.0012	0.0082	0.2418	0.0000	0.0000	0.0009

Table D.191: RIOTINTO, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0464	0.0870	0.2766	0.4444	0.7729	-
4-comp-exponential	0.5729	-	-	0.5224	-	-
6-comp-exponential	0.3594	0.8656	-	0.0324	0.2261	0.9978
10-comp-exponential	0.0208	0.0467	0.7227	0.0003	0.0003	0.0012

Table D.192: SSELN, N = 50: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0144	0.0172	0.0453	0.1343	0.3381	0.8520
4-comp-exponential	0.3963	0.6743	-	0.7622	-	-
6-comp-exponential	0.5103	-	-	0.1027	0.6595	-
10-comp-exponential	0.0789	0.3085	0.9547	0.0008	0.0023	0.1480

Table D.193: ABFLN, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0389	0.0864	0.3077	0.4547	0.8383	-
4-comp-exponential	0.6720	-	-	0.5427	-	-
6-comp-exponential	0.2831	0.8902	-	0.0026	0.1617	0.9983
10-comp-exponential	0.0059	0.0234	0.6923	0.0000	0.0000	0.0014

Table D.194: BARC, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0322	0.0412	0.1357	0.2693	0.5193	-
4-comp-exponential	0.4402	0.7439	-	0.6674	-	-
6-comp-exponential	0.4966	-	-	0.0630	0.4802	0.9968
10-comp-exponential	0.0309	0.2149	0.8643	0.0003	0.0004	0.0028

Table D.195: RRLN, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0283	0.0447	0.1398	0.2938	0.6709	-
4-comp-exponential	0.5480	-	-	0.6886	-	-
6-comp-exponential	0.4081	0.9133	-	0.0176	0.3291	0.9986
10-comp-exponential	0.0155	0.0420	0.8602	0.0000	0.0000	0.0011

Table D.196: VOD, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0355	0.0819	0.2961	0.4339	0.8370	-
4-comp-exponential	0.6854	-	-	0.5634	-	-
6-comp-exponential	0.2717	0.8885	-	0.0026	0.1630	0.9983
10-comp-exponential	0.0075	0.0296	0.7039	0.0000	0.0000	0.0014

Table D.197: RIOTINTO, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0090	0.0132	0.0305	0.1156	0.3132	0.8707
4-comp-exponential	0.4147	0.6814	-	0.7704	-	-
6-comp-exponential	0.5165	-	-	0.1135	0.6859	-
10-comp-exponential	0.0599	0.3054	0.9695	0.0006	0.0009	0.1293

Table D.198: SSELN, N = 100: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0041	0.0049	0.0082	0.0205	0.0492	0.3328
4-comp-exponential	0.1795	0.3795	0.9918	0.7320	-	-
6-comp-exponential	0.6615	-	-	0.2443	0.9451	-
10-comp-exponential	0.1549	0.6156	-	0.0033	0.0057	0.6672

Table D.199: ABFLN, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0068	0.0084	0.0479	0.1366	0.4375	0.9146
4-comp-exponential	0.4788	0.8493	-	0.8463	-	-
6-comp-exponential	0.4859	-	-	0.0171	0.5624	-
10-comp-exponential	0.0286	0.1423	0.9521	0.0000	0.0000	0.0854

Table D.200: BARC, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0075	0.0093	0.0117	0.0722	0.1476	0.5677
4-comp-exponential	0.2413	0.4722	0.9883	0.7180	-	-
6-comp-exponential	0.6659	-	-	0.2090	0.8506	-
10-comp-exponential	0.0853	0.5186	-	0.0009	0.0018	0.4323

Table D.201: RRLN, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0131	0.0155	0.0321	0.0667	0.2030	0.7610
4-comp-exponential	0.3125	0.6300	-	0.8519	-	-
6-comp-exponential	0.5994	-	-	0.0815	0.7969	-
10-comp-exponential	0.0749	0.3545	0.9679	0.0000	0.0001	0.2390

Table D.202: VOD, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0071	0.0125	0.0501	0.1255	0.4177	0.9121
4-comp-exponential	0.5000	-	-	0.8597	-	-
6-comp-exponential	0.4609	0.9160	-	0.0149	0.5823	-
10-comp-exponential	0.0321	0.0715	0.9499	0.0000	0.0000	0.0879

Table D.203: RIOTINTO, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0025	0.0043	0.0062	0.0155	0.0378	0.3123
4-comp-exponential	0.1735	0.3705	0.9938	0.7305	-	-
6-comp-exponential	0.6890	-	-	0.2528	0.9554	-
10-comp-exponential	0.1351	0.6252	-	0.0012	0.0068	0.6877

Table D.204: SSELN, N = 200: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0000	0.0000	0.0024	0.0000	0.0000	0.0072
4-comp-exponential	0.0359	0.0885	0.9976	0.2656	0.8445	-
6-comp-exponential	0.7153	-	-	0.7225	-	-
10-comp-exponential	0.2488	0.9115	-	0.0120	0.1555	0.9928

Table D.205: ABFLN, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0032	0.0042	0.0042	0.0073	0.0276	0.3372
4-comp-exponential	0.1930	0.4505	0.9958	0.8383	-	-
6-comp-exponential	0.6714	-	-	0.1544	0.9722	-
10-comp-exponential	0.1324	0.5453	-	0.0000	0.0001	0.6628

Table D.206: BARC, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0008	0.0008	0.0024	0.0032	0.0072	0.0620
4-comp-exponential	0.0747	0.1566	0.9976	0.3736	0.9030	-
6-comp-exponential	0.7130	-	-	0.6200	-	-
10-comp-exponential	0.2114	0.8426	-	0.0032	0.0898	0.9380

Table D.207: RRLN, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0070	0.0075	0.0080	0.0150	0.0228	0.0935
4-comp-exponential	0.0684	0.1660	0.9920	0.5683	-	-
6-comp-exponential	0.6736	-	-	0.4166	0.9762	-
10-comp-exponential	0.2509	0.8265	-	0.0000	0.0010	0.9065

Table D.208: VOD, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0038	0.0055	0.0105	0.0101	0.0352	0.3243
4-comp-exponential	0.2171	0.4975	-	0.8571	-	-
6-comp-exponential	0.6459	-	-	0.1327	0.9647	-
10-comp-exponential	0.1333	0.4970	0.9895	0.0000	0.0001	0.6757

Table D.209: RIOTINTO, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0017	0.0017	0.0052	0.0035	0.0035	0.0035
4-comp-exponential	0.0157	0.0507	0.9948	0.2972	0.8881	-
6-comp-exponential	0.6853	-	-	0.6923	-	-
10-comp-exponential	0.2972	0.9476	-	0.0070	0.1084	0.9965

Table D.210: SSELN, N = 500: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0000	0.0000	0.0075	0.0000	0.0000	0.0075
4-comp-exponential	0.0000	0.0000	0.9925	0.0602	0.2857	0.9925
6-comp-exponential	0.5564	-	-	0.9098	-	-
10-comp-exponential	0.4436	1.0000	-	0.0301	0.7143	-

Table D.211: ABFLN, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0026	0.0031	0.0031	0.0037	0.0055	0.0225
4-comp-exponential	0.0604	0.1618	0.9969	0.5133	-	-
6-comp-exponential	0.6659	-	-	0.4825	0.9935	-
10-comp-exponential	0.2711	0.8351	-	0.0004	0.0010	0.9775

Table D.212: BARC, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0000	0.0000	0.0018	0.0018	0.0018	0.0036
4-comp-exponential	0.0250	0.0660	0.9982	0.1907	0.5205	-
6-comp-exponential	0.6007	-	-	0.7950	-	-
10-comp-exponential	0.3743	0.9340	-	0.0125	0.4777	0.9964

Table D.213: RRLN, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0031	0.0041	0.0047	0.0083	0.0088	0.0171
4-comp-exponential	0.0124	0.0347	0.9953	0.1957	0.8106	-
6-comp-exponential	0.5487	-	-	0.7945	-	-
10-comp-exponential	0.4358	0.9612	-	0.0016	0.1806	0.9829

Table D.214: VOD, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0038	0.0050	0.0051	0.0072	0.0085	0.0297
4-comp-exponential	0.0721	0.1986	0.9949	0.5643	-	-
6-comp-exponential	0.6634	-	-	0.4284	0.9907	-
10-comp-exponential	0.2607	0.7963	-	0.0001	0.0008	0.9703

Table D.215: RIOTINTO, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

	AIC			BIC		
2-comp-exponential	0.0000	0.0000	0.0045	0.0000	0.0000	0.0045
4-comp-exponential	0.0000	0.0000	0.9955	0.0493	0.3498	0.9955
6-comp-exponential	0.5022	-	-	0.9238	-	-
10-comp-exponential	0.4978	1.0000	-	0.0269	0.6502	-

Table D.216: SSELN, N = 1000: mid-price waiting times, zero inflated (exponentially distributed).

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.091425	1.252567	0.000018	0.000007	0.000068	0.100	1.252	0.003	0.001	0.016	0.726	2.031	0.271	0.089	0.918	0.353	0.159	0.351	0.315	0.128	0.308	0.332	0.166	0.309	-402.435	-476.374	-332.856	0.000
100	0.064813	1.082220	0.000016	0.000007	0.000052	0.072	1.082	0.002	0.000	0.013	0.561	1.594	0.235	0.080	0.783	0.343	0.140	0.345	0.317	0.107	0.311	0.340	0.146	0.326	-811.917	-940.653	-691.112	0.000
200	0.046851	0.943415	0.000015	0.000006	0.000046	0.053	0.943	0.002	0.000	0.011	0.452	1.379	0.204	0.073	0.647	0.330	0.123	0.326	0.316	0.091	0.309	0.354	0.128	0.344	-1631.499	-1855.914	-1430.000	0.000
500	0.000025	0.000040	0.000014	0.000007	0.000037	0.004	0.008	0.001	0.000	0.009	0.300	0.418	0.173	0.077	0.506	0.316	0.100	0.304	0.315	0.070	0.306	0.369	0.098	0.364	-4078.312	-4531.438	-3677.984	0.000
1000	0.000026	0.000032	0.000016	0.000009	0.000038	0.004	0.006	0.001	0.000	0.008	0.309	0.361	0.195	0.086	0.538	0.306	0.097	0.305	0.313	0.067	0.311	0.381	0.091	0.373	-7947.279	-8774.127	-7085.675	0.000

Table D.217: 3-component exponential mixture on mid-price waiting time ABFLN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.096524	1.264367	0.000440	0.000179	0.001275	0.138	1.263	0.026	0.008	0.086	1.168	2.533	0.541	0.218	1.487	0.255	0.148	0.243	0.303	0.133	0.295	0.442	0.201	0.427	-259.258	-331.757	-184.068	0.000
100	0.097065	1.314782	0.000403	0.000181	0.000956	0.134	1.313	0.024	0.008	0.072	0.941	1.933	0.521	0.225	1.349	0.254	0.132	0.244	0.296	0.115	0.290	0.449	0.183	0.440	-524.785	-655.172	-392.193	0.000
200	0.099302	1.363776	0.000384	0.000184	0.000808	0.132	1.362	0.023	0.008	0.063	0.828	1.626	0.499	0.229	1.256	0.254	0.120	0.244	0.289	0.100	0.283	0.457	0.170	0.455	-1057.632	-1297.004	-814.541	0.001
500	0.098975	1.400181	0.000369	0.000190	0.000709	0.128	1.398	0.022	0.009	0.053	0.763	1.507	0.484	0.232	1.165	0.255	0.108	0.243	0.281	0.088	0.275	0.464	0.159	0.473	-2658.396	-3202.160	-2097.384	0.002
1000	0.096866	1.412820	0.000362	0.000193	0.000663	0.124	1.411	0.021	0.009	0.048	0.735	1.487	0.469	0.236	1.118	0.254	0.101	0.240	0.276	0.083	0.269	0.470	0.153	0.486	-5326.843	-6382.365	-4262.027	0.002

