Kimberly E. Becker, Michael K. Murray, Daniel Stevenson
The Weyl map and bundle gerbes
Journal of Geometry and Physics, 2020; 149:103572-1-103572-19

Final publication at: http://dx.doi.org/10.1016/j.geomphys.2019.103572

THE WEYL MAP AND BUNDLE GERBES

KIMBERLY E. BECKER, MICHAEL K. MURRAY, AND DANIEL STEVENSON

Abstract. We introduce the notion of a general cup product bundle gerbe and use it to define the Weyl bundle gerbe on $T \times SU(n)/T$. The Weyl map from $T \times SU(n)/T$ to $SU(n)$ is then used to show that the pullback of the basic bundle gerbe on $SU(n)$ defined by the second two authors is stably isomorphic to the Weyl bundle gerbe as $SU(n)$-equivariant bundle gerbes. Both bundle gerbes come equipped with connections and curvings and by considering the holonomy of these we show that these bundle gerbes are not D-stably isomorphic.

Contents

Date: December 13, 2019.
2010 *Mathematics Subject Classification.* 53C08, 55N45.
Key words and phrases. bundle gerbe, basic bundle gerbe, cup product bundle gerbe, stable isomorphism.
The second two authors acknowledge support under the Australian Research Council's *Discovery Projects* funding scheme (project number DP180100383). The first author acknowledges the support of an *Australian Government Research Training Program Scholarship*.

1. Introduction

The *basic bundle gerbe* is defined over a compact, simple, simply connected Lie group G and has Dixmier-Douady class equal to a generator of $H^3(G,\mathbb{Z}) \simeq \mathbb{Z}$. In this work, we will restrict our study to the case where $G = SU(n)$ and use the finite-dimensional construction of the basic bundle gerbe in this case given by the second two authors [16], see also [11]. The history of the various constructions of the basic bundle gerbe and its extensions to other groups can be found in the Introduction to [16]. As well as the construction of the basic bundle gerbe over $SU(n)$, [16] gives an explicit connection and curving on this bundle gerbe with three-curvature equal to $2\pi i$ times the basic 3-form

$$\nu = -\frac{1}{24\pi^2}\,\mathrm{tr}(g^{-1}dg)^3$$

on $SU(n)$. These explicit formulae will be used extensively in our work.

Our work here can be understood as a follow up to [16]. Namely, we study the pullback of the basic gerbe over $SU(n)$ from [16] by the Weyl map

$$p\colon T \times SU(n)/T \to SU(n)$$
$$p(t, hT) = hth^{-1}$$

and explain why the pulled back bundle gerbe decomposes into simpler objects. Our motivation for this is the following observation. The pullback of the basic 3-form by the Weyl map, $p^*\nu$, defines a class in $H^3\left(T \times SU(n)/T\right)$. By the Kunneth formula, and noting that the cohomology of $SU(n)/T$ vanishes in odd degree [4], we see that

$$[p^*\nu] \in H^3(T) \oplus \left(H^2\left(SU(n)/T\right) \otimes H^1(T)\right).$$

It follows from T being abelian that the restriction of ν to T vanishes. Therefore

$$[p^*\nu] \in H^2\left(SU(n)/T\right) \otimes H^1(T).$$

For this reason, we expect that the three-curvature of the pullback of the basic bundle gerbe by the Weyl map to equal a sum of wedge products of 1-forms and 2-forms (modulo exact forms). We are interested in the impact this fact has on the geometry of the pulled back basic bundle gerbe. It follows from work of Johnson in [7] that a bundle gerbe whose Dixmier-Douady class is the cup product of a one-class and a two-class can be realised by a geometric cup product construction from a $U(1)$-valued function whose winding class is the one-class and a line bundle whose chern class is the two-class. We expect therefore that the pullback of the basic bundle gerbe by the Weyl map is stably isomorphic to a particular product of these cup product bundle gerbes which we call the Weyl bundle gerbe.

Following Johnson [7] we introduce the *cup product bundle gerbe* construction. The *Weyl bundle gerbe* is then defined as a reduced product (see Section 2.2 below) of certain cup product bundle gerbes over $T \times SU(n)/T$ for T the maximal torus of $SU(n)$ consisting of diagonal matrices. We call any such bundle gerbe that is the reduced product of cup product bundle gerbes a *general cup product bundle gerbe*. In our main theorem (Theorem 6.8), we

(1) describe an explicit $SU(n)$-equivariant stable isomorphism of the Weyl bundle gerbe and the pullback of the basic bundle gerbe over $SU(n)$ via the Weyl map $p : T \times SU(n)/T \to SU(n)$;
(2) explicitly define a 2-form $\beta \in \Omega^2(T \times SU(n)/T)$ that satisfies $d\beta = \omega_{p^*b} - \omega_c$ for ω_{p^*b} and ω_c the three-curvatures associated to the pullback connective data on the pullback of the basic bundle gerbe and natural connective data on the Weyl bundle gerbe, respectively;
(3) show there is no general cup product bundle gerbe that is stably isomorphic to the pullback of the basic bundle gerbe such that the three-curvature ω'_c associated to the induced connective data on the general cup product bundle gerbe satisfies $\omega'_c = \omega_{p^*b}$; and
(4) consider the holonomies of our bundle gerbes to show that the pullback of the basic bundle gerbe and the Weyl bundle gerbe are not D-stably isomorphic with respect to their natural connective data.

There is a long history of exploiting the attractive features of the Weyl map going back to Weyl's proof of the Integral Formula [18] and the K-theory of compact Lie groups [1]. Cup product constructions similar to ours have been used by Brylinski [5] to construct projective unitary group bundles, in index theory [10] and in twisted K-theory [6, 2]. We note also that [9, Section 8.2] gives a general construction of the cup product bundle gerbe for a decomposable Dixmier-Douady class.

In summary, in Section 2 we briefly review basic results, definitions and notation from the theory of bundle gerbes. In Section 3, we introduce the notions of cup product and general cup product bundle gerbes, and study the geometry of the former. Criteria for general cup product bundle gerbes to be stably isomorphic are also considered. Next, in Section 4, we apply the theory from Section 3 to construct the Weyl bundle gerbe over $T \times SU(n)/T$. The pullback of the basic bundle gerbe is considered in Section 5, and we show that it is also a general cup product bundle gerbe over $T \times SU(n)/T$. The stable isomorphism between the pullback of the basic bundle gerbe and the Weyl bundle gerbe is given in Section 6, where we exploit the results of Section 3 using the fact that both bundle gerbes are general cup product bundle gerbes. We conclude Section 6 by demonstrating that these bundle gerbes, with their given connections and curvings, are not stably isomorphic, i.e. they are not D-stably isomorphic. Our results are summarised in Theorem 6.8.

2. Bundle gerbes

We review some notation and basic facts about bundle gerbes. For more detail on bundle gerbes see [12, 13, 3].

2.1. **Surjective submersions.** Let $\pi\colon Y \to M$ be a surjective submersion. Denote by $Y^{[p]}$ the p-fold fibre product of Y; note that the canonical map $Y^{[p]} \to M$ is also a surjective submersion. Define $\pi_i\colon Y^{[p+1]} \to Y^{[p]}$ by omitting the i-th entry for each $i = 1, \dots, p+1$. Notice that this means that the two maps $Y^{[2]} \to Y$ are (perhaps confusingly) $\pi_1((y_1, y_2)) = y_2$ and $\pi_2((y_1, y_2)) = y_1$. If $g\colon Y^{[p]} \to A$ is a map to an abelian group A, define $\delta(g)\colon Y^{[p+1]} \to A$ by $\delta(g) = g \circ \pi_1 - g \circ \pi_2 + \cdots$ and if $g\colon M \to A$ define $\delta(g)\colon Y \to A$ by $\delta(g) = g \circ \pi$.

Similarly, if $\omega \in \Omega^q(Y^{[p]})$ is a q-form, define $\delta(\omega) \in \Omega^q(Y^{[p+1]})$ by $\delta(\omega) = \pi_1^*(\omega) - \pi_2^*(\omega) + \cdots$ and likewise $\delta(\omega) = \pi^*(\omega)$ if $\omega \in \Omega^q(M)$. It is straightforward to check that the *fundamental complex* defined by

$$0 \to \Omega^q(M) \xrightarrow{\delta} \Omega^q(Y) \xrightarrow{\delta} \Omega^q(Y^{[2]}) \xrightarrow{\delta} \Omega^q(Y^{[3]}) \xrightarrow{\delta} \cdots \tag{2.1}$$

is exact [12, Section 8][1].

If $K \to Y^{[p]}$ is a Hermitian line bundle, define a Hermitian line bundle $\delta(K)$ over $Y^{[p+1]}$ by $\delta(K) = \pi_1^{-1}(K) \otimes \pi_2^{-1}(K)^* \otimes \cdots$. Note that the Hermitian line bundle $\delta\delta(K)$ over $Y^{[p+2]}$ has a canonical trivialisation. If K is equipped with a connection ∇_K, then there is an induced connection $\delta(\nabla_K)$ on $\delta(K)$.

2.2. **Bundle gerbes.** Let M be a manifold. Recall that a *bundle gerbe* over M, denoted (P, Y) or (P, Y, π), consists of a surjective submersion $\pi : Y \to M$ and a Hermitian line bundle $P \to Y^{[2]}$. The bundle P is equipped with a *bundle gerbe multiplication* $\pi_3^{-1}(P) \otimes \pi_1^{-1}(P) \to \pi_2^{-1}(P)$ which is associative in the sense that the two different ways of mapping

$$P_{(y_1,y_2)} \otimes P_{(y_2,y_3)} \otimes P_{(y_3,y_4)} \to P_{(y_1,y_4)}$$

agree for any $(y_1, y_2, y_3, y_4) \in Y^{[4]}$.

If (P, Y) is a bundle gerbe over M and (Q, X) is a bundle gerbe over N, then a morphism of bundle gerbes $(P, Y) \to (Q, X)$ is a triple of maps $f\colon M \to N$, $g\colon Y \to X$ and $h\colon P \to Q$. These have to satisfy: g covers f and thus induces a map $g^{[2]}\colon Y^{[2]} \to X^{[2]}$ and $h\colon P \to Q$ is a bundle morphism covering $g^{[2]}$.

A bundle gerbe is called *trivial* if there is a Hermitian line bundle $R \to Y$ and an isomorphism of P to $\delta(R) = \pi_1^{-1}(R) \otimes \pi_2^{-1}(R)^*$ such that the bundle gerbe product on P commutes with the obvious contraction

$$R_{y_2} \otimes R_{y_1}^* \otimes R_{y_3} \otimes R_{y_2}^* \to R_{y_3} \otimes R_{y_1}^*. \tag{2.2}$$

A *trivialisation* of (P, Y) is a choice of such a *trivialising line bundle* R and isomorphism (2.2).