Table D.218: 3-component exponential mixture on mid-price waiting time BARC data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.069098	1.074904	0.000042	0.000014	0.000249	0.084	1.085	0.004	0.001	0.027	0.674	1.722	0.323	0.148	0.855	0.260	0.174	0.242	0.266	0.137	0.254	0.474	0.239	0.442	-320.833	-432.016	-190.552	0.000
100	0.062864	1.060008	0.000036	0.000013	0.000137	0.075	1.060	0.003	0.001	0.023	0.556	1.367	0.304	0.150	0.743	0.255	0.163	0.240	0.263	0.120	0.256	0.482	0.226	0.453	-647.648	-855.617	-413.904	0.000
200	0.051767	0.987209	0.000032	0.000013	0.000098	0.062	0.987	0.003	0.001	0.018	0.472	1.132	0.291	0.155	0.626	0.250	0.154	0.234	0.258	0.106	0.251	0.492	0.216	0.463	-1299.856	-1695.714	-885.264	0.000
500	0.015996	0.565513	0.000027	0.000013	0.000076	0.023	0.566	0.002	0.001	0.013	0.382	0.715	0.270	0.154	0.489	0.234	0.140	0.210	0.251	0.092	0.247	0.515	0.204	0.504	-3224.430	-4172.507	-2268.331	0.000
1000	0.000041	0.000053	0.000025	0.000012	0.000058	0.005	0.012	0.002	0.001	0.009	0.323	0.347	0.259	0.142	0.416	0.213	0.126	0.190	0.245	0.087	0.239	0.542	0.190	0.558	-6321.717	-8198.213	-4649.818	0.000

Table D.219: 3-component exponential mixture on mid-price waiting time RRLN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\text{ ave}}$	$\pm\hat{\lambda}_{1\text{ ave}}$	$\hat{\lambda}_{1\text{ median}}$	$\hat{\lambda}_{1\text{ lower}}$	$\hat{\lambda}_{1\text{ upper}}$	$\hat{\lambda}_{2\text{ ave}}$	$\pm\hat{\lambda}_{2\text{ ave}}$	$\hat{\lambda}_{2\text{ median}}$	$\hat{\lambda}_{2\text{ lower}}$	$\hat{\lambda}_{2\text{ upper}}$	$\hat{\lambda}_{3\text{ ave}}$	$\pm\hat{\lambda}_{3\text{ ave}}$	$\hat{\lambda}_{3\text{ median}}$	$\hat{\lambda}_{3\text{ lower}}$	$\hat{\lambda}_{3\text{ upper}}$	$\hat{\pi}_{1\text{ ave}}$	$\pm\hat{\pi}_{1\text{ ave}}$	$\hat{\pi}_{1\text{ median}}$	$\hat{\pi}_{2\text{ ave}}$	$\pm\hat{\pi}_{2\text{ ave}}$	$\hat{\pi}_{2\text{ median}}$	$\hat{\pi}_{3\text{ ave}}$	$\pm\hat{\pi}_{3\text{ ave}}$	$\hat{\pi}_{3\text{ median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.123801	1.453891	0.000135	0.000053	0.000409	0.146	1.452	0.013	0.003	0.044	1.180	2.664	0.462	0.137	1.567	0.326	0.161	0.323	0.328	0.141	0.315	0.346	0.180	0.324	-325.869	-398.105	-256.070	0.000
100	0.124613	1.498476	0.000125	0.000054	0.000327	0.145	1.497	0.012	0.003	0.040	0.933	2.072	0.434	0.135	1.411	0.325	0.145	0.321	0.323	0.125	0.309	0.352	0.164	0.337	-658.582	-787.508	-538.136	0.000
200	0.126239	1.542894	0.000119	0.000053	0.000281	0.144	1.542	0.012	0.002	0.036	0.814	1.800	0.403	0.130	1.323	0.323	0.133	0.320	0.317	0.111	0.306	0.360	0.153	0.348	-1326.010	-1559.442	-1113.563	0.000
500	0.121003	1.551779	0.000112	0.000052	0.000247	0.137	1.551	0.011	0.002	0.032	0.725	1.609	0.368	0.122	1.249	0.319	0.119	0.315	0.312	0.096	0.302	0.370	0.140	0.359	-3334.151	-3860.977	-2865.703	0.002
1000	0.097326	1.419386	0.000107	0.000051	0.000218	0.112	1.418	0.010	0.002	0.029	0.672	1.555	0.328	0.114	1.166	0.313	0.110	0.307	0.306	0.082	0.299	0.381	0.135	0.369	-6685.573	-7644.661	-5829.999	0.002

Table D.220: 3-component exponential mixture on mid-price waiting time VOD data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\lambda}_{3\ \text{ave}}$	$\pm\hat{\lambda}_{3\ \text{ave}}$	$\hat{\lambda}_{3\ \text{median}}$	$\hat{\lambda}_{3\ \text{lower}}$	$\hat{\lambda}_{3\ \text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.150933	1.581379	0.000582	0.000225	0.001685	0.192	1.579	0.025	0.008	0.081	1.302	2.747	0.614	0.234	1.635	0.252	0.141	0.243	0.296	0.128	0.291	0.451	0.194	0.437	-252.223	-322.933	-185.448	0.000
100	0.152368	1.646100	0.000537	0.000226	0.001318	0.189	1.644	0.023	0.008	0.068	1.083	2.242	0.596	0.240	1.485	0.252	0.123	0.245	0.291	0.108	0.288	0.457	0.173	0.447	-510.634	-637.613	-395.640	0.000
200	0.155827	1.706101	0.000511	0.000227	0.001158	0.188	1.704	0.022	0.008	0.056	0.978	2.011	0.585	0.240	1.391	0.252	0.109	0.247	0.285	0.093	0.283	0.463	0.157	0.455	-1029.140	-1260.521	-823.695	0.000
500	0.160760	1.782406	0.000488	0.000230	0.001031	0.188	1.780	0.021	0.008	0.047	0.916	1.941	0.572	0.239	1.282	0.251	0.094	0.247	0.278	0.081	0.276	0.471	0.143	0.465	-2587.332	-3113.243	-2130.879	0.001
1000	0.160748	1.817642	0.000475	0.000230	0.000971	0.186	1.816	0.020	0.009	0.041	0.888	1.926	0.565	0.240	1.229	0.250	0.086	0.246	0.275	0.077	0.273	0.475	0.136	0.471	-5185.623	-6165.625	-4327.002	0.002

Table D.221: 3-component exponential mixture on mid-price waiting time RIOTINTO data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\lambda}_{3\ \text{ave}}$	$\pm\hat{\lambda}_{3\ \text{ave}}$	$\hat{\lambda}_{3\ \text{median}}$	$\hat{\lambda}_{3\ \text{lower}}$	$\hat{\lambda}_{3\ \text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.081372	1.181023	0.000021	0.000009	0.000066	0.095	1.218	0.003	0.001	0.016	0.688	1.845	0.261	0.092	0.952	0.343	0.153	0.341	0.321	0.130	0.312	0.336	0.159	0.316	-397.150	-465.916	-332.035	0.000
100	0.077147	1.181469	0.000019	0.000008	0.000050	0.083	1.181	0.002	0.001	0.012	0.546	1.529	0.223	0.085	0.787	0.332	0.135	0.331	0.322	0.109	0.313	0.346	0.142	0.333	-802.078	-920.227	-690.386	0.000
200	0.059151	1.058461	0.000017	0.000008	0.000043	0.064	1.058	0.001	0.000	0.009	0.446	1.367	0.195	0.084	0.612	0.323	0.117	0.318	0.322	0.094	0.318	0.355	0.124	0.349	-1613.498	-1809.791	-1427.951	0.000
500	0.035089	0.838660	0.000015	0.000009	0.000036	0.039	0.839	0.001	0.000	0.006	0.365	1.231	0.156	0.080	0.477	0.306	0.096	0.300	0.321	0.074	0.314	0.373	0.102	0.371	-4050.071	-4439.266	-3688.746	0.000
1000	0.000022	0.000026	0.000014	0.000010	0.000028	0.003	0.005	0.001	0.000	0.003	0.277	0.556	0.142	0.090	0.312	0.290	0.079	0.282	0.327	0.061	0.327	0.383	0.088	0.380	-8079.845	-8691.176	-7497.679	0.000

Table D.222: 3-component exponential mixture on mid-price waiting time SSELN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\lambda}_{3\ \text{ave}}$	$\pm\hat{\lambda}_{3\ \text{ave}}$	$\hat{\lambda}_{3\ \text{median}}$	$\hat{\lambda}_{3\ \text{lower}}$	$\hat{\lambda}_{3\ \text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.091362	1.252756	0.000018	0.000007	0.000068	0.100	1.252	0.003	0.001	0.016	0.792	2.042	0.313	0.091	0.938	0.354	0.158	0.351	0.316	0.128	0.309	0.330	0.167	0.307	-394.079	-470.152	-322.613	0.000
100	0.064882	1.082854	0.000017	0.000007	0.000052	0.072	1.083	0.002	0.000	0.013	0.567	1.390	0.264	0.081	0.818	0.344	0.139	0.346	0.316	0.108	0.309	0.340	0.145	0.325	-794.053	-927.163	-669.157	0.001
200	0.046851	0.943415	0.000015	0.000006	0.000046	0.053	0.943	0.002	0.000	0.011	0.480	1.318	0.232	0.075	0.712	0.333	0.123	0.327	0.316	0.092	0.309	0.352	0.129	0.341	-1594.369	-1827.630	-1379.354	0.000
500	0.000026	0.000039	0.000014	0.000007	0.000037	0.004	0.007	0.001	0.000	0.008	0.336	0.463	0.190	0.076	0.616	0.318	0.102	0.310	0.314	0.072	0.305	0.368	0.100	0.359	-3981.394	-4439.512	-3511.365	0.000
1000	0.000026	0.000032	0.000016	0.000009	0.000038	0.004	0.006	0.001	0.000	0.008	0.344	0.405	0.205	0.086	0.626	0.306	0.096	0.304	0.314	0.068	0.312	0.379	0.091	0.374	-7761.330	-8619.562	-6871.635	0.000

Table D.223: 3-component exponential mixture on mid-price waiting time ABFLN data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\lambda}_{3\ \text{ave}}$	$\pm\hat{\lambda}_{3\ \text{ave}}$	$\hat{\lambda}_{3\ \text{median}}$	$\hat{\lambda}_{3\ \text{lower}}$	$\hat{\lambda}_{3\ \text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	$\log L_{\text{ave}}$	$\log L_{\text{lower}}$	$\log L_{\text{upper}}$	perc-non-convergence
50	0.097404	1.266605	0.000436	0.000178	0.001265	0.138	1.266	0.025	0.008	0.081	1.080	1.890	0.550	0.228	1.703	0.253	0.147	0.241	0.298	0.136	0.289	0.450	0.205	0.435	-245.651	-320.807	-166.787	0.001
100	0.097311	1.315710	0.000399	0.000180	0.000945	0.132	1.314	0.022	0.008	0.067	0.919	1.570	0.528	0.238	1.473	0.252	0.131	0.242	0.289	0.116	0.282	0.459	0.186	0.449	-495.760	-630.966	-356.475	0.001
200	0.099314	1.363923	0.000380	0.000184	0.000795	0.130	1.362	0.021	0.008	0.056	0.837	1.494	0.513	0.243	1.319	0.252	0.119	0.242	0.281	0.100	0.274	0.467	0.171	0.465	-996.280	-1245.270	-741.319	0.001
500	0.098948	1.400268	0.000365	0.000190	0.000691	0.125	1.399	0.020	0.009	0.046	0.773	1.484	0.500	0.249	1.068	0.251	0.106	0.241	0.271	0.087	0.264	0.477	0.158	0.483	-2497.546	-3069.221	-1913.870	0.002
1000	0.096855	1.412908	0.000359	0.000194	0.000640	0.121	1.411	0.019	0.010	0.041	0.738	1.477	0.492	0.256	0.924	0.251	0.099	0.239	0.266	0.081	0.257	0.483	0.150	0.495	-4996.085	-6105.190	-3877.189	0.002

Table D.224: 3-component exponential mixture on mid-price waiting time BARC data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\lambda}_{3\ ave}$	$\pm\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{3\ median}$	$\hat{\lambda}_{3\ lower}$	$\hat{\lambda}_{3\ upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.069507	1.075855	0.000042	0.000014	0.000250	0.085	1.086	0.004	0.001	0.028	0.721	1.677	0.357	0.152	0.892	0.261	0.175	0.243	0.266	0.138	0.254	0.473	0.240	0.438	-308.874	-425.265	-171.747	0.001
100	0.062906	1.060204	0.000036	0.000013	0.000136	0.075	1.060	0.004	0.001	0.022	0.589	1.293	0.336	0.154	0.769	0.256	0.163	0.241	0.263	0.120	0.255	0.481	0.227	0.452	-622.632	-840.444	-378.546	0.000
200	0.051760	0.987355	0.000032	0.000013	0.000099	0.062	0.987	0.003	0.001	0.018	0.502	1.102	0.322	0.160	0.675	0.251	0.154	0.234	0.258	0.106	0.250	0.491	0.217	0.463	-1246.948	-1659.007	-807.459	0.000
500	0.015996	0.565513	0.000028	0.000013	0.000075	0.023	0.565	0.002	0.001	0.013	0.412	0.747	0.288	0.158	0.591	0.235	0.140	0.214	0.251	0.093	0.244	0.514	0.204	0.504	-3087.665	-4077.858	-2092.514	0.000
1000	0.000042	0.000052	0.000026	0.000012	0.000059	0.006	0.011	0.002	0.001	0.010	0.355	0.397	0.276	0.148	0.520	0.216	0.127	0.193	0.246	0.088	0.237	0.539	0.192	0.555	-6047.652	-8029.274	-4277.866	0.000

Table D.225: 3-component exponential mixture on mid-price waiting time RRLN data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\lambda}_{3\ ave}$	$\pm\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{3\ median}$	$\hat{\lambda}_{3\ lower}$	$\hat{\lambda}_{3\ upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.124157	1.454927	0.000135	0.000053	0.000407	0.146	1.453	0.013	0.003	0.044	1.163	2.269	0.524	0.146	1.828	0.326	0.161	0.323	0.328	0.141	0.314	0.347	0.182	0.325	-316.082	-387.737	-246.744	0.001
100	0.124709	1.499172	0.000125	0.000054	0.000326	0.144	1.498	0.012	0.003	0.039	0.937	1.783	0.504	0.145	1.567	0.325	0.144	0.322	0.322	0.125	0.308	0.353	0.165	0.338	-637.146	-764.506	-515.669	0.001
200	0.126316	1.543424	0.000119	0.000054	0.000279	0.144	1.542	0.012	0.002	0.034	0.832	1.674	0.487	0.139	1.378	0.324	0.132	0.321	0.316	0.111	0.303	0.361	0.152	0.349	-1280.177	-1511.019	-1066.239	0.001
500	0.120968	1.551585	0.000113	0.000053	0.000247	0.136	1.550	0.011	0.002	0.029	0.750	1.626	0.468	0.133	1.152	0.321	0.117	0.318	0.309	0.095	0.299	0.370	0.137	0.361	-3210.185	-3726.998	-2731.227	0.001
1000	0.097428	1.420119	0.000108	0.000052	0.000219	0.111	1.419	0.010	0.002	0.027	0.682	1.508	0.448	0.121	0.945	0.316	0.108	0.312	0.303	0.080	0.296	0.381	0.130	0.370	-6426.152	-7404.104	-5545.849	0.003

Table D.226: 3-component exponential mixture on mid-price waiting time VOD data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\lambda}_{3\ ave}$	$\pm\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{3\ median}$	$\hat{\lambda}_{3\ lower}$	$\hat{\lambda}_{3\ upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.153144	1.586497	0.000576	0.000223	0.001667	0.192	1.584	0.023	0.008	0.077	1.207	2.093	0.612	0.246	1.907	0.250	0.141	0.239	0.291	0.130	0.286	0.459	0.197	0.445	-239.769	-311.529	-170.121	0.001
100	0.153082	1.647594	0.000529	0.000224	0.001293	0.186	1.645	0.021	0.008	0.062	1.060	1.859	0.590	0.254	1.702	0.249	0.123	0.241	0.285	0.109	0.281	0.466	0.175	0.458	-483.844	-612.523	-365.428	0.001
200	0.156168	1.706731	0.000505	0.000226	0.001132	0.185	1.705	0.020	0.008	0.051	0.989	1.837	0.575	0.257	1.561	0.249	0.109	0.242	0.278	0.093	0.275	0.474	0.158	0.468	-972.427	-1206.173	-761.406	0.001
500	0.160894	1.782537	0.000484	0.000229	0.000997	0.186	1.781	0.019	0.008	0.040	0.932	1.868	0.563	0.257	1.384	0.248	0.094	0.243	0.270	0.080	0.268	0.483	0.143	0.480	-2438.236	-2969.442	-1973.058	0.001
1000	0.160804	1.817672	0.000470	0.000230	0.000937	0.183	1.816	0.018	0.009	0.035	0.901	1.883	0.559	0.258	1.259	0.246	0.086	0.241	0.266	0.074	0.263	0.488	0.134	0.486	-4880.708	-5871.829	-4007.538	0.002

Table D.227: 3-component exponential mixture on mid-price waiting time RIOTINTO data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\ ave}$	$\pm\hat{\lambda}_{1\ ave}$	$\hat{\lambda}_{1\ median}$	$\hat{\lambda}_{1\ lower}$	$\hat{\lambda}_{1\ upper}$	$\hat{\lambda}_{2\ ave}$	$\pm\hat{\lambda}_{2\ ave}$	$\hat{\lambda}_{2\ median}$	$\hat{\lambda}_{2\ lower}$	$\hat{\lambda}_{2\ upper}$	$\hat{\lambda}_{3\ ave}$	$\pm\hat{\lambda}_{3\ ave}$	$\hat{\lambda}_{3\ median}$	$\hat{\lambda}_{3\ lower}$	$\hat{\lambda}_{3\ upper}$	$\hat{\pi}_{1\ ave}$	$\pm\hat{\pi}_{1\ ave}$	$\hat{\pi}_{1\ median}$	$\hat{\pi}_{2\ ave}$	$\pm\hat{\pi}_{2\ ave}$	$\hat{\pi}_{2\ median}$	$\hat{\pi}_{3\ ave}$	$\pm\hat{\pi}_{3\ ave}$	$\hat{\pi}_{3\ median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.081732	1.181415	0.000021	0.000009	0.000066	0.095	1.218	0.003	0.001	0.016	0.727	1.748	0.305	0.093	0.971	0.345	0.153	0.342	0.321	0.130	0.312	0.334	0.160	0.314	-389.487	-459.859	-322.119	0.001
100	0.077263	1.181514	0.000019	0.000008	0.000050	0.083	1.181	0.002	0.001	0.012	0.577	1.445	0.265	0.086	0.801	0.333	0.135	0.332	0.323	0.110	0.315	0.344	0.142	0.330	-785.850	-905.830	-668.600	0.000
200	0.059137	1.058459	0.000017	0.000008	0.000044	0.064	1.058	0.001	0.001	0.010	0.479	1.274	0.221	0.084	0.711	0.326	0.117	0.320	0.323	0.094	0.317	0.351	0.125	0.344	-1579.384	-1784.670	-1390.212	0.000
500	0.035089	0.838660	0.000015	0.000009	0.000036	0.039	0.839	0.001	0.000	0.006	0.388	1.064	0.163	0.080	0.618	0.309	0.097	0.302	0.322	0.073	0.314	0.370	0.103	0.366	-3962.807	-4357.245	-3570.220	0.000
1000	0.000022	0.000026	0.000014	0.000010	0.000030	0.003	0.005	0.001	0.000	0.004	0.321	0.635	0.149	0.091	0.513	0.295	0.081	0.284	0.328	0.061	0.328	0.377	0.089	0.373	-7920.282	-8525.040	-7228.789	0.004

Table D.228: 3-component exponential mixture on mid-price waiting time SSELN data with censoring on regions [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\,\text{ave}}$	$\pm\hat{\lambda}_{1\,\text{ave}}$	$\hat{\lambda}_{1\,\text{median}}$	$\hat{\lambda}_{1\,\text{lower}}$	$\hat{\lambda}_{1\,\text{upper}}$	$\hat{\lambda}_{2\,\text{ave}}$	$\pm\hat{\lambda}_{2\,\text{ave}}$	$\hat{\lambda}_{2\,\text{median}}$	$\hat{\lambda}_{2\,\text{lower}}$	$\hat{\lambda}_{2\,\text{upper}}$	$\hat{\lambda}_{3\,\text{ave}}$	$\pm\hat{\lambda}_{3\,\text{ave}}$	$\hat{\lambda}_{3\,\text{median}}$	$\hat{\lambda}_{3\,\text{lower}}$	$\hat{\lambda}_{3\,\text{upper}}$	$\hat{\lambda}_{4\,\text{ave}}$	$\pm\hat{\lambda}_{4\,\text{ave}}$	$\hat{\lambda}_{4\,\text{median}}$	$\hat{\lambda}_{4\,\text{lower}}$	$\hat{\lambda}_{4\,\text{upper}}$	$\hat{\pi}_{1\,\text{ave}}$	$\pm\hat{\pi}_{1\,\text{ave}}$	$\hat{\pi}_{1\,\text{median}}$	$\hat{\pi}_{2\,\text{ave}}$	$\pm\hat{\pi}_{2\,\text{ave}}$	$\hat{\pi}_{2\,\text{median}}$	$\hat{\pi}_{3\,\text{ave}}$	$\pm\hat{\pi}_{3\,\text{ave}}$	$\hat{\pi}_{3\,\text{median}}$	$\hat{\pi}_{4\,\text{ave}}$	$\pm\hat{\pi}_{4\,\text{ave}}$	$\hat{\pi}_{4\,\text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.092717	1.261563	0.000014	0.000005	0.000051	0.094	1.261	0.000	0.000	0.002	0.144	1.304	0.028	0.011	0.089	1.777	3.625	0.657	0.226	1.959	0.256	0.137	0.250	0.273	0.127	0.265	0.244	0.110	0.233	0.227	0.157	0.195	-397.629	-470.308	-321.946	0.014
100	0.065939	1.091611	0.000012	0.000005	0.000037	0.067	1.092	0.000	0.000	0.001	0.110	1.148	0.023	0.010	0.071	1.352	3.012	0.567	0.223	1.622	0.246	0.120	0.241	0.273	0.105	0.271	0.240	0.093	0.229	0.241	0.143	0.213	-800.948	-928.773	-668.294	0.017
200	0.047582	0.950837	0.000011	0.000005	0.000029	0.048	0.951	0.000	0.000	0.001	0.094	1.096	0.019	0.009	0.056	0.951	2.155	0.512	0.213	1.307	0.234	0.102	0.231	0.277	0.091	0.275	0.236	0.077	0.228	0.253	0.126	0.231	-1609.315	-1831.295	-1391.266	0.016
500	0.000017	0.000029	0.000010	0.000005	0.000024	0.000	0.001	0.000	0.000	0.001	0.024	0.024	0.018	0.010	0.040	0.648	1.025	0.430	0.211	1.103	0.221	0.081	0.219	0.277	0.069	0.276	0.233	0.060	0.234	0.269	0.098	0.252	-4010.932	-4444.498	-3542.128	0.019
1000	0.000017	0.000019	0.000012	0.000007	0.000024	0.000	0.000	0.000	0.000	0.001	0.022	0.015	0.017	0.011	0.033	0.587	0.459	0.469	0.226	0.975	0.209	0.076	0.205	0.274	0.065	0.273	0.233	0.050	0.233	0.284	0.085	0.266	-7824.731	-8677.530	-6891.732	0.030