If (P, Y) and (Q, X) are bundle gerbes over M, we define the *dual* of (P, Y) by (P^*, Y) with the obvious multiplication and their *product* $(P, Y) \otimes (Q, X)$ by $(P \otimes Q, Y \times_M X)$, where $Y \times_M X$ is the fibre product of $Y \to M$ and $X \to M$ and

$$(P \otimes Q)_{((y_1,x_1),(y_2,x_2))} = P_{(y_1,y_2)} \otimes Q_{(x_1,x_2)}$$

with the obvious multiplication. In the case that $X = Y$ we have the diagonal inclusion of Y into $Y \times_M Y$ and we may use this to pull back $P \otimes Q$ to define the *reduced product* $(P \otimes_R Q, Y)$.

We say that (P, Y) and (Q, X) are *stably isomorphic* if there exists a trivialisation of $(P, Y)^* \otimes (Q, X)$. Similarly a *stable isomorphism* from (P, Y) to (Q, X) is a choice of such a trivialisation. It is shown in [17] that stable isomorphisms can be composed. The notion of morphism introduced above of course leads to a notion of isomorphism, which is much stronger than the notion of stable isomorphism. We use the notations $\cong$ and $\cong_{\text{stab}}$ for the notions of isomorphism and stable isomorphism respectively.

[1]The proof given there is actually for $Y \to M$ a fibration but can be adapted to the case where $Y \to M$ is a surjective submersion using the fact that surjective submersions admit local sections.

Associated to a bundle gerbe (P, Y) is a characteristic class called its *Dixmier-Douady class* or *DD-class*, $\mathrm{DD}(P, Y) \in H^3(M, \mathbb{Z})$, which determines exactly the stable isomorphism class of the bundle gerbe [15].

If (P, Y) is a bundle gerbe over M and $f \colon N \to M$, then $f^{-1}(Y) \to N$ is a surjective submersion and we have $f^{[2]} \colon f^{-1}(Y)^{[2]} \to Y^{[2]}$. The pullback of P by $f^{[2]}$, more conveniently called $f^{-1}(P)$, then inherits a natural bundle gerbe multiplication. The bundle gerbe $(f^{-1}(P), f^{-1}(Y))$ over N is called the *pullback* of (P, Y) by f. Pulling back preserves products and duals and is natural for the Dixmier-Douady class. For more details see [3].

Similarly, we can consider the pullback of bundle gerbes over M by morphisms of surjective submersions over M. These are maps $f \colon X \to Y$ of surjective submersions $X \to M$ and $Y \to M$ covering the identity map $M \to M$. If (P, Y) is a bundle gerbe we can pull back using $f^{[2]}$ to obtain a bundle gerbe we denote by $(f^{-1}(P), X)$ over M. A basic fact [15, Proposition 3.4] is that $(f^{-1}(P), X)$ is stably isomorphic to (P, Y). Moreover, there is a canonical choice of stable isomorphism. This implies immediately that the product and reduced product of two bundle gerbes are canonically stably isomorphic.

Finally, if G is a Lie group we can consider (strongly) equivariant bundle gerbes where G acts on $Y \to M$ and P, preserving all relevant structure, for a precise definition see, for example, [14]. The notion of stable isomorphism extends to this case by requiring that R also have a G-action and that the isomorphism $\delta(R) \cong P$ be G-equivariant. We use the obvious notation $\cong_G$ and $\cong_{G\text{-stab}}$ for G-equivariant isomorphisms and stable isomorphism between G-equivariant bundle gerbes.

2.3. **Connections, curvings and holonomy.** Any bundle gerbe (P, Y, π) admits a *bundle gerbe connection* ∇ which is a Hermitian connection on P preserving the bundle gerbe multiplication. As a result of this condition, the curvature $F_\nabla \in \Omega^2(Y^{[2]})$ of a bundle gerbe connection satisfies $\delta(F_\nabla) = 0$. It follows from exactness of the fundamental complex (2.1) that there exist imaginary 2-forms $f \in \Omega^2(Y)$ satisfying $\delta(f) = F_\nabla$. A choice of such an $f \in \Omega^2(Y)$ is called a *curving* for ∇, and the pair (∇, f) is called *connective data* for (P, Y). Notice that the curving is determined only up to addition of the pullback of a form in $\Omega^2(M)$. Commutativity of the maps δ in the fundamental complex with exterior differentiation implies $\delta(df) = d\delta(f) = dF_\nabla = 0$, so that $df = \pi^*(\omega)$ for a unique $\omega \in \Omega^3(M)$ called the *three-curvature* of (∇, f). We have that $d\omega = 0$ and the class of $\omega/2\pi i$ in de Rham cohomology is the image of $DD(P, Y)$ in real cohomology under the de Rham isomorphism.

Connective data on bundle gerbes naturally induce connective data on dual, pullback, reduced product and product bundle gerbes. For more on this see [3].

There is a notion of stable isomorphisms of bundle gerbes with connective data. Following Johnson [7] we say two bundle gerbes (P, Y) and (Q, X) with connective data are *D-stably isomorphic*, denoted $(P, Y) \cong_{D\text{-stab}} (Q, X)$, if $(P, Y) \cong_{\text{stab}} (Q, X)$ and this stable isomorphism preserves connections and curvings. Here the trivial bundle gerbe $P^* \otimes Q$ is assumed to have *trivial connective data*, i.e. connective data of the form $(\delta(\nabla_R), F_{\nabla_R})$ for ∇_R a connection on a line bundle $R \to Y$ with curvature F_{∇_R}. The D-stable isomorphism classes of bundle gerbes over M (or *Deligne classes*) are in bijective correspondence with the Deligne cohomology group $H^3(M, \mathbb{Z}(3)_D)$ [15, Theorem 4.1]. For more on this, see [7, 15].

If (P, Y) is a bundle gerbe over an oriented two-dimensional manifold Σ, then $H^3(\Sigma, \mathbb{Z}) = 0$ and there exists a Hermitian line bundle $R \to Y$ trivialising P. Suppose ∇ is a bundle gerbe connection on P and ∇_R is a connection on R. Denote by $\delta(\nabla_R)$ the connection on (P, Y) induced by the isomorphism $P \cong \delta(R)$. As both ∇ and $\delta(\nabla_R)$ are bundle gerbe connections, it follows that $\nabla = \delta(\nabla_R) + \alpha$ for $\alpha \in \Omega^1(Y^{[2]})$ satisfying $\delta(\alpha) = 0$. From exactness of the fundamental complex we can solve $\alpha = \delta(\beta)$ for some $\beta \in \Omega^1(Y)$ and thus $\nabla = \delta(\nabla_R + \beta)$. It follows that we may suppose without loss of generality that $\delta(\nabla_R) = \nabla$. Denote by F_{∇_R} the curvature of ∇_R and note that $\delta(F_{\nabla_R}) = F_\nabla$. Then $\delta(f - F_{\nabla_R}) = F_\nabla - F_\nabla = 0$ so we have $f - F_{\nabla_R} = \pi^*(\mu)$ for $\mu \in \Omega^2(\Sigma)$. We define the *holonomy* of (∇, f) over Σ, $\mathrm{hol}(\nabla, f)$, by $\exp\left(\int_\Sigma \mu\right)$. This is independent of the choice of R and ∇_R. Similarly, if $\chi\colon \Sigma \to M$ and (P, Y) is a bundle gerbe over M with connective data (∇, f), we can define the *holonomy* of (∇, f) over χ, $\mathrm{hol}(\nabla, f, \chi)$, to be the holonomy of the pullback of (∇, f) by χ. Notice that the holonomy depends on the choice of curving. It is a straightforward calculation that if bundle gerbes over M are D-stably isomorphic they have the same holonomy.

3. Cup product bundle gerbes

3.1. **The cup product bundle gerbe construction.** Our aim is to show that, after pulling back by the Weyl map $p\colon T \times SU(n)/T \to SU(n)$, the basic bundle gerbe decomposes into a product of simpler objects. In this section, we define these simpler objects (namely *cup product bundle gerbes*), and consider more generally the class of bundle gerbes that decompose into such objects (namely *general cup product bundle gerbes*).

In [7], Johnson constructed the *cup product bundle gerbe* $(f \cup L, f^{-1}(\mathbb{R}))$ over M from a smooth map $f : M \to S^1$ and a line bundle $L \to M$. The motivating idea for this construction was that the Dixmier-Douady class of the cup product bundle gerbe should be the cup product of the winding class of f and the chern class of L. The definition is as follows: let $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = U(1)$ and note that $f^{-1}(\mathbb{R}) \to M$ is a surjective submersion, in fact a principal $\mathbb{Z}$-bundle. As a result there is a well-defined map $d\colon f^{-1}(\mathbb{R})^{[2]} \to \mathbb{Z}$ given by $d(x, y) = y - x$ and satisfying $\delta(d) = 0$, in the sense of Section 2.1. Johnson's cup product bundle gerbe is then defined by $(f \cup L)_{(m,x,y)} = L_m^{d(x,y)} = L_m^{(y-x)}$. Tensor product gives rise to a bundle gerbe product via the obvious identification $L^{(y-x)} \otimes L^{(z-y)} = L^{(z-x)}$. Note that if $n < 0$ then we define $L^n = (L^*)^{-n}$ and L^0 is the trivial bundle.

For our purposes we need more general surjective submersions. Notice first that if $K \to X$ is a Hermitian line bundle and $h\colon X \to \mathbb{Z}$ is a smooth function (that is, locally constant), then there is a Hermitian line bundle $K^h \to X$ defined fibrewise by

$$(K^h)_x = (K_x)^{h(x)}$$

for every $x \in X$. In other words, K^h on each connected component of X is just K raised to the tensor power determined by the constant value of h on that component.

We consider $Y \to M$ a surjective submersion, $g\colon Y^{[2]} \to \mathbb{Z}$ with $\delta(g) = 0$ and $L \to M$ a line bundle. Let $L^g \to Y^{[2]}$ be $(\pi^{[2]})^{-1}(L)^g$ where $\pi^{[2]}\colon Y^{[2]} \to M$. We will often abuse notation like this and omit obvious projections. The existence of the bundle gerbe product follows from the fact that $\delta(g) = 0$. We have the following definition.

Definition 3.1. Let $Y \to M$ be a surjective submersion, $L \to M$ be a line bundle, and $g : Y^{[2]} \to \mathbb{Z}$ be a smooth map satisfying $\delta(g) = 0$. The bundle gerbe (L^g, Y) over M is the *cup product bundle gerbe over M of $L \to M$ and $g : Y^{[2]} \to \mathbb{Z}$.*

Any use of the term 'cup product bundle gerbe' henceforth shall refer to Definition 3.1, rather than Johnson's definition mentioned above. More generally we have the following definition:

Definition 3.2. If $(L_i^{g_i}, Y)$ are cup product bundle gerbes over M for $i = 1, \dots, n$, we call the reduced product $\otimes_R (L_i^{g_i}, Y)$ the *general cup product bundle gerbe of $L_i \to M$ and $g_i : Y^{[2]} \to \mathbb{Z}$ for $i = 1, \dots, n$.*

Associated to $g \colon Y^{[2]} \to \mathbb{Z}$ is a class in $H^1(M, \mathbb{Z})$. This is defined by using the fundamental complex (2.1) to solve $\delta(\psi) = g$ for some $\psi \colon Y \to \mathbb{R}$ and taking the class to be the winding class of the map $q : M \to U(1)$ whose value at $m \in M$ is $\exp(2\pi i \psi(y))$, where $\pi(y) = m$. This class is represented in de Rham cohomology by $\frac{1}{2\pi i} q^{-1} dq$. A straightforward calculation shows that the Dixmier-Douady class of the cup product bundle gerbe (L^g, Y) is the cup product of this class and the chern class of L. Hence it is decomposable in the sense of [**?**]. More generally, the Dixmier-Douady class of a general cup product bundle gerbe is a sum of such cup products.