Table D.229: 4-component exponential mixture on mid-price waiting time ABFLN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\,\text{ave}}$	$\pm\hat{\lambda}_{1\,\text{ave}}$	$\hat{\lambda}_{1\,\text{median}}$	$\hat{\lambda}_{1\,\text{lower}}$	$\hat{\lambda}_{1\,\text{upper}}$	$\hat{\lambda}_{2\,\text{ave}}$	$\pm\hat{\lambda}_{2\,\text{ave}}$	$\hat{\lambda}_{2\,\text{median}}$	$\hat{\lambda}_{2\,\text{lower}}$	$\hat{\lambda}_{2\,\text{upper}}$	$\hat{\lambda}_{3\,\text{ave}}$	$\pm\hat{\lambda}_{3\,\text{ave}}$	$\hat{\lambda}_{3\,\text{median}}$	$\hat{\lambda}_{3\,\text{lower}}$	$\hat{\lambda}_{3\,\text{upper}}$	$\hat{\lambda}_{4\,\text{ave}}$	$\pm\hat{\lambda}_{4\,\text{ave}}$	$\hat{\lambda}_{4\,\text{median}}$	$\hat{\lambda}_{4\,\text{lower}}$	$\hat{\lambda}_{4\,\text{upper}}$	$\hat{\pi}_{1\,\text{ave}}$	$\pm\hat{\pi}_{1\,\text{ave}}$	$\hat{\pi}_{1\,\text{median}}$	$\hat{\pi}_{2\,\text{ave}}$	$\pm\hat{\pi}_{2\,\text{ave}}$	$\hat{\pi}_{2\,\text{median}}$	$\hat{\pi}_{3\,\text{ave}}$	$\pm\hat{\pi}_{3\,\text{ave}}$	$\hat{\pi}_{3\,\text{median}}$	$\hat{\pi}_{4\,\text{ave}}$	$\pm\hat{\pi}_{4\,\text{ave}}$	$\hat{\pi}_{4\,\text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.097569	1.272208	0.000367	0.000146	0.001188	0.111	1.271	0.008	0.002	0.027	0.235	1.270	0.109	0.047	0.238	2.172	4.083	0.844	0.337	2.179	0.204	0.135	0.187	0.199	0.126	0.182	0.283	0.133	0.273	0.314	0.189	0.278	-257.698	-329.599	-178.563	0.013
100	0.098362	1.324506	0.000321	0.000139	0.000829	0.109	1.324	0.007	0.002	0.021	0.220	1.321	0.100	0.046	0.201	1.777	3.495	0.832	0.370	1.910	0.195	0.119	0.182	0.200	0.111	0.188	0.279	0.115	0.272	0.326	0.177	0.297	-521.325	-650.325	-380.375	0.015
200	0.100772	1.374796	0.000292	0.000133	0.000663	0.109	1.374	0.005	0.002	0.016	0.205	1.370	0.091	0.042	0.166	1.364	2.659	0.797	0.385	1.704	0.185	0.106	0.175	0.199	0.099	0.188	0.274	0.097	0.269	0.342	0.165	0.318	-1050.222	-1286.532	-790.314	0.017
500	0.100901	1.414675	0.000264	0.000127	0.000557	0.107	1.414	0.004	0.001	0.012	0.186	1.410	0.080	0.036	0.137	1.072	1.887	0.737	0.384	1.567	0.172	0.091	0.161	0.195	0.088	0.186	0.270	0.078	0.266	0.363	0.152	0.348	-2637.142	-3176.440	-2029.638	0.022
1000	0.098832	1.428019	0.000248	0.000125	0.000510	0.103	1.428	0.003	0.001	0.010	0.175	1.424	0.073	0.033	0.123	0.988	1.689	0.698	0.371	1.515	0.164	0.082	0.154	0.194	0.081	0.184	0.267	0.067	0.265	0.376	0.143	0.370	-5281.976	-6320.345	-4118.059	0.024

Table D.230: 4-component exponential mixture on mid-price waiting time BARC data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\,\text{ave}}$	$\pm\hat{\lambda}_{1\,\text{ave}}$	$\hat{\lambda}_{1\,\text{median}}$	$\hat{\lambda}_{1\,\text{lower}}$	$\hat{\lambda}_{1\,\text{upper}}$	$\hat{\lambda}_{2\,\text{ave}}$	$\pm\hat{\lambda}_{2\,\text{ave}}$	$\hat{\lambda}_{2\,\text{median}}$	$\hat{\lambda}_{2\,\text{lower}}$	$\hat{\lambda}_{2\,\text{upper}}$	$\hat{\lambda}_{3\,\text{ave}}$	$\pm\hat{\lambda}_{3\,\text{ave}}$	$\hat{\lambda}_{3\,\text{median}}$	$\hat{\lambda}_{3\,\text{lower}}$	$\hat{\lambda}_{3\,\text{upper}}$	$\hat{\lambda}_{4\,\text{ave}}$	$\pm\hat{\lambda}_{4\,\text{ave}}$	$\hat{\lambda}_{4\,\text{median}}$	$\hat{\lambda}_{4\,\text{lower}}$	$\hat{\lambda}_{4\,\text{upper}}$	$\hat{\pi}_{1\,\text{ave}}$	$\pm\hat{\pi}_{1\,\text{ave}}$	$\hat{\pi}_{1\,\text{median}}$	$\hat{\pi}_{2\,\text{ave}}$	$\pm\hat{\pi}_{2\,\text{ave}}$	$\hat{\pi}_{2\,\text{median}}$	$\hat{\pi}_{3\,\text{ave}}$	$\pm\hat{\pi}_{3\,\text{ave}}$	$\hat{\pi}_{3\,\text{median}}$	$\hat{\pi}_{4\,\text{ave}}$	$\pm\hat{\pi}_{4\,\text{ave}}$	$\hat{\pi}_{4\,\text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.069647	1.081040	0.000032	0.000011	0.000244	0.072	1.091	0.001	0.000	0.003	0.149	1.103	0.053	0.017	0.143	1.596	3.345	0.622	0.293	1.723	0.189	0.146	0.166	0.213	0.135	0.195	0.251	0.120	0.235	0.348	0.222	0.294	-316.900	-426.874	-183.514	0.012
100	0.063542	1.066413	0.000026	0.000010	0.000107	0.065	1.066	0.001	0.000	0.002	0.126	1.065	0.045	0.016	0.112	1.204	2.615	0.575	0.300	1.462	0.180	0.130	0.159	0.214	0.122	0.198	0.240	0.100	0.231	0.367	0.213	0.319	-639.197	-844.673	-400.005	0.012
200	0.052683	0.996479	0.000023	0.000010	0.000073	0.054	0.996	0.001	0.000	0.002	0.105	0.995	0.040	0.015	0.094	0.923	1.951	0.533	0.304	1.238	0.172	0.117	0.152	0.213	0.107	0.202	0.232	0.083	0.226	0.384	0.204	0.338	-1280.843	-1674.906	-846.655	0.019
500	0.016101	0.567523	0.000020	0.000010	0.000050	0.017	0.568	0.001	0.000	0.001	0.059	0.567	0.034	0.014	0.075	0.692	1.278	0.469	0.298	1.006	0.158	0.099	0.138	0.211	0.093	0.204	0.222	0.066	0.221	0.409	0.190	0.386	-3177.348	-4110.986	-2216.751	0.008
1000	0.000028	0.000028	0.000019	0.000010	0.000044	0.001	0.001	0.001	0.000	0.001	0.040	0.027	0.033	0.014	0.072	0.576	0.450	0.441	0.302	0.879	0.145	0.085	0.123	0.207	0.085	0.194	0.216	0.056	0.216	0.432	0.179	0.446	-6217.921	-8072.588	-4547.205	0.009

Table D.231: 4-component exponential mixture on mid-price waiting time RRLN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\,\text{ave}}$	$\pm\hat{\lambda}_{1\,\text{ave}}$	$\hat{\lambda}_{1\,\text{median}}$	$\hat{\lambda}_{1\,\text{lower}}$	$\hat{\lambda}_{1\,\text{upper}}$	$\hat{\lambda}_{2\,\text{ave}}$	$\pm\hat{\lambda}_{2\,\text{ave}}$	$\hat{\lambda}_{2\,\text{median}}$	$\hat{\lambda}_{2\,\text{lower}}$	$\hat{\lambda}_{2\,\text{upper}}$	$\hat{\lambda}_{3\,\text{ave}}$	$\pm\hat{\lambda}_{3\,\text{ave}}$	$\hat{\lambda}_{3\,\text{median}}$	$\hat{\lambda}_{3\,\text{lower}}$	$\hat{\lambda}_{3\,\text{upper}}$	$\hat{\lambda}_{4\,\text{ave}}$	$\pm\hat{\lambda}_{4\,\text{ave}}$	$\hat{\lambda}_{4\,\text{median}}$	$\hat{\lambda}_{4\,\text{lower}}$	$\hat{\lambda}_{4\,\text{upper}}$	$\hat{\pi}_{1\,\text{ave}}$	$\pm\hat{\pi}_{1\,\text{ave}}$	$\hat{\pi}_{1\,\text{median}}$	$\hat{\pi}_{2\,\text{ave}}$	$\pm\hat{\pi}_{2\,\text{ave}}$	$\hat{\pi}_{2\,\text{median}}$	$\hat{\pi}_{3\,\text{ave}}$	$\pm\hat{\pi}_{3\,\text{ave}}$	$\hat{\pi}_{3\,\text{median}}$	$\hat{\pi}_{4\,\text{ave}}$	$\pm\hat{\pi}_{4\,\text{ave}}$	$\hat{\pi}_{4\,\text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.125362	1.463637	0.000100	0.000038	0.000347	0.128	1.463	0.002	0.001	0.007	0.197	1.462	0.054	0.023	0.126	2.469	4.316	0.994	0.324	2.578	0.228	0.142	0.213	0.237	0.127	0.226	0.298	0.142	0.282	0.237	0.167	0.203	-323.112	-394.383	-248.889	0.014
100	0.126396	1.508759	0.000088	0.000037	0.000254	0.129	1.509	0.001	0.001	0.006	0.190	1.510	0.051	0.023	0.109	1.880	3.489	0.975	0.348	2.084	0.219	0.126	0.208	0.240	0.111	0.230	0.296	0.127	0.283	0.244	0.152	0.217	-652.384	-780.209	-523.860	0.016
200	0.128523	1.557078	0.000081	0.000036	0.000210	0.131	1.557	0.001	0.000	0.005	0.187	1.564	0.049	0.022	0.094	1.430	2.654	0.924	0.345	1.804	0.212	0.112	0.202	0.241	0.097	0.233	0.295	0.113	0.283	0.252	0.138	0.231	-1312.549	-1542.718	-1082.093	0.019
500	0.123264	1.566515	0.000074	0.000036	0.000175	0.125	1.566	0.001	0.000	0.003	0.172	1.563	0.044	0.021	0.079	1.128	1.938	0.852	0.326	1.607	0.201	0.096	0.191	0.242	0.084	0.236	0.294	0.098	0.285	0.263	0.122	0.250	-3296.713	-3817.749	-2790.118	0.020
1000	0.099336	1.434283	0.000070	0.000036	0.000159	0.101	1.434	0.001	0.000	0.003	0.145	1.431	0.041	0.022	0.073	1.025	1.718	0.803	0.328	1.503	0.195	0.081	0.188	0.243	0.073	0.237	0.295	0.087	0.287	0.267	0.111	0.255	-6609.630	-7551.640	-5660.554	0.023