3.2. **Stable isomorphisms of general cup product bundle gerbes.** In this section we will consider sufficient criteria for cup product bundle gerbes and general cup product bundle gerbes to be stably isomorphic. These results, in particular Corollary 3.5, will simplify calculations in Section 6. The proofs of the following are straightforward:

Proposition 3.3. *Let (L^g, Y) be a cup product bundle gerbe. If there exists a smooth map $h : Y \to \mathbb{Z}$ such that $g = \delta(h)$, then (L^g, Y) is trivialised by $L^h \to Y$.*

Corollary 3.4. *Let $\left(L^f, Y\right)$ and (L^g, X) be cup product bundle gerbes over M. If there exists a smooth map $h : X \times_M Y \to \mathbb{Z}$ such that $f - g = \delta(h)$, then*

$$\left(L^f, Y\right) \cong_{\text{stab}} (L^g, X)$$

with trivialising line bundle $L^h \to X \times_M Y$.

Corollary 3.5. *Let $(L_i^{f_i}, Y)$ and $(L_i^{g_i}, X)$ be cup product bundle gerbes over M for $i = 1, \dots, n$. If there exist smooth maps $h_i : X \times_M Y \to \mathbb{Z}$ for each $i = 1, \dots, n$ satisfying $f_i - g_i = \delta(h_i)$ for all $i = 1, \dots, n$, then*

$$\bigotimes_{i=1}^{n}{}_{\mathrm{R}} (L_i^{f_i}, Y) \cong_{\text{stab}} \bigotimes_{i=1}^{n}{}_{\mathrm{R}} (L_i^{g_i}, X)$$

with trivialising line bundle $\otimes_i L_i^{h_i} \to X \times_M Y$.

3.3. **Geometry of cup product bundle gerbes.** We next describe connective data (∇, f) and compute the associated three-curvature ω on a cup product bundle gerbe. The induced connective data on a general cup product bundle gerbe can then be easily inferred.

Let (L^g, Y) be a cup product bundle gerbe over M and ∇ be a connection on $L \to M$ with curvature F_∇. Then $L^g \to Y^{[2]}$ restricted to each connected component of $Y^{[2]}$ is a tensor power of L (pulled back to $Y^{[2]}$) with the power determined by the corresponding value of $g\colon Y^{[2]} \to \mathbb{Z}$. Taking appropriate tensor powers of ∇ gives a natural connection for L^g which we denote by ∇^g. It is easy to check that this is a bundle gerbe connection and that it has curvature:

$$F_{\nabla^g} = g\, \pi^{[2]^*} F_\nabla .$$

To construct a curving for this connection we need a small amount of additional data as follows:

Proposition 3.6. *Let (L^g, Y, π) be a cup product bundle gerbe over M, ∇ be a connection on $L \to M$, and ∇^g be the bundle gerbe connection defined above. Then*

1. *there exists a smooth function $\varphi : Y \to \mathbb{R}$ such that $\delta(\varphi) = g$;*
2. *the 2-form $f \in \Omega^2(Y)$ defined by*
$$f = \varphi\, \pi^* F_\nabla$$
satisfies $\delta(f) = F_{\nabla^g}$, so f is a curving for the connection ∇^g;
3. *if $q\colon M \to U(1)$ is defined by $q = \exp(2\pi i \varphi)$, the three-curvature $\omega \in \Omega^3(M)$ of (∇^g, f) is given by*
$$\omega = \frac{q^{-1}dq}{2\pi i} \wedge F_\nabla ;$$
4. *the real DD-class of (L^g, Y) is represented by*
$$-\frac{1}{4\pi^2} q^{-1} dq \wedge F_\nabla .$$

Proof. The existence of φ follows by exactness of the fundamental complex and $\delta(g) = 0$. For (2) we have

$$\delta(f) = \delta(\psi)\pi^{[2]^*}(F_\nabla) = g\pi^{[2]^*}(F_\nabla) = F_{\nabla^g}.$$

Equations (3) and (4) follow from definitions. □

Similarly in the general cup product gerbe case we have the following Proposition.

Proposition 3.7. *For each $i = 1, \ldots, n$ let $(L_i^{g_i}, Y, \pi)$ be a cup product bundle gerbe over M, ∇_i be a connection on $L_i \to M$, and $\nabla_i^{g_i}$ be the corresponding bundle gerbe connection defined above. Then*

1. *there exist smooth functions $\varphi_i : Y \to \mathbb{R}$ such that $\delta(\varphi_i) = g_i$;*
2. *the 2-form $f \in \Omega^2(Y)$ defined by*
$$f = \sum_{i=1}^{n} \varphi_i\, \pi^* F_{\nabla_i}$$
satisfies $\delta(f) = \sum_{i=1}^n F_{\nabla_i^{g_i}}$, so f is a curving for the product connection induced by the $\nabla_i^{g_i}$;
3. *if $q_i\colon M \to U(1)$ is defined by $q_i = \exp(2\pi i \varphi_i)$, the three-curvature $\omega \in \Omega^3(M)$ of the general cup product gerbe of the $(L_i^{g_i}, Y, \pi)$ is given by*
$$\omega = \sum_{i=1}^{n} \frac{q_i^{-1} dq_i}{2\pi i} \wedge F_{\nabla_i};$$
4. *the real DD-class of the general cup product bundle gerbe of the $(L_i^{g_i}, Y, \pi)$ is represented by*
$$-\frac{1}{4\pi^2} \sum_{i=1}^{n} q_i^{-1} dq_i \wedge F_{\nabla_i}.$$

4. Cup product bundle gerbes over $T \times SU(n)/T$

4.1. **The i-th cup product bundle gerbes.** In this section, we will define cup product bundle gerbes over $T \times SU(n)/T$ called the *i-th cup product bundle gerbes*, for T the subgroup of $SU(n)$ consisting of diagonal matrices. Our aim is to construct their reduced product, which we call the *Weyl bundle gerbe*. We begin with some preliminaries. For $n \in \mathbb{N}$, let Proj_n be the set of n-tuples of orthogonal projections $(P_1, \dots, P_n)$, where, for each i, $P_i : \mathbb{C}^n \to W_i$ for W_i mutually orthogonal one-dimensional subspaces of $\mathbb{C}^n$. It follows from the characterisation of homogeneous spaces [8, Theorem 21.18] that there is a bijection $SU(n)/T \cong \mathrm{Proj}_n$, which implies Proj_n is a smooth manifold diffeomorphic to $SU(n)/T$.

For each $i = 1, \dots, n$, let $p_i \colon T \to S^1$ be the homomorphism $p_i(t_1, \dots, t_n) = t_i$. Define $J_i \to SU(n)/T$ to be the (homogeneous) Hermitian line bundle associated to the principal T-bundle $SU(n) \to SU(n)/T$ via the action of T on $\mathbb{C}$ by $t \cdot z = p_i(t^{-1})z$. Define $K_i \to \mathrm{Proj}_n$, a subbundle of the trivial bundle $\mathbb{C}^n \times \mathrm{Proj}_n$, by $(K_i)_{(P_1,\dots,P_n)} = \mathrm{im}(P_i) \times \{(P_1, \dots, P_n)\}$. It can be verified easily that J_i and K_i are $SU(n)$-equivariant line bundles with respect to the $SU(n)$-action on $SU(n)/T$ defined by left multiplication and the $SU(n)$-action on Proj_n defined by

$$g \cdot (v, P_1, \dots, P_n) = (gv, gP_1g^{-1}, \dots, gP_ng^{-1}).$$

By the equivalence of linear representations and equivariant line bundles, there is an $SU(n)$-equivariant isomorphism $J_i \cong K_i$. Throughout this work, we will continue to write $J_i \to SU(n)/T$, but will in practice work with the line bundles $K_i \to \mathrm{Proj}_n$.

Denote by X_T the subset of $(x_1, \dots, x_n) \in \mathbb{R}^n$ which sum to zero. Define a surjective submersion $\pi \colon X_T \to T$ by

$$\pi(x_1, \dots, x_n) = \mathrm{diag}(e^{2\pi i x_1}, \dots, e^{2\pi i x_n}).$$

Note that this defines a surjective submersion $\pi_c \colon X_T \times SU(n)/T \to T \times SU(n)/T$ and that $(X_T \times SU(n)/T)^{[2]} = X_T^{[2]} \times SU(n)/T$. Define $d_i : X_T^{[2]} \to \mathbb{Z}$ by $d_i(x, y) = x_i - y_i$ and extend it to $X_T^{[2]} \times SU(n)/T$ by projection with the same name.

The i-th cup product bundle gerbe is defined as follows.

Definition 4.1. The *i-th cup product bundle gerbe over $T \times SU(n)/T$ for $i = 1, \dots, n$* is the cup product bundle gerbe

$$\left(J_i^{d_i}, X_T \times SU(n)/T, \pi_c\right).$$

Proposition 4.2. *The i-th cup product bundle gerbe is $SU(n)$-equivariant for the action of $SU(n)$ on $T \times SU(n)/T$ defined by multiplication in the $SU(n)/T$ factor.*

Proof. This follows easily by $SU(n)$-homogeneity of $J_i \to SU(n)/T$ and by noting that the $SU(n)$-action on each of the spaces in the bundle gerbe is given by multiplication on the $SU(n)/T$ factor. □

4.2. **Geometry of the i-th cup product bundle gerbes.** We will now apply the results from Section 3.3 to the i-th cup product bundle gerbes. The following standard fact will be used repeatedly: let $L \to M$ be a line bundle that is a subbundle of the trivial bundle of rank n. Let $P : \mathbb{C}^n \times M \to L$ be orthogonal projection. Then the induced connection $\nabla = P \circ d$ on L (for d the trivial connection) has curvature $F_\nabla = \mathrm{tr}(PdPdP)$.

Proposition 4.3. *There is a canonical line bundle connection ∇_{J_i} on $J_i \to SU(n)/T$ with curvature $F_{\nabla_{J_i}} = \mathrm{tr}(P_i dP_i dP_i)$ for $P_i : SU(n)/T \times \mathbb{C}^n \to J_i$ orthogonal projection.*

Proof. By the standard fact above, we need only show that J_i is a subbundle of the trivial bundle of rank n. This follows by noting that J_i is a subbundle of the $SU(n)$-homogeneous vector bundle $(\mathbb{C}^n \times SU(n))/T \to SU(n)/T$, which is isomorphic to the trivial bundle $\mathbb{C}^n \times SU(n)/T \to SU(n)/T$ by the equivalence of linear representations and equivariant bundles. □

Proposition 4.4. *Let ∇_{J_i} be the connection on J_i from* Proposition 4.3. *Let*

$$\pi_c^{[2]} : X_T^{[2]} \times SU(n)/T \to T \times SU(n)/T$$

be projection. Then there is a bundle gerbe connection ∇_{c_i} on the i-th cup product bundle gerbe with curvature

$$F_{\nabla_{c_i}} = d_i\,(\pi_c^{[2]})^* \mathrm{tr}(P_i dP_i dP_i)$$

for $P_i : T \times SU(n)/T \times \mathbb{C}^n \to J_i$ orthogonal projection.