Table D.232: 4-component exponential mixture on mid-price waiting time VOD data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\,\text{ave}}$	$\pm\hat{\lambda}_{1\,\text{ave}}$	$\hat{\lambda}_{1\,\text{median}}$	$\hat{\lambda}_{1\,\text{lower}}$	$\hat{\lambda}_{1\,\text{upper}}$	$\hat{\lambda}_{2\,\text{ave}}$	$\pm\hat{\lambda}_{2\,\text{ave}}$	$\hat{\lambda}_{2\,\text{median}}$	$\hat{\lambda}_{2\,\text{lower}}$	$\hat{\lambda}_{2\,\text{upper}}$	$\hat{\lambda}_{3\,\text{ave}}$	$\pm\hat{\lambda}_{3\,\text{ave}}$	$\hat{\lambda}_{3\,\text{median}}$	$\hat{\lambda}_{3\,\text{lower}}$	$\hat{\lambda}_{3\,\text{upper}}$	$\hat{\lambda}_{4\,\text{ave}}$	$\pm\hat{\lambda}_{4\,\text{ave}}$	$\hat{\lambda}_{4\,\text{median}}$	$\hat{\lambda}_{4\,\text{lower}}$	$\hat{\lambda}_{4\,\text{upper}}$	$\hat{\pi}_{1\,\text{ave}}$	$\pm\hat{\pi}_{1\,\text{ave}}$	$\hat{\pi}_{1\,\text{median}}$	$\hat{\pi}_{2\,\text{ave}}$	$\pm\hat{\pi}_{2\,\text{ave}}$	$\hat{\pi}_{2\,\text{median}}$	$\hat{\pi}_{3\,\text{ave}}$	$\pm\hat{\pi}_{3\,\text{ave}}$	$\hat{\pi}_{3\,\text{median}}$	$\hat{\pi}_{4\,\text{ave}}$	$\pm\hat{\pi}_{4\,\text{ave}}$	$\hat{\pi}_{4\,\text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.152588	1.591517	0.000488	0.000186	0.001573	0.165	1.591	0.008	0.002	0.026	0.302	1.586	0.114	0.048	0.264	2.383	4.307	0.949	0.364	2.370	0.202	0.131	0.185	0.206	0.123	0.193	0.267	0.127	0.257	0.325	0.191	0.298	-250.802	-320.832	-180.116	0.013
100	0.154490	1.658544	0.000426	0.000176	0.001122	0.164	1.658	0.007	0.002	0.020	0.287	1.653	0.104	0.047	0.221	1.986	3.783	0.938	0.392	2.057	0.192	0.115	0.180	0.208	0.107	0.200	0.261	0.107	0.255	0.338	0.175	0.321	-507.442	-632.947	-384.833	0.015
200	0.158241	1.720261	0.000387	0.000168	0.000937	0.166	1.720	0.006	0.002	0.016	0.271	1.714	0.094	0.043	0.182	1.597	3.111	0.914	0.398	1.841	0.183	0.101	0.173	0.208	0.093	0.202	0.255	0.089	0.250	0.355	0.162	0.346	-1022.416	-1250.489	-803.067	0.017
500	0.164018	1.801543	0.000354	0.000159	0.000821	0.170	1.801	0.004	0.002	0.012	0.256	1.795	0.083	0.037	0.144	1.295	2.465	0.877	0.388	1.697	0.170	0.087	0.161	0.205	0.080	0.201	0.248	0.070	0.247	0.377	0.147	0.372	-2568.084	-3085.294	-2071.316	0.022
1000	0.164415	1.839168	0.000331	0.000155	0.000760	0.169	1.839	0.004	0.001	0.010	0.245	1.833	0.076	0.034	0.129	1.195	2.254	0.853	0.375	1.631	0.161	0.077	0.153	0.205	0.073	0.201	0.246	0.059	0.245	0.389	0.138	0.386	-5142.966	-6103.212	-4201.118	0.025

Table D.233: 4-component exponential mixture on mid-price waiting time RIOTINTO data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\,ave}$	$\pm\hat{\lambda}_{1\,ave}$	$\hat{\lambda}_{1\,median}$	$\hat{\lambda}_{1\,lower}$	$\hat{\lambda}_{1\,upper}$	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\lambda}_{3\,ave}$	$\pm\hat{\lambda}_{3\,ave}$	$\hat{\lambda}_{3\,median}$	$\hat{\lambda}_{3\,lower}$	$\hat{\lambda}_{3\,upper}$	$\hat{\lambda}_{4\,ave}$	$\pm\hat{\lambda}_{4\,ave}$	$\hat{\lambda}_{4\,median}$	$\hat{\lambda}_{4\,lower}$	$\hat{\lambda}_{4\,upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	$\hat{\pi}_{4\,ave}$	$\pm\hat{\pi}_{4\,ave}$	$\hat{\pi}_{4\,median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.082694	1.190581	0.000016	0.000007	0.000053	0.086	1.209	0.001	0.000	0.002	0.136	1.226	0.032	0.012	0.090	1.878	3.662	0.752	0.255	2.088	0.256	0.133	0.248	0.274	0.129	0.264	0.249	0.113	0.236	0.221	0.147	0.195	-392.253	-460.258	-320.984	0.016
100	0.078318	1.190380	0.000014	0.000007	0.000039	0.079	1.190	0.000	0.000	0.001	0.118	1.188	0.028	0.011	0.073	1.489	3.092	0.695	0.262	1.732	0.247	0.116	0.242	0.278	0.108	0.271	0.246	0.095	0.238	0.229	0.132	0.208	-791.574	-908.585	-669.963	0.015
200	0.060155	1.067405	0.000013	0.000007	0.000033	0.061	1.067	0.000	0.000	0.001	0.095	1.066	0.026	0.011	0.062	1.052	2.166	0.626	0.252	1.469	0.242	0.100	0.241	0.278	0.088	0.272	0.243	0.081	0.233	0.236	0.115	0.220	-1593.340	-1788.313	-1391.289	0.017
500	0.035502	0.843464	0.000012	0.000007	0.000026	0.036	0.843	0.000	0.000	0.001	0.064	0.843	0.022	0.011	0.045	0.823	1.653	0.551	0.250	1.227	0.233	0.080	0.232	0.278	0.068	0.275	0.240	0.064	0.233	0.249	0.095	0.238	-3992.301	-4382.983	-3596.110	0.012
1000	0.000016	0.000018	0.000011	0.000008	0.000023	0.000	0.000	0.000	0.000	0.001	0.025	0.016	0.022	0.012	0.038	0.720	1.416	0.525	0.273	0.989	0.224	0.069	0.223	0.279	0.054	0.272	0.241	0.056	0.232	0.256	0.083	0.235	-7958.039	-8598.540	-7339.221	0.013

Table D.234: 4-component exponential mixture on mid-price waiting time SSELN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\,ave}$	$\pm\hat{\lambda}_{1\,ave}$	$\hat{\lambda}_{1\,median}$	$\hat{\lambda}_{1\,lower}$	$\hat{\lambda}_{1\,upper}$	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\lambda}_{3\,ave}$	$\pm\hat{\lambda}_{3\,ave}$	$\hat{\lambda}_{3\,median}$	$\hat{\lambda}_{3\,lower}$	$\hat{\lambda}_{3\,upper}$	$\hat{\lambda}_{4\,ave}$	$\pm\hat{\lambda}_{4\,ave}$	$\hat{\lambda}_{4\,median}$	$\hat{\lambda}_{4\,lower}$	$\hat{\lambda}_{4\,upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	$\hat{\pi}_{4\,ave}$	$\pm\hat{\pi}_{4\,ave}$	$\hat{\pi}_{4\,median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.093006	1.263943	0.000014	0.000005	0.000051	0.094	1.264	0.000	0.000	0.002	0.143	1.307	0.027	0.010	0.087	1.607	3.012	0.625	0.229	2.385	0.255	0.137	0.249	0.271	0.127	0.263	0.241	0.109	0.230	0.233	0.159	0.201	-391.218	-467.223	-311.467	0.018
100	0.065676	1.089251	0.000012	0.000005	0.000036	0.066	1.089	0.000	0.000	0.001	0.109	1.146	0.021	0.010	0.067	1.148	2.003	0.554	0.227	1.903	0.245	0.121	0.240	0.272	0.105	0.271	0.237	0.091	0.226	0.245	0.144	0.218	-788.112	-921.332	-654.352	0.013
200	0.047780	0.952819	0.000011	0.000005	0.000029	0.048	0.953	0.000	0.000	0.001	0.093	1.098	0.018	0.009	0.054	0.870	1.503	0.505	0.228	1.525	0.232	0.101	0.229	0.276	0.092	0.275	0.233	0.076	0.225	0.259	0.127	0.237	-1582.570	-1808.279	-1354.923	0.020
500	0.000017	0.000029	0.000010	0.000006	0.000023	0.000	0.001	0.000	0.000	0.001	0.023	0.023	0.017	0.009	0.035	0.659	0.686	0.461	0.228	0.996	0.222	0.081	0.218	0.276	0.069	0.274	0.231	0.059	0.230	0.272	0.096	0.254	-3950.371	-4411.053	-3478.391	0.012
1000	0.000018	0.000021	0.000012	0.000007	0.000024	0.000	0.001	0.000	0.000	0.001	0.022	0.016	0.017	0.010	0.033	0.641	0.569	0.496	0.257	0.932	0.211	0.077	0.205	0.271	0.064	0.267	0.231	0.050	0.229	0.288	0.084	0.273	-7678.270	-8513.961	-6754.012	0.015

Table D.235: 4-component exponential mixture on mid-price waiting time ABFLN data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\,ave}$	$\pm\hat{\lambda}_{1\,ave}$	$\hat{\lambda}_{1\,median}$	$\hat{\lambda}_{1\,lower}$	$\hat{\lambda}_{1\,upper}$	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\lambda}_{3\,ave}$	$\pm\hat{\lambda}_{3\,ave}$	$\hat{\lambda}_{3\,median}$	$\hat{\lambda}_{3\,lower}$	$\hat{\lambda}_{3\,upper}$	$\hat{\lambda}_{4\,ave}$	$\pm\hat{\lambda}_{4\,ave}$	$\hat{\lambda}_{4\,median}$	$\hat{\lambda}_{4\,lower}$	$\hat{\lambda}_{4\,upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	$\hat{\pi}_{4\,ave}$	$\pm\hat{\pi}_{4\,ave}$	$\hat{\pi}_{4\,median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.098552	1.274652	0.000365	0.000145	0.001192	0.112	1.274	0.008	0.002	0.028	0.247	1.274	0.113	0.048	0.267	1.740	2.692	0.842	0.332	2.718	0.204	0.135	0.186	0.200	0.127	0.184	0.280	0.132	0.270	0.316	0.189	0.279	-242.728	-318.812	-157.710	0.014
100	0.098560	1.325111	0.000320	0.000138	0.000825	0.109	1.324	0.007	0.002	0.021	0.229	1.322	0.102	0.045	0.223	1.467	2.085	0.830	0.365	2.381	0.195	0.119	0.181	0.201	0.111	0.189	0.275	0.114	0.268	0.330	0.178	0.299	-491.561	-628.714	-340.596	0.015
200	0.100617	1.373676	0.000290	0.000132	0.000658	0.109	1.373	0.005	0.001	0.017	0.212	1.370	0.090	0.040	0.181	1.274	1.731	0.789	0.381	2.117	0.185	0.106	0.174	0.199	0.099	0.189	0.269	0.098	0.262	0.347	0.170	0.323	-990.455	-1242.243	-715.112	0.015
500	0.100382	1.411255	0.000261	0.000124	0.000550	0.106	1.411	0.003	0.001	0.012	0.188	1.409	0.075	0.033	0.141	1.115	1.574	0.698	0.374	1.898	0.170	0.092	0.159	0.195	0.088	0.186	0.260	0.078	0.255	0.375	0.159	0.362	-2487.664	-3069.376	-1850.116	0.017
1000	0.098014	1.422243	0.000244	0.000122	0.000499	0.102	1.422	0.003	0.001	0.010	0.173	1.418	0.067	0.029	0.121	1.043	1.534	0.646	0.368	1.800	0.161	0.083	0.150	0.194	0.081	0.184	0.255	0.066	0.251	0.391	0.151	0.389	-4986.320	-6107.545	-3776.119	0.016