Proof. This follows from our discussion in Section 3.3 and Proposition 4.3. □

Proposition 4.5. *Let ∇_{c_i} be the connection on the i-th cup product bundle gerbe from* Proposition 4.4. *Let $P_i : T \times SU(n)/T \times \mathbb{C}^n \to J_i$ be orthogonal projection, and define a 2-form $f_{c_i} \in \Omega^2(X_T \times SU(n)/T)$ by*

$$f_{c_i}(t, gT, x) = -x_i\,\pi_c^* \mathrm{tr}(P_i dP_i dP_i).$$

Abuse notation and denote the pullback of p_i to $T \times SU(n)/T$ by p_i. Then

1. *the 2-form f_{c_i} satisfies $\delta(f_{c_i}) = F_{\nabla_{c_i}}$, so f_{c_i} is a curving for ∇_{c_i};*
2. *the three-curvature $\omega_{c_i} \in \Omega^3(T \times SU(n)/T)$ of (∇_{c_i}, f_{c_i}) is given by*

$$\omega_{c_i} = -\frac{1}{2\pi i}\, p_i^{-1} dp_i \mathrm{tr}(P_i dP_i dP_i);$$

3. *the real DD-class of the i-th cup product bundle gerbe is represented by*

$$\frac{1}{4\pi^2}\, p_i^{-1} dp_i \mathrm{tr}(P_i dP_i dP_i).$$

Proof. Consider the proof of Proposition 3.6. If $\varphi(y) = -y_i$ then $\delta(\varphi)(x,y) = \varphi(y) - \varphi(x) = x_i - y_i = d_i(x,y)$ and $q(x) = \exp(-2\pi i x_i) = p_i^{-1}$. The results then follow by substitution into the formula in Proposition 3.6 □

By comparing this result to Proposition 3.6, we see that the real Dixmier-Douady class of the i-th cup product bundle gerbe is the cup product of the winding class of the map $p_i^{-1} \colon T \to U(1)$ and the chern class of the line bundle $J_i \to SU(n)/T$.

4.3. **The Weyl bundle gerbe.** We can now define the *Weyl bundle gerbe*, and compute its connective data and associated three-curvature using results from Section 4.2.

Definition 4.6. The *Weyl bundle gerbe* over $T \times SU(n)/T$ is the reduced product of the i-th cup product bundle gerbes, denoted

$$(P_c, X_T \times SU(n)/T, \pi_c) := \bigotimes_{i=1}^{n}{}_{\mathrm{R}} \left(J_i^{d_i}, X_T \times SU(n)/T\right).$$

Proposition 4.7. *The Weyl bundle gerbe is $SU(n)$-equivariant for the action of $SU(n)$ on $T \times SU(n)/T$ defined by multiplication in the $SU(n)/T$ factor.*

Proof. This follows from Proposition 4.2 and the fact that the reduced product of equivariant bundle gerbes is again equivariant. □

We compute connective data and the curvature of the Weyl bundle gerbe as follows.

Proposition 4.8. *Let $P_i : SU(n)/T \times T \times \mathbb{C}^n \to J_i$ be orthogonal projection and $\pi^{[2]} : X_T^{[2]} \times SU(n)/T \to T \times SU(n)/T$ be projection. The i-th cup product bundle gerbe connections ∇_{c_i} from* Proposition 4.4 *induce a bundle gerbe connection ∇_c on the Weyl bundle gerbe with curvature*

$$F_{\nabla_c} = \sum_{i=1}^{n} d_i (\pi^{[2]})^* \mathrm{tr}(P_i dP_i dP_i).$$

Proof. This follows from elementary bundle gerbe theory and Proposition 4.4. □

Using Proposition 4.5 and elementary facts about products of bundle gerbe connections and curvings we similarly obtain the connective data on the Weyl bundle gerbe.

Proposition 4.9. *Let ∇_c be the connection on the Weyl bundle gerbe from* Proposition 4.8*. Let $P_i : T \times SU(n)/T \times \mathbb{C}^n \to J_i$ be orthogonal projection, and define a 2-form $f_c \in \Omega^2 (X_T \times SU(n)/T)$ by*

$$f_c(x_1, \dots, x_n, gT) := -\sum_{i=1}^{n} x_i \pi_c^* \mathrm{tr}(P_i dP_i dP_i). \tag{4.1}$$

Abuse notation and denote the pullback of p_i to $T \times SU(n)/T$ by p_i. Then

(1) *the 2-form f_c satisfies $\delta(f_c) = F_{\nabla_c}$, so f_c is a curving for ∇_c;*
(2) *the three-curvature $\omega_c \in \Omega^3 (T \times SU(n)/T)$ of (∇_c, f_c) is given by*

$$\omega_c = -\frac{1}{2\pi i} \sum_{i=1}^{n} p_i^{-1} dp_i \mathrm{tr}(P_i dP_i dP_i);$$

(3) *the real DD-class of (P_c, X_T) is represented by*

$$\frac{1}{4\pi^2} \sum_{i=1}^{n} p_i^{-1} dp_i \mathrm{tr}(P_i dP_i dP_i).$$

5. The basic bundle gerbe and the Weyl map

5.1. **The basic bundle gerbe.** We review the construction of the basic bundle gerbe over $SU(n)$ in [16]. Our aim is to show that, when pulled back to $T \times SU(n)/T$ by the Weyl map, the basic bundle gerbe is a general cup product bundle gerbe different to the one defined in 4.6. We will then exploit the techniques from the first section to construct a stable isomorphism between them.

Let $Z := U(1) \backslash \{1\}$ and define the manifold

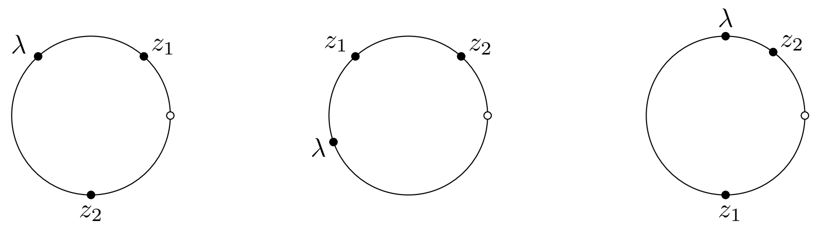
$$Y := \{(z, g) \in Z \times SU(n) \mid z \notin \mathrm{spec}(g)\}.$$

Let $\pi_b : Y \to SU(n)$ be the surjective submersion defined by projection onto the second factor. Note that $Y^{[2]}$ can be identified with triples (z_1, z_2, g) with

$z_1, z_2 \notin \mathrm{spec}(g)$, the set of eigenvalues of g. Order the set Z by anti-clockwise rotation and define the following notions.

Definition 5.1. Let $(z_1, z_2, g) \in Y^{[2]}$ and λ be an eigenvalue of g. Say that $\lambda \in Z$ is *between* z_1 and z_2 if $z_1 < \lambda < z_2$ or $z_2 < \lambda < z_1$. Call $(z_1, z_2, g) \in Y^{[2]}$ *positive* if there exist eigenvalues of g between $z_1 > z_2$, *null* if there are no eigenvalues of g between z_1 and z_2, and *negative* if there exist eigenvalues of g between $z_1 < z_2$.

Denote the set of all positive, null, and negative triplets in $Y^{[2]}$ by $Y^{[2]}_+, Y^{[2]}_0$ and $Y^{[2]}_-$ respectively. Note that $(z_1, z_2, g) \in Y^{[2]}_+$ if and only if $(z_2, z_1, g) \in Y^{[2]}_-$. Elements in each of these sets are depicted in Figure 5.1, where we assume for simplicity that all eigenvalues of g are in the connected component of $Z\backslash\{z_1, z_2\}$ containing λ.



$(z_1, z_2, g) \in Y^{[2]}_-$ $\quad$ $(z_1, z_2, g) \in Y^{[2]}_0$ $\quad$ $(z_1, z_2, g) \in Y^{[2]}_+$

FIGURE 5.1. Components of $Y^{[2]}$

We define a Hermitian line bundle $P_b \to Y^{[2]}$ as follows. For λ an eigenvalue of $g \in SU(n)$, let $E_{(g,\lambda)}$ denote the λ-eigenspace of g. Define the vector bundle $L \to Y^{[2]}_+$ fibrewise by

$$L_{(z_1,z_2,g)} = \bigoplus_{z_1 > \lambda > z_2} E_{(g,\lambda)}. \tag{5.1}$$

For a proof that this is indeed a vector bundle see [16]. Note that $L_{(z_1,z_2,g)}$ has finite dimension as a finite sum of finite-dimensional spaces. Therefore we can define

$$(P_b)_{(z_1,z_2,g)} = \begin{cases} \det(L_{(z_1,z_2,g)}) & \text{if } (z_1, z_2, g) \in Y^{[2]}_+ \\ \mathbb{C} & \text{if } (z_1, z_2, g) \in Y^{[2]}_0 \\ \det(L_{(z_2,z_1,g)})^* & \text{if } (z_1, z_2, g) \in Y^{[2]}_-. \end{cases}$$

By [16], $P_b \to Y^{[2]}$ is a smooth locally trivial Hermitian line bundle, and there is an associative multiplication operation endowing $(P_b, Y, SU(n))$ with a bundle gerbe structure.

Definition 5.2. Call the bundle gerbe (P_b, Y, π_b) over $SU(n)$ constructed above the *basic bundle gerbe over* $SU(n)$, or simply the *basic bundle gerbe.*

5.2. **The pullback of the basic bundle gerbe by the Weyl map.** Recall that, for G a compact, connected Lie group and T a maximal torus of G, the *Weyl map*[2] is the G-equivariant map defined by

$$p : T \times G/T \to G, \; (t, gT) \mapsto gtg^{-1}.$$

[2]So-called because it is used in the Weyl integral formulae in [18].

The Weyl map has a number of attractive features for our purposes. Firstly, the action of $SU(n)$ by conjugation on itself lifts to an action of $SU(n)$ on $T \times SU(n)/T$ where it acts only on the left of $SU(n)/T$. Secondly, if $g \in SU(n)$, we can decompose $\mathbb{C}^n$ into a direct-sum of distinct eigenspaces of g. These eigenspaces may change in dimension as g varies and thus do not extend to vector bundles over the whole of $SU(n)$. However, on $T \times SU(n)/T$ things are much more pleasant. If we consider (t, hT) which maps to $g = hth^{-1}$, then we can write $\mathbb{C}^n$ as a direct sum of the one-dimensional spaces which are multiples of the standard basis vector or eigenspaces of t. Moreover, if we act on these by h, we decompose $\mathbb{C}^n$ into a direct sum of one-dimensional spaces which are subspaces of the eigenspaces of g. In fact, these one-dimensional spaces are a decomposition of the trivial $\mathbb{C}^n$ bundle over $T \times SU(n)/T$ into homogeneous vector bundles $J_1 \oplus J_2 \oplus \cdots \oplus J_n$ pulled back from $SU(n)/T$. We will make extensive use of these basic geometric facts.

It is straightforward to see that under the identification $SU(n)/T \cong \mathrm{Proj}_n$ the Weyl map $p : T \times \mathrm{Proj}_n \to SU(n)$ is given by

$$p : (t, P_1, \ldots, P_n) \mapsto \sum_{i=1}^{n} p_i(t) P_i.$$

Notice that $p^{-1}(Y)$ is the collection of all $(t, z, gT) \in T \times Z \times SU(n)/T$ with $z \neq t_i$ for any $i = 1, \ldots, n$. If we let $Y_T \subset T \times Z$ be all (t, z) with $z \neq t_i$ for any $i = 1, \ldots, n$ then we have

$$p^{-1}(Y) = Y_T \times SU(n)/T.$$

Our aim in this section is to prove the following Proposition for particular ε_i defined below in Definition 5.6, thereby realising the pullback of the basic bundle gerbe as a general cup product bundle gerbe.

Proposition 5.3. *There is an $SU(n)$-equivariant isomorphism over $T \times SU(n)/T$*

$$p^{-1}\left(P_b, Y\right) \cong_{SU(n)} \bigotimes_{i=1}^{n}{}_{\mathrm{R}} \left(J_i^{\varepsilon_i}, Y_T \times SU(n)/T\right).$$

The proof of Proposition 5.3 relies on the following intermediary isomorphisms over $T \times SU(n)/T$:

$$\begin{aligned} p^{-1}\left(P_b, Y\right) &\overset{\text{Prop 5.5}}{\cong_{SU(n)}} \left(P_{b,T} \times_T SU(n), Y_T \times SU(n)/T\right) \\ &\overset{\text{Prop 5.9}}{\cong_{SU(n)}} \left(\bigotimes_{i=1}^{n} J_i^{\varepsilon_i}, Y_T \times SU(n)/T\right). \end{aligned}$$

We begin with the following proposition and leave the proof of this result as an exercise. Let $P_{b,T} := (P_b)|_{Y_T^{[2]}}$. The restriction of the basic bundle gerbe to T is $(P_{b,T}, Y_T)$.

Proposition 5.4 ([16, p. 1582])**.** *Define $P_{b,T} \times_T SU(n)$ to be the set of equivalence classes in $P_{b,T} \times SU(n)$ under the relation*

$$(v_1 \wedge \cdots \wedge v_k, g) \sim (tv_1 \wedge \cdots \wedge tv_k, gt^{-1})$$

for all $t \in T$ where k is the rank of L in (5.1)*. Then $P_{b,T} \times_T SU(n)$ is a line bundle over $Y_T^{[2]} \times SU(n)/T$, and there is an associative multiplication induced by that on*

$(P_{b,T}, Y_T)$ *making*

$$(P_{b,T} \times_T SU(n), Y_T \times SU(n)/T) \tag{5.2}$$

an $SU(n)$-equivariant bundle gerbe over $T \times SU(n)/T$ with respect to the $SU(n)$-action on $T \times SU(n)/T$ defined by multiplication on the $SU(n)/T$ component.

The following proposition is shown in [16, Proposition 7.1].

Proposition 5.5 ([16, Proposition 7.3])**.** *There is an $SU(n)$-equivariant bundle gerbe isomorphism over $T \times SU(n)/T$*

$$(P_{b,T} \times_T SU(n), Y_T \times SU(n)/T) \cong_{SU(n)} p^{-1}(P_b, Y).$$

Recall that T is the maximal torus of $SU(n)$ consisting of diagonal matrices, and $p_i : T \to S^1$ is the homomorphism sending $t \in T$ to its i-th diagonal. To define ε_i we use the ordering on Z from Section 5.1. Let $i \in \{1, \dots, n\}$ throughout.

Definition 5.6. Define $\varepsilon_i : Y_T^{[2]} \to \mathbb{Z}$ by

$$\varepsilon_i(z_1, z_2, t) = \begin{cases} 1 & \text{if } z_1 > p_i(t) > z_2 \\ -1 & \text{if } z_2 > p_i(t) > z_1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that ε_i is a smooth function (that is, locally constant) on $Y_T^{[2]}$.

Definition 5.7. Let $\mathbb{C}_{p_i}$ be the space $\mathbb{C}$ equipped with the T-action $v \cdot t := p_i(t^{-1})v$.

Throughout this section, let $\mathbb{C}^1_{p_i} := \mathbb{C}_{p_i}, \mathbb{C}^{-1}_{p_i} := \mathbb{C}^*_{p_i}$, and $\mathbb{C}^0_{p_i} := \mathbb{C}$, where $\mathbb{C}^0_{p_i}$ is equipped with the identity action. The space $\mathbb{C}^*_{p_i}$ can be understood as the dual of $\mathbb{C}_{p_i}$, or equivalently as the space $\mathbb{C}$ equipped with the dual action $v \cdot t = p_i(t)v$.

Recall that $J_i \to SU(n)/T$ is the $SU(n)$-homogeneous line bundle defined by setting $J_i := \mathbb{C} \times_{p_i} SU(n)$ under the relation $(z, s) \sim_{p_i} (p_i(t^{-1})z, st)$ for all $t \in T$.

Proposition 5.8. *There is an associative multiplication making*

$$(J_i^{\varepsilon_i}, Y_T \times SU(n)/T) \tag{5.3}$$

a cup product bundle gerbe over $T \times SU(n)/T$. Moreover, this bundle gerbe is $SU(n)$-equivariant with respect to the $SU(n)$-action on $T \times SU(n)/T$ defined by multiplication on the $SU(n)/T$ component.

Proof. To see that this is a cup product bundle gerbe, and hence a bundle gerbe, it suffices to show that $\varepsilon_i : Y_T^{[2]} \to \mathbb{Z}$ satisfies the cocycle condition $\delta(\varepsilon_i) = 0$. This is trivial to verify on the connected components of $Y_T^{[2]}$. The equivariance result follows easily. □

Proposition 5.9. *There exists an $SU(n)$-equivariant bundle gerbe isomorphism over $T \times SU(n)/T$*

$$(P_{b,T} \times_T SU(n), Y_T \times SU(n)/T) \cong_{SU(n)} \left(\bigotimes_{i=1}^{n} J_i^{\varepsilon_i}, Y_T \times SU(n)/T \right).$$

Proof. First, we show there is a line bundle isomorphism $P_{b,T} \cong \bigotimes \mathbb{C}^{\varepsilon_i}_{p_i} \times Y_T^{[2]}$. Let $(z_1, z_2, t) \in Y_T^{[2]}$ with $z_1 > z_2$. Suppose there are eigenvalues of t between z_1 and z_2. Denote these eigenvalues by $p_{k_1}(t), \dots, p_{k_m}(t)$ for $1 \leq k_1 \leq \cdots \leq k_m \leq n$. Then

$$L_{(z_1,z_2,t)} = E_{(t, p_{k_1}(t))} \oplus \cdots \oplus E_{(t, p_{k_m}(t))} \tag{5.4}$$

and

$$(P_{b,T})_{(z_1,z_2,t)} = \det(L_{(z_1,z_2,t)}) = \bigotimes_{z_1>\lambda>z_2} \det\left(E_{(t,\lambda)}\right) = E_{(t,p_{k_1}(t))} \otimes \cdots \otimes E_{(t,p_{k_m}(t))}.$$

Each eigenspace $E_{(t,p_k(t))} \cong \mathbb{C}$ is equipped with a T-action $v \cdot s := p_k(s^{-1})v$, hence $E_{(t,p_k(t))} \cong \mathbb{C}_{p_k}$ for each k. Since $\varepsilon_{k_i}(z_1, z_2, t) = 1$ for $i = 1, \ldots, m$ and $\varepsilon_k = 0$ otherwise

$$(P_{b,T})_{(z_1,z_2,t)} \cong \mathbb{C}_{p_{k_1}} \otimes \cdots \otimes \mathbb{C}_{p_{k_m}} \cong \mathbb{C}_{p_1}^{\varepsilon_1(z_1,z_2,t)} \otimes \cdots \otimes \mathbb{C}_{p_n}^{\varepsilon_n(z_1,z_2,t)}.$$

By almost identical arguments, this holds over the other components of $Y_T^{[2]}$. Therefore we have an isomorphism $P_{b,T} \cong \bigotimes_{i=1}^n \mathbb{C}_{p_i}^{\varepsilon_i} \times Y_T^{[2]}$, as claimed. This implies we have an isomorphism

$$P_{b,T} \times_T SU(n) \cong \left(\bigotimes \mathbb{C}_{p_i}^{\varepsilon_i} \times Y_T^{[2]}\right) \times_T SU(n), \tag{5.5}$$

where the latter line bundle is $SU(n)$-equivariant with T-action defined by

$$(z_1, \ldots, z_n, u, g) \cdot t = (p_1(t^{-1})z_1, \ldots, p_n(t^{-1})z_n, u, gt). \tag{5.6}$$

It can be verified that the line bundle isomorphism (5.5) is $SU(n)$-equivariant. This will act as our 'intermediary isomorphism'. Next, consider $\left(\bigotimes_{i=1}^n \mathbb{C}_{p_i}^{\varepsilon_i}\right) \times_T SU(n)$, the space of equivalence classes under the T-action defined similarly to (5.6). This is an $SU(n)$-homogeneous line bundle over $SU(n)/T$, and it can be verified that the natural map

$$\left(\mathbb{C}_{p_1}^{\varepsilon_1} \otimes \cdots \otimes \mathbb{C}_{p_n}^{\varepsilon_n} \times Y_T^{[2]}\right) \times_T SU(n) \to Y_T^{[2]} \times \left(\mathbb{C}_{p_1}^{\varepsilon_1} \otimes \cdots \otimes \mathbb{C}_{p_n}^{\varepsilon_n} \times_T SU(n)\right) \tag{5.7}$$

is a well-defined, $SU(n)$-equivariant line bundle isomorphism over $Y_T^{[2]} \times SU(n)/T$. It follows by the equivalence of linear representations and equivariant bundles that there are $SU(n)$-equivariant line bundle isomorphisms

$$\begin{aligned}\mathbb{C}_{p_1}^{\varepsilon_1} \otimes \cdots \otimes \mathbb{C}_{p_n}^{\varepsilon_n} \times_T SU(n) &\cong (\mathbb{C}_{p_1}^{\varepsilon_1} \times_T SU(n)) \otimes \cdots \otimes (\mathbb{C}_{p_n}^{\varepsilon_n} \times_T SU(n)) \\ &\cong J_1^{\varepsilon_1} \otimes \cdots \otimes J_n^{\varepsilon_n}.\end{aligned}$$