Table D.236: 4-component exponential mixture on mid-price waiting time BARC data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\,ave}$	$\pm\hat{\lambda}_{1\,ave}$	$\hat{\lambda}_{1\,median}$	$\hat{\lambda}_{1\,lower}$	$\hat{\lambda}_{1\,upper}$	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\lambda}_{3\,ave}$	$\pm\hat{\lambda}_{3\,ave}$	$\hat{\lambda}_{3\,median}$	$\hat{\lambda}_{3\,lower}$	$\hat{\lambda}_{3\,upper}$	$\hat{\lambda}_{4\,ave}$	$\pm\hat{\lambda}_{4\,ave}$	$\hat{\lambda}_{4\,median}$	$\hat{\lambda}_{4\,lower}$	$\hat{\lambda}_{4\,upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	$\hat{\pi}_{4\,ave}$	$\pm\hat{\pi}_{4\,ave}$	$\hat{\pi}_{4\,median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.070179	1.080845	0.000032	0.000010	0.000237	0.073	1.091	0.001	0.000	0.003	0.151	1.108	0.052	0.017	0.146	1.413	2.612	0.604	0.292	2.081	0.189	0.146	0.166	0.213	0.135	0.195	0.248	0.121	0.234	0.350	0.221	0.300	-306.345	-422.924	-164.832	0.010
100	0.063606	1.066132	0.000026	0.000010	0.000106	0.065	1.066	0.001	0.000	0.002	0.124	1.065	0.043	0.015	0.111	1.086	1.854	0.560	0.305	1.733	0.179	0.130	0.158	0.213	0.122	0.198	0.236	0.100	0.227	0.371	0.214	0.325	-617.695	-837.144	-362.466	0.011
200	0.052180	0.991659	0.000023	0.000010	0.000068	0.053	0.992	0.001	0.000	0.001	0.102	0.990	0.037	0.014	0.089	0.881	1.404	0.528	0.313	1.425	0.171	0.116	0.151	0.213	0.108	0.202	0.226	0.083	0.219	0.390	0.206	0.345	-1237.543	-1656.703	-784.968	0.009
500	0.016192	0.569121	0.000020	0.000010	0.000051	0.017	0.569	0.001	0.000	0.001	0.057	0.569	0.032	0.013	0.072	0.689	0.860	0.479	0.318	0.966	0.156	0.099	0.137	0.210	0.092	0.203	0.214	0.065	0.211	0.420	0.193	0.400	-3052.182	-4042.265	-2024.576	0.014
1000	0.000027	0.000028	0.000019	0.000010	0.000042	0.001	0.001	0.001	0.000	0.001	0.037	0.025	0.031	0.014	0.065	0.606	0.531	0.455	0.324	0.770	0.145	0.085	0.122	0.206	0.084	0.196	0.208	0.056	0.205	0.441	0.183	0.465	-5981.805	-7983.138	-4183.377	0.007

Table D.237: 4-component exponential mixture on mid-price waiting time RRLN data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\,ave}$	$\pm\hat{\lambda}_{1\,ave}$	$\hat{\lambda}_{1\,median}$	$\hat{\lambda}_{1\,lower}$	$\hat{\lambda}_{1\,upper}$	$\hat{\lambda}_{2\,ave}$	$\pm\hat{\lambda}_{2\,ave}$	$\hat{\lambda}_{2\,median}$	$\hat{\lambda}_{2\,lower}$	$\hat{\lambda}_{2\,upper}$	$\hat{\lambda}_{3\,ave}$	$\pm\hat{\lambda}_{3\,ave}$	$\hat{\lambda}_{3\,median}$	$\hat{\lambda}_{3\,lower}$	$\hat{\lambda}_{3\,upper}$	$\hat{\lambda}_{4\,ave}$	$\pm\hat{\lambda}_{4\,ave}$	$\hat{\lambda}_{4\,median}$	$\hat{\lambda}_{4\,lower}$	$\hat{\lambda}_{4\,upper}$	$\hat{\pi}_{1\,ave}$	$\pm\hat{\pi}_{1\,ave}$	$\hat{\pi}_{1\,median}$	$\hat{\pi}_{2\,ave}$	$\pm\hat{\pi}_{2\,ave}$	$\hat{\pi}_{2\,median}$	$\hat{\pi}_{3\,ave}$	$\pm\hat{\pi}_{3\,ave}$	$\hat{\pi}_{3\,median}$	$\hat{\pi}_{4\,ave}$	$\pm\hat{\pi}_{4\,ave}$	$\hat{\pi}_{4\,median}$	$\log L_{ave}$	$\log L_{lower}$	$\log L_{upper}$	perc-non-convergence
50	0.125783	1.464602	0.000099	0.000037	0.000345	0.129	1.464	0.002	0.001	0.007	0.199	1.463	0.054	0.022	0.129	2.055	3.174	1.024	0.316	3.128	0.227	0.142	0.212	0.237	0.127	0.226	0.294	0.142	0.279	0.241	0.168	0.207	-313.646	-388.483	-234.128	0.014
100	0.126423	1.508986	0.000087	0.000037	0.000251	0.129	1.509	0.001	0.000	0.006	0.191	1.510	0.050	0.022	0.110	1.619	2.308	0.987	0.342	2.569	0.219	0.126	0.208	0.240	0.111	0.230	0.292	0.127	0.278	0.249	0.154	0.222	-633.815	-767.914	-496.158	0.016
200	0.127624	1.551667	0.000080	0.000036	0.000203	0.130	1.551	0.001	0.000	0.004	0.186	1.559	0.046	0.021	0.093	1.357	1.878	0.907	0.349	2.187	0.210	0.112	0.201	0.241	0.097	0.233	0.290	0.113	0.278	0.259	0.140	0.238	-1274.561	-1520.611	-1036.940	0.012
500	0.122512	1.561755	0.000073	0.000035	0.000170	0.124	1.562	0.001	0.000	0.003	0.170	1.559	0.041	0.020	0.077	1.175	1.697	0.774	0.341	1.921	0.200	0.096	0.189	0.241	0.084	0.235	0.288	0.097	0.278	0.271	0.123	0.258	-3201.620	-3746.981	-2664.690	0.014
1000	0.098656	1.429391	0.000069	0.000035	0.000154	0.100	1.429	0.001	0.000	0.002	0.142	1.427	0.038	0.020	0.070	1.076	1.557	0.690	0.348	1.792	0.193	0.082	0.185	0.242	0.073	0.237	0.289	0.085	0.279	0.277	0.113	0.266	-6411.013	-7421.187	-5431.618	0.016

Table D.238: 4-component exponential mixture on mid-price waiting time VOD data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\ \mathrm{ave}}$	$\pm\hat{\lambda}_{1\ \mathrm{ave}}$	$\hat{\lambda}_{1\ \mathrm{median}}$	$\hat{\lambda}_{1\ \mathrm{lower}}$	$\hat{\lambda}_{1\ \mathrm{upper}}$	$\hat{\lambda}_{2\ \mathrm{ave}}$	$\pm\hat{\lambda}_{2\ \mathrm{ave}}$	$\hat{\lambda}_{2\ \mathrm{median}}$	$\hat{\lambda}_{2\ \mathrm{lower}}$	$\hat{\lambda}_{2\ \mathrm{upper}}$	$\hat{\lambda}_{3\ \mathrm{ave}}$	$\pm\hat{\lambda}_{3\ \mathrm{ave}}$	$\hat{\lambda}_{3\ \mathrm{median}}$	$\hat{\lambda}_{3\ \mathrm{lower}}$	$\hat{\lambda}_{3\ \mathrm{upper}}$	$\hat{\lambda}_{4\ \mathrm{ave}}$	$\pm\hat{\lambda}_{4\ \mathrm{ave}}$	$\hat{\lambda}_{4\ \mathrm{median}}$	$\hat{\lambda}_{4\ \mathrm{lower}}$	$\hat{\lambda}_{4\ \mathrm{upper}}$	$\hat{\pi}_{1\ \mathrm{ave}}$	$\pm\hat{\pi}_{1\ \mathrm{ave}}$	$\hat{\pi}_{1\ \mathrm{median}}$	$\hat{\pi}_{2\ \mathrm{ave}}$	$\pm\hat{\pi}_{2\ \mathrm{ave}}$	$\hat{\pi}_{2\ \mathrm{median}}$	$\hat{\pi}_{3\ \mathrm{ave}}$	$\pm\hat{\pi}_{3\ \mathrm{ave}}$	$\hat{\pi}_{3\ \mathrm{median}}$	$\hat{\pi}_{4\ \mathrm{ave}}$	$\pm\hat{\pi}_{4\ \mathrm{ave}}$	$\hat{\pi}_{4\ \mathrm{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.155071	1.597366	0.000485	0.000185	0.001587	0.168	1.596	0.008	0.002	0.026	0.316	1.593	0.120	0.049	0.299	1.904	2.834	0.995	0.358	2.947	0.201	0.131	0.185	0.206	0.124	0.194	0.266	0.127	0.257	0.326	0.191	0.296	-237.464	-309.858	-162.120	0.015
100	0.155198	1.660036	0.000424	0.000176	0.001121	0.165	1.659	0.007	0.002	0.021	0.299	1.657	0.108	0.047	0.252	1.644	2.337	0.992	0.387	2.577	0.192	0.115	0.180	0.209	0.107	0.201	0.260	0.108	0.253	0.339	0.176	0.320	-481.094	-610.927	-351.754	0.016
200	0.158208	1.718665	0.000386	0.000167	0.000931	0.166	1.718	0.006	0.002	0.017	0.280	1.714	0.096	0.041	0.203	1.451	2.051	0.965	0.392	2.275	0.182	0.102	0.173	0.208	0.093	0.203	0.252	0.091	0.247	0.357	0.164	0.347	-969.552	-1206.322	-740.031	0.015
500	0.163117	1.795718	0.000350	0.000155	0.000805	0.169	1.795	0.004	0.001	0.012	0.258	1.790	0.079	0.034	0.154	1.302	1.944	0.897	0.380	2.054	0.169	0.088	0.159	0.206	0.080	0.201	0.242	0.070	0.239	0.384	0.151	0.379	-2437.306	-2975.836	-1926.410	0.016
1000	0.162802	1.829821	0.000323	0.000151	0.000742	0.168	1.829	0.003	0.001	0.010	0.245	1.825	0.069	0.031	0.134	1.236	1.937	0.828	0.370	1.949	0.159	0.078	0.150	0.205	0.074	0.201	0.237	0.059	0.235	0.398	0.143	0.397	-4888.319	-5896.321	-3941.167	0.015