This, combined with (5.7), implies there is an $SU(n)$-equivariant line bundle isomorphism

$$\left(\bigotimes_{i=1}^n \mathbb{C}_{p_i}^{\varepsilon_i} \times Y_T^{[2]}\right) \times_T SU(n) \cong \bigotimes_{i=1}^n J_i^{\varepsilon_i}$$

and hence, by (5.5), we obtain an $SU(n)$-equivariant isomorphism of line bundles $P_{b,T} \times_T SU(n) \cong \bigotimes_{i=1}^n J_i^{\varepsilon_i}$. It remains to show that this isomorphism preserves the bundle gerbe product. Suppose $(z_1, z_2, z_3, g) \in Y^{[3]}$ with $z_1 > z_2 > z_3$, and that there are eigenvalues of g between z_1 and z_2 and also between z_2 and z_3. Then $L_{(z_1,z_2,g)} \oplus L_{(z_2,z_3,g)} = L_{(z_1,z_3,g)}$ and the basic bundle gerbe product is induced from

$$\det(L_{(z_1,z_2,t)}) \otimes \det(L_{(z_2,z_3,t)}) \cong \det(L_{(z_1,z_2,t)} \oplus L_{(z_2,z_3,t)}) \cong \det(L_{(z_1,z_3,t)}).$$

From the discussion above and equation (5.4), each $L_{(z_i,z_j,t)}$ decomposes into appropriate sums of powers of the J_l, so this becomes

$$\bigotimes_{i=1}^n J_i^{\varepsilon_i(z_1,z_2,t)} \otimes \bigotimes_{i=1}^n J_i^{\varepsilon_i(z_2,z_3,t)} \cong \bigotimes_{i=1}^n J_i^{\varepsilon_i(z_1,z_3,t)},$$

which is the cup product multiplication. The other cases proceed similarly. $\square$

Clearly, the reduced product of the $SU(n)$-equivariant bundle gerbes (5.3) will be an $SU(n)$-equivariant bundle gerbe. This leads us to our final isomorphism, which follows from Propositions 5.5 and 5.9.

Proposition 5.3. *There is an $SU(n)$-equivariant isomorphism over $T \times SU(n)/T$*

$$p^{-1}(P_b, Y) \cong_{SU(n)} \bigotimes_{i=1}^{n}{}_{\mathrm{R}} (J_i^{\varepsilon_i}, Y_T \times SU(n)/T).$$

5.3. **Geometry of the pullback of the basic bundle gerbe.** In [16], connective data (∇_b, f_b) was defined on the basic bundle gerbe as follows. First, a connection on the bundle L defined in equation (5.1) was constructed using orthogonal projection of the flat connection. Taking the highest exterior power of this connection gave rise to a bundle gerbe connection ∇_b on the basic bundle gerbe. Second, the curving f_b was constructed using holomorphic functional calculus. We will not detail this construction here as we only need the connective data on the pullback of the basic bundle gerbe which is given by Proposition 5.10 below from [16].

Recall from Section 4.2 that the cup product bundle gerbes $(J_i^{\varepsilon_i}, Y_T \times SU(n)/T)$ can be endowed with a so-called cup product bundle gerbe connection using orthogonal projection, similar to Proposition 4.4. This in turn induces a general cup product bundle gerbe connection on $\otimes_R (J_i^{\varepsilon_i}, Y_T \times SU(n)/T)$ in the obvious way. We see that this general cup product bundle gerbe connection and ∇_b are both tensor products of or the determinant of connections defined by orthogonal projection of the flat connection onto subbundles. By the naturality of these constructions it follows that the pulled back connection on $p^{-1}(P_b, Y)$ under the isomorphism in Proposition 5.3 is the general cup product connection on $\otimes_R (J_i^{\varepsilon_i}, Y_T \times SU(n)/T)$.

Proposition 5.10 ([16, Appendix B])**.** *Let ∇_{p^*b} be the connection on the pullback of the basic bundle gerbe by the Weyl map induced by ∇_b. The pulled back curving and curvature are given by*

$$f_{p^*b} = \frac{i}{4\pi} \sum_{\substack{i,k=1 \\ i \neq k}}^{n} (\log_z p_i - \log_z p_k + (p_k - p_i) p_k^{-1}) \mathrm{tr}(P_i dP_k dP_k) \tag{5.8}$$

and

$$\omega_{p^*b} = \frac{i}{4\pi} \sum_{\substack{i,k=1 \\ i \neq k}}^{n} \left(p_i^{-1} dp_i - p_k^{-1} dp_k - p_k^{-1} dp_i + p_k^{-1} dp_k p_k^{-1} p_i\right) \mathrm{tr}(P_i dP_k dP_k) - \frac{i}{4\pi} \sum_{\substack{i,k=1 \\ i \neq k}}^{n} p_i p_k^{-1} \mathrm{tr}(dP_i dP_k dP_k). \tag{5.9}$$

Here $\log_z$ is the branch of the logarithm defined by cutting along the ray through $z \neq 1$ and requiring $\log_z(1) = 0$. Note that here and in the remainder of this work, we abuse notation and let the homomorphisms p_i and projections P_i be defined on the spaces $Y_T \times SU(n)/T, X_T \times SU(n)/T$, or $(X_T \times_T Y_T) \times SU(n)/T$ depending on the context.

The formulae in Proposition 5.10 can be simplified in a way that makes them more comparable to the Weyl bundle gerbe data as follows.

Proposition 5.11. *Let ∇_{p^*b} be the connection on the pullback of the basic bundle gerbe and $\pi_{p^*b} : Y_T \times SU(n)/T \to T \times SU(n)/T$ be projection. Define $\beta \in \Omega^2\left(T \times SU(n)/T\right)$ by*

$$\beta = -\frac{i}{4\pi} \sum_{\substack{i,k=1 \\ i \neq k}}^{n} p_i p_k^{-1} \operatorname{tr}(P_i dP_k dP_k).$$

Then

$$f_{p^*b} = \sum_{k=1}^{n} \left(\frac{-1}{2\pi i} \log_z p_k \right) \operatorname{tr}(P_k dP_k dP_k) + (\pi_{p^*b})^* \beta \tag{5.10}$$

and consequently

$$\omega_{p^*b} = -\frac{1}{2\pi i} \sum_{k=1}^{n} p_k^{-1} dp_k \operatorname{tr}(P_k dP_k dP_k) + d\beta \tag{5.11}$$

Proof. The proof needs a number of ingredients, some of which are proved in the Appendix. Firstly, we know that $\sum_{k=1}^{n} P_k = I$ and thus $\sum_{k=1}^{n} dP_k = 0$. Also as shown in the Appendix if $i \neq k$ then $\operatorname{tr}(P_i dP_k dP_k) = -\operatorname{tr}(P_k dP_i dP_i)$. Again from the Appendix $\sum_{k=1}^{n} \operatorname{tr}(P_k dP_k dP_k) = 0$. Using these it is straightforward to show that (5.8) reduces to (5.10) and (5.9) reduces to (5.11). □

5.4. **Other choices of general cup product bundle gerbes.** By comparing the curving and three-curvature of the Weyl bundle gerbe with the curving and three-curvature of the pullback of the basic bundle gerbe from Proposition 5.11, we can begin to establish a relationship between these bundle gerbes. To do so, we require the following key observation.

Lemma 5.12. *For each $i = 1, \ldots, n$ and $(z, w, t) \in Y_T^{[2]}$,*

$$\varepsilon_i(z, w, t) = \frac{1}{2\pi i} \left(\log_z p_i(t) - \log_w p_i(t) \right).$$

Proof. Recall Definition 5.6. Let $(z, w, t) \in Y_T^{[2]}$ with $z > w$. If $w < p_i(t) < z$, $\log_z p_i(t) - \log_w p_i(t) = 2\pi i$. Otherwise, this difference is zero. Therefore in general

$$\log_z p_i(t) - \log_w p_i(t) = \begin{cases} 2\pi i & \text{if } z > p_i(t) > w \\ -2\pi i & \text{if } w > p_i(t) > z \\ 0 & \text{otherwise.} \end{cases}$$

Dividing through by $2\pi i$, we see that this is precisely the definition of ε_i. □

It follows from Propositions 4.3 and 3.7 and equations (5.10) and (5.11) that if we let $\varphi_i = -\frac{1}{2\pi i} \log_z(p_i)\colon Y_T \to \mathbb{R}$, then $\delta(\varphi_i) = \varepsilon_i$ and we can construct a general cup product curving f and curvature ω for the pullback of the basic bundle gerbe which would be

$$f = \sum_{k=1}^{n} \left(\frac{-1}{2\pi i} \log_z p_k \right) \operatorname{tr}(P_k dP_k dP_k) = f_{p^*b} - (\pi_{p^*b})^* \beta$$

$$\omega = -\frac{1}{2\pi i} \sum_{k=1}^{n} p_k^{-1} dp_k \operatorname{tr}(P_k dP_k dP_k) = \omega_{p^*b} - d\beta.$$

We can ask more generally if there is a choice of functions $f_i : Y_T^{[2]} \to \mathbb{Z}$ and $\varphi_i \colon Y_T \to \mathbb{R}$ satisfying $\delta(\varphi_i) = f_i$ such that the curving f and three-curvature ω of the resulting general cup product bundle gerbe of J_i and f_i satisfy $f_{p^*b} = f$ and $\omega_{p^*b} = \omega$. For this to hold we would require functions $\alpha_i : T \times SU(n)/T \to \mathbb{R}$ for $i = 1, \dots, n$ such that $\beta = \sum_{i=1}^n \alpha_i \mathrm{tr}(P_i dP_i dP_i)$. We claim that for $n > 2$ this is not possible.

Proposition 5.13. *If $n > 2$, there do not exist functions $\alpha_i : T \times SU(n)/T \to \mathbb{R}$ for $i = 1, \dots, n$ such that $\beta = \sum_{i=1}^n \alpha_i \mathrm{tr}(P_i dP_i dP_i)$.*

Proof. By Lemma A.2, there exists β_{ij} such that β decomposes into the sums

$$\beta = \sum_{i<j<n} (\beta_{ij} - \beta_{in} + \beta_{jn}) \mathrm{tr}(P_j dP_i dP_i) - \sum_{i<n} \beta_{in} \mathrm{tr}(P_i dP_i dP_i).$$

Moreover, the first of these summations is non-zero by Lemma A.2 (3). By Lemma A.1 (4), $\sum_{k=1}^n \alpha_k \mathrm{tr}(P_k dP_k dP_k) = \sum_{k<n} (\alpha_k - \alpha_n) \mathrm{tr}(P_k dP_k dP_k)$. Therefore it suffices to show that

$$\mathrm{span}\, \{\mathrm{tr}(P_j dP_i dP_i) \mid i < j < n\} \cap \mathrm{span}\, \{\mathrm{tr}(P_k dP_k dP_k) \mid k < n\} = \{0\}. \tag{5.12}$$

Let E_{ij} be the $n \times n$ matrix with a 1 in the (i, j) entry and zeros elsewhere. Set $O_i := E_{ii}$. Then

$$E_{ij} E_{kl} = \delta_{jk} E_{il} \tag{5.13}$$

$$O_i E_{kl} = \delta_{ik} E_{kl} \tag{5.14}$$

$$E_{kl} O_i = \delta_{il} E_{kl}. \tag{5.15}$$

The root spaces for the Lie algebra $LSU(n)$ are spanned by matrices of the form $A^\mu_{ij} = \mu E_{ij} - \overline{\mu} E_{ji}$ for $\mu \in \mathbb{C}$. Let $\gamma(t) = g\exp(tX)T$ be a curve in $SU(n)/T$ through gT. So

$$P_i(\gamma(t)) = g \exp(tX) O_i \exp(-tX) g^{-1}$$

and $dP_i(gX) = g[X, O_i]g^{-1}$. Using this, it can be verified easily that

$$\begin{aligned}
\mathrm{tr}(P_j dP_i dP_i)(gX, gY) &= -\mathrm{tr}(O_j X O_i Y) + \mathrm{tr}(O_j Y O_i X) \\
\mathrm{tr}(P_i dP_i dP_i)(gX, gY) &= \mathrm{tr}(-O_i XY) + \mathrm{tr}(O_i X O_j Y) \\
&\quad + \mathrm{tr}(O_j Y X) - \mathrm{tr}(O_i Y O_j X).
\end{aligned}$$

In particular, using equations (5.13) – (5.15), a simple computation yields

$$\mathrm{tr}(P_j dP_i dP_i)(gA^\mu_{in}, gA^\lambda_{in}) = 0 \tag{5.16}$$

$$\text{and } \mathrm{tr}(P_i dP_i dP_i)(gA^\mu_{in}, gA^\lambda_{in}) = \delta_{ki}(\overline{\lambda}\mu - \mu\overline{\lambda}). \tag{5.17}$$

Consider an element

$$\sum_{i<j<n} b_{ij} \mathrm{tr}(P_j dP_i dP_i) = \sum_{k<n} \alpha_k \mathrm{tr}(P_k dP_k dP_k)$$

in the intersection from (5.12). By equations (5.16) and (5.17), evaluating this element at $(gA^\mu_{kn}, gA^\lambda_{kn})$ yields $0 = \alpha_k(\lambda\overline{\mu} - \mu\overline{\lambda})$. Choosing λ and μ so that $\alpha_k = 0$ for all k proves (5.12). □

By the earlier discussion, the following corollary is immediate.

Corollary 5.14. *Let $n > 2$. There does not exist a choice of functions $f_i : Y_T^{[2]} \to \mathbb{Z}$ and $\varphi_i \colon Y_T \to \mathbb{R}$ satisfying $\delta(\varphi_i) = f_i$ such that the curving f and three-curvature ω of the resulting general cup product bundle gerbe of J_i and f_i satisfy $f_{p^*b} = f$ and $\omega_{p^*b} = \omega$.*

6. The stable isomorphism

6.1. **Set up of the problem.** Our central aim in this section is to prove that the pullback of the basic bundle gerbe by the Weyl map is $SU(n)$-stably isomorphic to the Weyl bundle gerbe, i.e.

$$p^{-1}(P_b, Y) \cong_{SU(n)\text{-stab}} (P_c, X). \tag{6.1}$$

By Definition 4.6 and Proposition 5.3, (6.1) is equivalent to

$$\bigotimes_{i=1}^{n}{}_{\mathrm{R}} \left(J_i^{\varepsilon_i}, Y_T \times SU(n)/T\right) \cong_{SU(n)\text{-stab}} \bigotimes_{i=1}^{n}{}_{\mathrm{R}} \left(J_i^{d_i}, X_T \times SU(n)/T\right)$$

Since both of these bundle gerbes are general cup product bundle gerbes, Corollary 3.5 applies to give us the following result.

Proposition 6.1. *The pullback of the basic bundle gerbe is $SU(n)$-stably isomorphic to the Weyl bundle gerbe if, for all $i = 1, \ldots, n$, there exist smooth functions $h_i : (X_T \times_T Y_T) \times SU(n)/T \to \mathbb{Z}$ such that*

$$\varepsilon_i(z, w, t) - (x_i - y_i) = h_i(y, w, t, gT) - h_i(x, z, t, gT) \tag{6.2}$$

for all $(x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), z, w, t, gT) \in (X_T \times_T Y_T)^{[2]} \times SU(n)/T$.

6.2. **Finding the stable isomorphism.** It follows from a standard fact in bundle gerbe theory that, if equation (6.1) holds with respect to the connective data $(\nabla_{p^*b}, f_{p^*b}), (\nabla_c, f_c)$ from Propositions 5.11 and 4.9, there exists a trivialising line bundle R with connection ∇_R and $\beta \in \Omega^2(T \times SU(n)/T)$ such that

$$\begin{aligned} f_{p^*b} - f_c &= F_{\nabla_R} + \pi^*\beta \\ \omega_{p^*b} - \omega_c &= d\beta \end{aligned} \tag{6.3}$$

for $\pi : (X_T \times_T Y_T) \times SU(n)/T \to T \times SU(n)/T$ projection. As in Corollary 3.5 we take R to be the line bundle

$$R := \bigotimes_{i=1}^{n} J_i^{h_i} \to (X_T \times_T Y_T) \times SU(n)/T$$

where we implicitly pull J_i back from $SU(n)/T$ to $(X_T \times_T Y_T) \times SU(n)/T$. Here, the functions $h_i : (X_T \times_T Y_T) \times SU(n)/T$ are parameters that we aim to define. In this situation we can take ∇_R to be the product connection and thus

$$F_{\nabla_R} = \sum_{i=1}^{n} h_i \operatorname{tr}(P_i dP_i dP_i). \tag{6.4}$$

First we compare the curving and curvature for the two bundle gerbes.

Proposition 6.2. *The pulled back connective data for $(P_{p^*b}, Y_T \times SU(n)/T)$ from* Proposition 5.11 *and the Weyl bundle gerbe connective data for $(P_c, X_T \times SU(n)/T)$*

from Proposition 3.6 *satisfy*

$$f_{p^*b} - f_c = \sum_{k=1}^{n} \left(-\frac{1}{2\pi i} \log_z p_k + x_i \right) \operatorname{tr}(P_k dP_k dP_k) + \pi^* \beta$$

$$\omega_{p^*b} - \omega_c = -\frac{1}{2\pi i} \sum_{k=1}^{n} p_k^{-1} dp_k \operatorname{tr}(P_k dP_k dP_k) + d\beta.$$

It follows by comparison with equations (6.3) and (6.4) that we want to take

$$h_i(x, z, t, gT) = x_i - \frac{1}{2\pi i} \log_z p_i(t)$$

for all $i = 1, \ldots, n$. It remains to be shown that these h_i satisfy equation (6.2) and hence define the required stable isomorphism.

Proposition 6.3. *For $i = 1, \ldots, n$ define $h_i : (X_T \times_T Y_T) \times SU(n)/T \to \mathbb{Z}$ by*

$$h_i(x, z, t, gT) = x_i - \frac{1}{2\pi i} \log_z p_i(t)$$

for $(x = (x_1, \ldots, x_n), z, w, t, gT) \in (X_T \times_T Y_T) \times SU(n)/T$. Then h_i is smooth and

$$\varepsilon_i(z, w, t) - x_i + y_i = h_i(y, w, t, gT) - h_i(x, z, t, gT)$$

for all $(x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), z, w, t, gT) \in (X_T \times_T Y_T)^{[2]} \times SU(n)/T$.

Proof. First, note that $h_i(x, z, t, gT) \in \mathbb{Z}$ since $e^{2\pi i x_i} = p_i(t)$, so upon exponentiating h_i we obtain $e^{2\pi i h_i} = e^{2\pi i x_i} p_i(t)^{-1} = 1$. Smoothness of h_i follows by noting that log is smooth over the given range as $z \neq p_i(t)$. By Lemma 5.12,

$$\begin{aligned} h_i(y, w, t, gT) - h_i(x, z, t, gT) &= y_i - x_i + \frac{1}{2\pi i} \left(\log_z p_i(t) - \log_w p_i(t) \right) \\ &= y_i - x_i + \varepsilon_i(z, w, t), \end{aligned}$$

so these are the desired functions h_i. □

The next result then follows immediately from Propositions 6.1 and 6.3. A more precise statement of this result will be provided in Theorem 6.8.

Proposition 6.4. *The Weyl bundle gerbe is $SU(n)$-stably isomorphic to the pullback of the basic bundle gerbe by the Weyl map, i.e.*

$$(P_c, X) \cong_{SU(n)\text{-stab}} p^{-1} (P_b, Y).$$

6.3. **Comparing holonomies.** Recall that bundle gerbes are *D-stably isomorphic* if they are stably isomorphic as bundle gerbes *with connective data*. It is a standard fact that, if two bundles gerbes over a surface are D-stably isomorphic, then they have the same holonomy. Therefore if we can show our bundle gerbes have different holonomies on a surface $\Sigma \subset T \times SU(n)/T$, then the restriction of our bundle gerbes to Σ (and hence the original bundle gerbes) cannot be D-stably isomorphic, and their D-stable isomorphism classes (Deligne classes) will not be equal.

By our choice of trivialising line bundle, the curvings of the pullback of the basic bundle gerbe and Weyl bundle gerbe satisfy

$$f_{p^*b} = f_c + F_{\nabla_R} + \pi^* \beta_n \tag{6.5}$$

$$\text{for} \quad \beta_n = -\frac{i}{4\pi} \sum_{\substack{i,k=1 \\ i \neq k}}^{n} p_i p_k^{-1} \operatorname{tr}(P_i dP_k dP_k). \tag{6.6}$$

Here we introduce the notation β_n to emphasise that β_n is defined on $T\times SU(n)/T$. It follows from Proposition 6.4, equation (6.5) and standard facts in holonomy that, for $\Sigma\subset T\times SU(n)/T$ a surface, the holonomies of the pullback of the basic bundle gerbe and Weyl bundle gerbe satisfy

$$\mathrm{hol}((\nabla_{p^*b}, f_{p^*b}), \Sigma) = \exp\left(\int_\Sigma \beta_n\right)\mathrm{hol}((\nabla_c, f_c), \Sigma). \tag{6.7}$$

It could be the case that $\int_\Sigma \beta_n = k2\pi i$ for some $k\in\mathbb{Z}$, implying these holonomies are equal. We next show that there exists a surface $\Sigma_2\subset T\times SU(2)/T$ for which $\int_{\Sigma_2}\beta_2 \neq k2\pi i$ for any $k\in\mathbb{Z}$. We will then generalise this result to obtain a surface $\Sigma_n\subset T\times SU(n)/T$ for which $\mathrm{hol}((\nabla_{p^*b}, f_{p^*b}), \Sigma_n)\neq \mathrm{hol}((\nabla_c, f_c), \Sigma_n)$.