Table D.239: 4-component exponential mixture on mid-price waiting time RIOTINTO data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\ \mathrm{ave}}$	$\pm\hat{\lambda}_{1\ \mathrm{ave}}$	$\hat{\lambda}_{1\ \mathrm{median}}$	$\hat{\lambda}_{1\ \mathrm{lower}}$	$\hat{\lambda}_{1\ \mathrm{upper}}$	$\hat{\lambda}_{2\ \mathrm{ave}}$	$\pm\hat{\lambda}_{2\ \mathrm{ave}}$	$\hat{\lambda}_{2\ \mathrm{median}}$	$\hat{\lambda}_{2\ \mathrm{lower}}$	$\hat{\lambda}_{2\ \mathrm{upper}}$	$\hat{\lambda}_{3\ \mathrm{ave}}$	$\pm\hat{\lambda}_{3\ \mathrm{ave}}$	$\hat{\lambda}_{3\ \mathrm{median}}$	$\hat{\lambda}_{3\ \mathrm{lower}}$	$\hat{\lambda}_{3\ \mathrm{upper}}$	$\hat{\lambda}_{4\ \mathrm{ave}}$	$\pm\hat{\lambda}_{4\ \mathrm{ave}}$	$\hat{\lambda}_{4\ \mathrm{median}}$	$\hat{\lambda}_{4\ \mathrm{lower}}$	$\hat{\lambda}_{4\ \mathrm{upper}}$	$\hat{\pi}_{1\ \mathrm{ave}}$	$\pm\hat{\pi}_{1\ \mathrm{ave}}$	$\hat{\pi}_{1\ \mathrm{median}}$	$\hat{\pi}_{2\ \mathrm{ave}}$	$\pm\hat{\pi}_{2\ \mathrm{ave}}$	$\hat{\pi}_{2\ \mathrm{median}}$	$\hat{\pi}_{3\ \mathrm{ave}}$	$\pm\hat{\pi}_{3\ \mathrm{ave}}$	$\hat{\pi}_{3\ \mathrm{median}}$	$\hat{\pi}_{4\ \mathrm{ave}}$	$\pm\hat{\pi}_{4\ \mathrm{ave}}$	$\hat{\pi}_{4\ \mathrm{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
50	0.082893	1.189736	0.000016	0.000006	0.000052	0.087	1.208	0.001	0.000	0.002	0.135	1.225	0.030	0.011	0.087	1.633	2.855	0.705	0.253	2.480	0.254	0.133	0.246	0.273	0.129	0.263	0.246	0.113	0.234	0.227	0.149	0.200	-386.229	-457.297	-312.045	0.015
100	0.078500	1.191146	0.000014	0.000007	0.000039	0.079	1.191	0.000	0.000	0.001	0.118	1.189	0.026	0.011	0.072	1.263	2.106	0.641	0.271	2.059	0.246	0.116	0.241	0.276	0.108	0.269	0.243	0.095	0.234	0.234	0.132	0.214	-779.038	-897.890	-649.230	0.016
200	0.060022	1.066398	0.000013	0.000007	0.000033	0.061	1.066	0.000	0.000	0.001	0.093	1.065	0.024	0.011	0.058	0.989	1.518	0.580	0.271	1.745	0.239	0.099	0.238	0.277	0.087	0.271	0.240	0.079	0.231	0.243	0.116	0.227	-1563.504	-1764.105	-1357.215	0.015
500	0.035755	0.846466	0.000012	0.000007	0.000027	0.036	0.846	0.000	0.000	0.001	0.063	0.846	0.020	0.011	0.044	0.825	1.189	0.533	0.276	1.495	0.232	0.080	0.229	0.277	0.067	0.273	0.237	0.063	0.231	0.255	0.095	0.244	-3914.526	-4289.594	-3477.851	0.019
1000	0.000016	0.000018	0.000011	0.000008	0.000021	0.000	0.000	0.000	0.000	0.001	0.023	0.015	0.019	0.011	0.036	0.679	0.702	0.502	0.316	0.933	0.223	0.068	0.223	0.278	0.053	0.271	0.237	0.055	0.229	0.262	0.081	0.242	-7828.917	-8442.229	-7165.238	0.013

Table D.240: 4-component exponential mixture on mid-price waiting time SSELN data with censoring on region [0, 0.5, 1.5, 2.5, 10].

n	$\hat{\lambda}_{1\ \mathrm{ave}}$	$\pm\hat{\lambda}_{1\ \mathrm{ave}}$	$\hat{\lambda}_{1\ \mathrm{median}}$	$\hat{\lambda}_{1\ \mathrm{lower}}$	$\hat{\lambda}_{1\ \mathrm{upper}}$	$\hat{\lambda}_{2\ \mathrm{ave}}$	$\pm\hat{\lambda}_{2\ \mathrm{ave}}$	$\hat{\lambda}_{2\ \mathrm{median}}$	$\hat{\lambda}_{2\ \mathrm{lower}}$	$\hat{\lambda}_{2\ \mathrm{upper}}$	$\hat{\alpha}_{\mathrm{ave}}$	$\pm\hat{\alpha}_{\mathrm{ave}}$	$\hat{\alpha}_{\mathrm{median}}$	$\hat{\alpha}_{\mathrm{lower}}$	$\hat{\alpha}_{\mathrm{upper}}$	$\hat{\beta}_{\mathrm{ave}}$	$\pm\hat{\beta}_{\mathrm{ave}}$	$\hat{\beta}_{\mathrm{median}}$	$\hat{\beta}_{\mathrm{lower}}$	$\hat{\beta}_{\mathrm{upper}}$	$\hat{\pi}_{1\ \mathrm{ave}}$	$\pm\hat{\pi}_{1\ \mathrm{ave}}$	$\hat{\pi}_{1\ \mathrm{median}}$	$\hat{\pi}_{2\ \mathrm{ave}}$	$\pm\hat{\pi}_{2\ \mathrm{ave}}$	$\hat{\pi}_{2\ \mathrm{median}}$	$\hat{\pi}_{3\ \mathrm{ave}}$	$\pm\hat{\pi}_{3\ \mathrm{ave}}$	$\hat{\pi}_{3\ \mathrm{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	0.294	0.989	0.158	0.067	0.580	15.648	1.378	15.671	14.537	17.068	8472.889	7252.567	6208.087	2919.148	17834.572	0.357	0.080	0.343	0.288	0.466	0.232	0.110	0.216	0.027	0.065	0.001	0.741	0.121	0.754	-1656.522	-1835.472	-1408.544	0.069
500	0.240	0.190	0.174	0.092	0.461	16.297	1.168	16.312	15.045	17.725	6388.179	4313.251	5247.152	2760.923	11971.243	0.334	0.059	0.325	0.281	0.397	0.231	0.098	0.215	0.020	0.043	0.000	0.749	0.102	0.763	-4072.704	-4453.194	-3586.702	0.041
1000	0.270	0.182	0.221	0.108	0.494	16.847	1.225	16.871	15.491	18.226	4418.001	2586.986	3782.157	2092.478	7259.799	0.330	0.055	0.324	0.273	0.383	0.237	0.095	0.213	0.022	0.041	0.000	0.741	0.093	0.748	-7840.177	-8653.153	-6971.041	0.015
2000	0.251	0.132	0.219	0.132	0.445	17.285	1.302	17.148	15.708	18.932	3364.827	1235.938	3249.414	2210.109	4774.357	0.341	0.038	0.331	0.307	0.383	0.253	0.073	0.245	0.024	0.038	0.000	0.723	0.069	0.732	-15194.879	-16178.528	-14147.915	0.000

Table D.241: exponential-exponential-weibull mixture on mid-price waiting time ABFLN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ \mathrm{ave}}$	$\pm\hat{\lambda}_{1\ \mathrm{ave}}$	$\hat{\lambda}_{1\ \mathrm{median}}$	$\hat{\lambda}_{1\ \mathrm{lower}}$	$\hat{\lambda}_{1\ \mathrm{upper}}$	$\hat{\lambda}_{2\ \mathrm{ave}}$	$\pm\hat{\lambda}_{2\ \mathrm{ave}}$	$\hat{\lambda}_{2\ \mathrm{median}}$	$\hat{\lambda}_{2\ \mathrm{lower}}$	$\hat{\lambda}_{2\ \mathrm{upper}}$	$\hat{\alpha}_{\mathrm{ave}}$	$\pm\hat{\alpha}_{\mathrm{ave}}$	$\hat{\alpha}_{\mathrm{median}}$	$\hat{\alpha}_{\mathrm{lower}}$	$\hat{\alpha}_{\mathrm{upper}}$	$\hat{\beta}_{\mathrm{ave}}$	$\pm\hat{\beta}_{\mathrm{ave}}$	$\hat{\beta}_{\mathrm{median}}$	$\hat{\beta}_{\mathrm{lower}}$	$\hat{\beta}_{\mathrm{upper}}$	$\hat{\pi}_{1\ \mathrm{ave}}$	$\pm\hat{\pi}_{1\ \mathrm{ave}}$	$\hat{\pi}_{1\ \mathrm{median}}$	$\hat{\pi}_{2\ \mathrm{ave}}$	$\pm\hat{\pi}_{2\ \mathrm{ave}}$	$\hat{\pi}_{2\ \mathrm{median}}$	$\hat{\pi}_{3\ \mathrm{ave}}$	$\pm\hat{\pi}_{3\ \mathrm{ave}}$	$\hat{\pi}_{3\ \mathrm{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	0.461	1.457	0.248	0.136	0.896	16.495	1.293	16.411	15.307	18.301	701.670	746.733	481.730	175.491	1646.589	0.433	0.105	0.420	0.346	0.577	0.396	0.181	0.396	0.035	0.073	0.002	0.569	0.179	0.568	-1094.288	-1292.837	-810.566	0.082
500	0.366	0.310	0.263	0.154	0.779	17.160	1.295	17.190	15.825	18.831	568.021	565.488	408.848	161.485	1182.902	0.402	0.071	0.395	0.338	0.491	0.387	0.175	0.397	0.025	0.043	0.000	0.588	0.165	0.578	-2727.808	-3190.939	-2095.658	0.063
1000	0.453	1.235	0.274	0.162	0.715	17.652	1.340	17.761	16.208	19.161	500.074	462.360	372.491	152.476	955.482	0.386	0.064	0.383	0.334	0.454	0.384	0.172	0.399	0.025	0.064	0.000	0.591	0.166	0.580	-5395.337	-6347.324	-4258.861	0.043
2000	0.438	1.122	0.280	0.171	0.685	18.116	1.400	18.254	16.607	19.629	452.352	390.519	345.164	146.203	841.301	0.375	0.055	0.374	0.331	0.431	0.383	0.168	0.402	0.022	0.058	0.000	0.596	0.160	0.582	-10738.821	-12562.312	-8529.814	0.035

Table D.242: exponential-exponential-weibull mixture on mid-price waiting time BARC data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ \mathrm{ave}}$	$\pm\hat{\lambda}_{1\ \mathrm{ave}}$	$\hat{\lambda}_{1\ \mathrm{median}}$	$\hat{\lambda}_{1\ \mathrm{lower}}$	$\hat{\lambda}_{1\ \mathrm{upper}}$	$\hat{\lambda}_{2\ \mathrm{ave}}$	$\pm\hat{\lambda}_{2\ \mathrm{ave}}$	$\hat{\lambda}_{2\ \mathrm{median}}$	$\hat{\lambda}_{2\ \mathrm{lower}}$	$\hat{\lambda}_{2\ \mathrm{upper}}$	$\hat{\alpha}_{\mathrm{ave}}$	$\pm\hat{\alpha}_{\mathrm{ave}}$	$\hat{\alpha}_{\mathrm{median}}$	$\hat{\alpha}_{\mathrm{lower}}$	$\hat{\alpha}_{\mathrm{upper}}$	$\hat{\beta}_{\mathrm{ave}}$	$\pm\hat{\beta}_{\mathrm{ave}}$	$\hat{\beta}_{\mathrm{median}}$	$\hat{\beta}_{\mathrm{lower}}$	$\hat{\beta}_{\mathrm{upper}}$	$\hat{\pi}_{1\ \mathrm{ave}}$	$\pm\hat{\pi}_{1\ \mathrm{ave}}$	$\hat{\pi}_{1\ \mathrm{median}}$	$\hat{\pi}_{2\ \mathrm{ave}}$	$\pm\hat{\pi}_{2\ \mathrm{ave}}$	$\hat{\pi}_{2\ \mathrm{median}}$	$\hat{\pi}_{3\ \mathrm{ave}}$	$\pm\hat{\pi}_{3\ \mathrm{ave}}$	$\hat{\pi}_{3\ \mathrm{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	0.352	1.094	0.235	0.112	-	16.370	1.077	16.234	15.563	-	4290.536	4120.522	3026.111	1276.539	-	0.356	0.087	0.341	0.287	-	0.349	0.200	0.302	0.024	0.057	0.000	0.627	0.196	0.666	-1391.989	-1680.037	-	0.164
500	0.297	0.218	0.255	0.137	1.026	16.830	0.996	16.802	15.845	19.244	3268.886	2787.225	2468.641	1109.819	11633.011	0.333	0.065	0.320	0.281	0.537	0.360	0.200	0.305	0.017	0.044	0.000	0.623	0.192	0.669	-3392.124	-4106.751	-2110.791	0.146
1000	0.303	0.187	0.275	0.151	0.674	17.095	1.114	17.011	15.976	19.062	2644.345	1970.790	2128.158	1049.377	6553.596	0.321	0.053	0.311	0.278	0.419	0.392	0.209	0.344	0.012	0.036	0.000	0.596	0.200	0.636	-6498.862	-8077.330	-4419.077	0.112
2000	0.328	0.174	0.300	0.213	0.830	17.196	1.136	16.979	16.063	19.580	1927.680	1214.933	1688.789	905.444	4061.773	0.316	0.047	0.310	0.280	0.421	0.446	0.212	0.536	0.010	0.026	0.000	0.544	0.200	0.464	-12028.606	-15439.282	-8755.339	0.130

Table D.243: exponential-exponential-weibull mixture on mid-price waiting time RRLN data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	0.371	1.483	0.146	0.074	0.681	16.203	1.385	16.191	15.104	17.851	2056.418	1967.805	1467.433	524.686	4526.452	0.415	0.109	0.400	0.321	0.562	0.295	0.132	0.284	0.043	0.085	0.018	0.662	0.146	0.674	-1351.197	-1547.531	-1103.267	0.070
500	0.379	1.607	0.156	0.084	0.602	16.974	1.239	16.976	15.779	18.378	1695.533	1506.939	1282.496	464.523	3372.088	0.386	0.085	0.375	0.312	0.489	0.284	0.125	0.275	0.038	0.079	0.015	0.677	0.134	0.689	-3358.584	-3827.214	-2842.179	0.049
1000	0.254	0.259	0.161	0.089	0.557	17.481	1.227	17.556	16.270	18.743	1510.613	1255.589	1183.634	456.810	2805.481	0.373	0.068	0.365	0.310	0.453	0.282	0.121	0.271	0.030	0.045	0.012	0.688	0.117	0.695	-6722.436	-7567.923	-5771.678	0.038
2000	0.310	1.078	0.166	0.092	0.488	18.022	1.225	18.110	16.782	19.219	1315.256	949.765	1069.927	461.620	2395.943	0.358	0.059	0.353	0.307	0.421	0.278	0.113	0.268	0.029	0.064	0.008	0.693	0.114	0.704	-13349.450	-14973.134	-11719.606	0.030

Table D.244: exponential-exponential-weibull mixture on mid-price waiting time VOD data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	0.616	1.610	0.327	0.168	1.062	16.895	1.170	16.789	15.773	18.382	505.800	526.956	344.448	113.773	1104.118	0.434	0.101	0.424	0.351	0.554	0.367	0.162	0.353	0.040	0.088	0.001	0.593	0.167	0.604	-1055.935	-1257.464	-821.856	0.062
500	0.635	1.573	0.357	0.191	0.999	17.473	1.202	17.443	16.196	18.887	417.246	393.243	303.457	101.431	830.975	0.405	0.077	0.402	0.342	0.487	0.356	0.155	0.340	0.034	0.083	0.000	0.610	0.157	0.622	-2621.418	-3102.180	-2125.252	0.040
1000	0.508	0.347	0.372	0.203	0.962	17.904	1.275	17.921	16.524	19.350	379.316	332.063	287.579	98.072	728.017	0.396	0.060	0.392	0.339	0.463	0.353	0.151	0.336	0.025	0.055	0.000	0.622	0.143	0.630	-5259.064	-6136.013	-4312.616	0.034
2000	0.512	0.339	0.376	0.210	0.947	18.308	1.337	18.353	16.797	19.763	350.994	289.668	275.262	97.216	663.238	0.387	0.054	0.385	0.335	0.445	0.350	0.147	0.332	0.022	0.051	0.000	0.627	0.138	0.635	-10487.737	-12152.268	-8714.847	0.028

Table D.245: exponential-exponential-weibull mixture on mid-price waiting time RIOTINTO data with censoring on region [0, 0.5].

n	$\hat{\lambda}_{1\ \text{ave}}$	$\pm\hat{\lambda}_{1\ \text{ave}}$	$\hat{\lambda}_{1\ \text{median}}$	$\hat{\lambda}_{1\ \text{lower}}$	$\hat{\lambda}_{1\ \text{upper}}$	$\hat{\lambda}_{2\ \text{ave}}$	$\pm\hat{\lambda}_{2\ \text{ave}}$	$\hat{\lambda}_{2\ \text{median}}$	$\hat{\lambda}_{2\ \text{lower}}$	$\hat{\lambda}_{2\ \text{upper}}$	$\hat{\alpha}_{\text{ave}}$	$\pm\hat{\alpha}_{\text{ave}}$	$\hat{\alpha}_{\text{median}}$	$\hat{\alpha}_{\text{lower}}$	$\hat{\alpha}_{\text{upper}}$	$\hat{\beta}_{\text{ave}}$	$\pm\hat{\beta}_{\text{ave}}$	$\hat{\beta}_{\text{median}}$	$\hat{\beta}_{\text{lower}}$	$\hat{\beta}_{\text{upper}}$	$\hat{\pi}_{1\ \text{ave}}$	$\pm\hat{\pi}_{1\ \text{ave}}$	$\hat{\pi}_{1\ \text{median}}$	$\hat{\pi}_{2\ \text{ave}}$	$\pm\hat{\pi}_{2\ \text{ave}}$	$\hat{\pi}_{2\ \text{median}}$	$\hat{\pi}_{3\ \text{ave}}$	$\pm\hat{\pi}_{3\ \text{ave}}$	$\hat{\pi}_{3\ \text{median}}$	log L_{ave}	log L_{lower}	log L_{upper}	perc-non-convergence
200	0.309	1.102	0.149	0.072	0.577	15.727	1.395	15.760	14.773	17.064	7135.524	5947.669	5556.112	2334.148	13794.068	0.356	0.080	0.344	0.290	0.452	0.233	0.108	0.220	0.028	0.063	0.007	0.739	0.121	0.754	-1626.460	-1790.331	-1404.444	0.058
500	0.280	0.873	0.160	0.086	0.475	16.405	1.146	16.434	15.197	17.628	5649.383	3822.113	4761.498	2270.258	9778.413	0.334	0.057	0.324	0.285	0.396	0.224	0.094	0.217	0.024	0.055	0.004	0.752	0.101	0.762	-4031.517	-4386.310	-3630.907	0.033
1000	0.244	0.196	0.178	0.107	0.450	16.999	1.240	16.994	15.728	18.294	4551.623	2522.478	4149.080	2326.430	7148.363	0.322	0.045	0.312	0.284	0.369	0.216	0.084	0.209	0.018	0.044	0.000	0.767	0.084	0.778	-8005.763	-8593.328	-7343.429	0.022
2000	0.298	0.239	0.221	0.144	0.571	18.037	1.003	18.152	16.894	19.084	2199.797	1213.460	1727.321	1083.923	4086.093	0.320	0.038	0.318	0.282	0.361	0.197	0.080	0.189	0.035	0.046	0.023	0.768	0.064	0.772	-14877.704	-15688.117	-13736.585	0.045

Table D.246: exponential-exponential-weibull mixture on mid-price waiting time SSELN data with censoring on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0205	0.1238	0.1098	0.1975
4-comp exponential	0.5820	-	0.3172	0.8025
exponential/exponential/weibull	0.3975	0.8762	0.5730	-

Table D.247: ABFLN, N = 200: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.2065	0.5119	0.7026	-
4-comp exponential	0.6187	-	0.0672	0.4486
exponential/exponential/weibull	0.1748	0.4881	0.2302	0.5508

Table D.248: BARC, N = 200: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0464	0.2769	0.2488	0.3398
4-comp exponential	0.6692	-	0.3090	0.6602
exponential/exponential/weibull	0.2844	0.7231	0.4422	-

Table D.249: RRLN, N = 200: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0656	0.2627	0.3511	0.4881
4-comp exponential	0.6579	-	0.1921	0.5115
exponential/exponential/weibull	0.2764	0.7372	0.4567	-

Table D.250: VOD, N = 200: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.2051	0.4822	0.7196	-
4-comp exponential	0.5974	-	0.0477	0.3561
exponential/exponential/weibull	0.1975	0.5177	0.2326	0.6432

Table D.251: RIOTINTO, N = 200: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0204	0.1382	0.1128	0.2020
4-comp exponential	0.5787	-	0.2875	0.7980
exponential/exponential/weibull	0.4009	0.8618	0.5998	-

Table D.252: SSELN, N = 200: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0048	0.0239	0.0096	0.0311
4-comp exponential	0.4952	0.9761	0.3373	0.9689
exponential/exponential/weibull	0.5000	-	0.6531	-

Table D.253: ABFLN, N = 500: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0446	0.3266	0.3105	0.4609
4-comp exponential	0.7730	-	0.3618	-
exponential/exponential/weibull	0.1822	0.6732	0.3275	0.5389

Table D.254: BARC, N = 500: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0079	0.1820	0.0413	0.2027
4-comp exponential	0.6494	-	0.4825	-
exponential/exponential/weibull	0.3426	0.8180	0.4762	0.7973

Table D.255: RRLN, N = 500: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0163	0.1143	0.0659	0.1065
4-comp exponential	0.6287	-	0.4046	0.8922
exponential/exponential/weibull	0.3547	0.8852	0.5292	-

Table D.256: VOD, N = 500: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0384	0.2584	0.3085	0.4042
4-comp exponential	0.7345	-	0.2882	0.5950
exponential/exponential/weibull	0.2270	0.7414	0.4032	-

Table D.257: RIOTINTO, N = 500: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0017	0.0140	0.0052	0.0297
4-comp exponential	0.4528	0.9860	0.3077	0.9703
exponential/exponential/weibull	0.5455	-	0.6871	-

Table D.258: SSELN, N = 500: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0000	0.0301	0.0000	0.0301
4-comp exponential	0.3759	0.9699	0.3233	0.9699
exponential/exponential/weibull	0.6241	-	0.6767	-

Table D.259: ABFLN, N = 1000: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0180	0.2237	0.0861	0.2914
4-comp exponential	0.7888	-	0.6104	-
exponential/exponential/weibull	0.1930	0.7761	0.3033	0.7083

Table D.260: BARC, N = 1000: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0018	0.1248	0.0071	0.1283
4-comp exponential	0.6203	-	0.5312	-
exponential/exponential/weibull	0.3779	0.8752	0.4617	0.8717

Table D.261: RRLN, N = 1000: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0072	0.0611	0.0150	0.0399
4-comp exponential	0.5678	-	0.4348	0.9581
exponential/exponential/weibull	0.4249	0.9389	0.5502	-

Table D.262: VOD, N = 1000: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0154	0.1443	0.0748	0.2188
4-comp exponential	0.7365	-	0.5236	-
exponential/exponential/weibull	0.2477	0.8555	0.4012	0.7809

Table D.263: RIOTINTO, N = 1000: mid-price waiting times, censored on region [0, 0.5].

	AIC		BIC	
3-comp exponential	0.0000	0.0135	0.0000	0.0135
4-comp exponential	0.3049	0.9865	0.2422	0.9865
exponential/exponential/weibull	0.6951	-	0.7578	-

Table D.264: SSELN, N = 1000: mid-price waiting times, censored on region [0, 0.5].

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