Proposition 6.5. *Define a surface $\Sigma_2\subset T\times SU(2)/T\cong S^1\times S^2$ by $\Sigma_2 := \{e^{\pi i/4}\}\times S^2$. Then the holonomies of the pullback of the basic bundle gerbe over $SU(2)$ and the Weyl bundle gerbe over $T\times SU(2)/T$ are not equal over Σ_2.*

Proof. By equation (6.7), we need only show that $\int_{\Sigma_2}\beta_2\neq k2\pi i$ for any $k\in\mathbb{Z}$. Since $P_1+P_2=1$ and $p_2=p_1^{-1}$, by setting $P:=P_1$ and $p:=p_1$ in equation (6.6) we obtain

$$\beta_2 = \frac{i}{4\pi}(p^2-p^{-2})\mathrm{tr}(PdPdP).$$

It is a standard fact that $\mathrm{tr}(PdPdP)$ is the curvature of the tautological line bundle over S^2, which has chern class minus one, i.e. $\frac{i}{2\pi}\int_{S^2}\mathrm{tr}(PdPdP)=-1$. Therefore

$$\begin{aligned}\int_{\Sigma_2}\beta_2 &= \frac{ie^{\frac{i\pi}{2}} - ie^{-\frac{i\pi}{2}}}{4\pi}\int_{S^2}\mathrm{tr}(PdPdP)\\ &= \frac{-e^{\frac{i\pi}{2}}+e^{-\frac{i\pi}{2}}}{2\pi}\\ &= \frac{1}{\pi i}\neq k2\pi i\ \forall\ k\in\mathbb{Z},\end{aligned}$$

hence $\exp\left(\int_{\Sigma_2}\beta_2\right)\neq 1$ and the holonomies are not equal over this surface. □

Corollary 6.6. *There exists a surface $\Sigma_n\subset T\times SU(n)/T$ such that*

$$\mathrm{hol}((\nabla_{p^*b}, f_{p^*b}), \Sigma_n)\neq \mathrm{hol}((\nabla_c, f_c), \Sigma_n).$$

Proof. First, note that surface $\Sigma_2=\{e^{\pi i/4}\}\times S^2$ from Proposition 6.5 is an embedded submanifold of $T\times SU(n)/T$ with respect to the inclusion $\iota: SU(2)/T_1\hookrightarrow SU(n)/T_{n-1}$ defined by

$$XT_1\mapsto \left[\begin{array}{c|c} X & 0\\ \hline 0 & \mathbf{I}_{n-2}\end{array}\right]T_{n-1}.$$

Here, T_1, T_{n-1} denote the subgroups of diagonal matrices in $SU(2)$ and $SU(n)$ respectively, and $\mathbf{I}_{n-2}$ is the $(n-2)\times(n-2)$ identity matrix. Let $\Sigma_n := \iota(\Sigma)$. By equation (6.7) it suffices to show that

$$\int_{\Sigma_n}\beta_n = \int_{\Sigma_2}\iota^*\beta_n\neq k2\pi i$$

for any $k \in \mathbb{Z}$. To do so, we prove that $\iota^*\beta_n = \beta_2$, hence $\int_{\Sigma_n} \beta_n \neq k2\pi i$ by the proof of Proposition 6.5. We compute $\iota^*\beta_n$ as follows. Recall that the maps $p_i : T \to S^1$ were defined as projection onto the i-th diagonal. Clearly

$$p_i \circ \iota = \begin{cases} p_i & \text{if } i = 1, 2 \\ 1 & \text{if } 2 < i \leq n. \end{cases}$$

Further recall that P_i was defined to be orthogonal projection onto $J_i := \mathbb{C} \times_{p_i} SU(n)$, where p_i was the relation $(z, s) \sim_{p_i} (p_i(t^{-1})z, st)$ for all $(z, s) \in \mathbb{C} \times SU(n)$. Now, when the maps p_i are the constant value 1, this relation is equality, and $J_i \to SU(n)/T$ is isomorphic to the trivial line bundle over $SU(n)/T$. In this case, P_i will be the constant projection onto the span of e_i, the i-th standard basis vector of $\mathbb{C}^n$. That is, $P_i = O_i$ for O_i the matrix with a 1 in the (i, i) position and zeros elsewhere. Therefore

$$P_i \circ \iota = \begin{cases} P_i & \text{if } i = 1, 2 \\ O_i & \text{if } 2 < i \leq n. \end{cases}$$

Of course, $dO_i = 0$, so any term of the form $\mathrm{tr}(P_k dP_i dP_i)$ for $i > 2$ in our expression for β_n in (6.6) will equal zero. Furthermore, any term of the form $\mathrm{tr}(P_i dP_k dP_k)$ for $i > 2$ will also be zero, by Lemma A.1 (2). So $\iota^*\beta_n = \beta_2$ as required. □

The following corollary is immediate from our earlier discussion.

Corollary 6.7. *There does not exist a D-stable isomorphism of (P_c, X) and $p^{-1}(P_b, Y)$ with respect to the connective data (∇_c, f_c) and $(\nabla_{p^*b}, f_{p^*b})$.*

The results of Sections 5 and 6 culminate in the following theorem.

Theorem 6.8. *Let $p^{-1}(P_b, Y)$ be the pullback of the basic bundle gerbe* (Definition 5.2) *by the Weyl map with connective data $(\nabla_{p^*b}, f_{p^*b})$ and three-curvature ω_{p^*b}* (Proposition 5.10). *Let (P_c, X) be the Weyl bundle gerbe* (Definition 4.6) *with connective data (∇_c, f_c) and three-curvature ω_c* (Propositions 4.8 - 4.9). *Then*

(1) *there is an $SU(n)$-equivariant stable isomorphism over $T \times SU(n)/T$*
$$(P_c, X) \cong_{SU(n)\text{-stab}} p^{-1}(P_b, Y),$$
with trivialising line bundle
$$R := \bigotimes_{i=1}^{n} \pi_2^{-1}(J_i)^{h_i} \to (X_T \times_T Y_T) \times SU(n)/T$$
for $\pi_2 : (X_T \times_T Y_T) \times SU(n)/T \to SU(n)/T$ projection and
$$h_i : (X_T \times_T Y_T) \times SU(n)/T \to \mathbb{Z}$$
$$((x_1, \ldots, x_n), z, t, gT) \mapsto x_i - \frac{1}{2\pi i} \log_z p_i(t);$$

(2) *if ∇_R is the connection on R induced by ∇_{J_i}* (Proposition 4.3), *then*
$$f_{p^*b} - f_c = F_{\nabla_R} + \pi^*\beta$$
$$\text{and } \ \omega_{p^*b} - \omega_c = d\beta$$
for $\pi : (X_T \times_T Y_T) \times SU(n)/T \to T \times SU(n)/T$ projection and
$$\beta = -\frac{i}{4\pi} \sum_{\substack{i,k=1 \\ i \neq k}}^{n} p_i p_k^{-1} \mathrm{tr}(P_i dP_k dP_k)$$

where $P_i : T \times SU(n)/T \times \mathbb{C}^n \to J_i$ *is orthogonal projection;*

(3) *there does not exist a general cup product bundle gerbe of* J_i *and some functions* $f_i : X_T^{[2]} \to \mathbb{Z}$ *and* $\varphi_i \colon X_T \to \mathbb{R}$ *with* $\delta(\varphi_i) = f_i$ *whose induced connective data (following* Proposition 4.9) *has associated three-curvature* $\omega = \omega_{p^*b}$;

(4) *there does not exist a* D*-stable isomorphism of* (P_c, X) *and* $p^{-1}(P_b, Y)$ *with respect to the connective data* (∇_c, f_c) *and* $(\nabla_{p^*b}, f_{p^*b})$.

Appendix A. Computational lemmas

Here, we present the lemmas used to prove various results in Section 6.

Lemma A.1. *Let* $i, j, k = 1, \dots, n$. *Then*

(1) *for distinct* i, j, k, $\mathrm{tr}(P_i dP_j dP_k) = 0$;
(2) *if* $i \neq j$, $\mathrm{tr}(P_i dP_j dP_j) = -\mathrm{tr}(P_j dP_i dP_i)$;
(3) $\sum_{k=1}^n \mathrm{tr}(P_k dP_k dP_k) = 0$;
(4) $\sum_{i=1}^n \alpha_i \mathrm{tr}(P_i dP_i dP_i) = \sum_{i=1}^{n-1} (\alpha_i - \alpha_n) \mathrm{tr}(P_i dP_i dP_i)$.

Proof. To prove (1), note that $P_i P_j = 0$ if $i \neq j$, and $dP_i = dP_i P_i + P_i dP_i$ (where we obtain the second equation by differentiating $P_i^2 = P_i$). So for distinct i, j and k we have

$$\begin{aligned}
\mathrm{tr}(P_i dP_j dP_k) &= \mathrm{tr}(P_i(P_j dP_j + dP_j P_j) dP_k) \\
&= \mathrm{tr}(P_i dP_j P_j dP_k) \\
&= \mathrm{tr}(P_i dP_j P_j (P_k dP_k + dP_k P_k)) \\
&= \mathrm{tr}(P_i dP_j P_j dP_k P_k) \\
&= \mathrm{tr}(P_k P_i dP_j P_j dP_k) = 0,
\end{aligned}$$

thereby proving (1). Next, by differentiating the identity $P_i P_j = 0$, we obtain $dP_i P_j = -P_i dP_j$ for $i \neq j$. Therefore, using (1) and that $\sum_{i=1}^n dP_i = 0$, we obtain

$$\begin{aligned}
\mathrm{tr}(P_i dP_j dP_j) &= -\mathrm{tr}(dP_i P_j dP_j) \\
&= \mathrm{tr}(P_j dP_j dP_i) \\
&= \mathrm{tr}\left(P_j \left(-\textstyle\sum_{k \neq j} dP_k\right) dP_i\right) \\
&= -\sum_{k \neq j} \mathrm{tr}(P_j dP_k dP_i) \\
&= -\mathrm{tr}(P_j dP_i dP_i),
\end{aligned}$$

thereby proving (2). For (3) we use (2). We have

$$
\begin{aligned}
\sum_{k=1}^{n} \operatorname{tr}(P_k dP_k dP_k) &= -\sum_{i \neq k} \operatorname{tr}(P_i dP_k dP_k) \\
&= -\sum_{i<k} \operatorname{tr}(P_i dP_k dP_k) - \sum_{k<i} \operatorname{tr}(P_i dP_k dP_k) \\
&= -\sum_{i<k} \operatorname{tr}(P_i dP_k dP_k) - \sum_{i<k} \operatorname{tr}(P_k dP_i dP_i) \\
&= -\sum_{i<k} \operatorname{tr}(P_i dP_k dP_k) + \sum_{i<k} \operatorname{tr}(P_i dP_k dP_k) \\
&= 0.
\end{aligned}
$$

Lastly, by (2), $\operatorname{tr}(P_k dP_l dP_l) + \operatorname{tr}(P_l dP_l dP_k) = 0$. Using this, together with (1) we obtain

$$
\begin{aligned}
\sum_{i=1}^{n} \alpha_i \operatorname{tr}(P_i dP_i dP_i) &= \sum_{i=1}^{n-1} \alpha_i \operatorname{tr}(P_i dP_i dP_i) - \alpha_n \operatorname{tr}\left(\left(\sum_{m=1}^{n-1} P_m\right)\left(\sum_{k=1}^{n-1} dP_k\right)\left(\sum_{l=1}^{n-1} dP_l\right)\right) \\
&= \sum_{i=1}^{n-1} \alpha_i \operatorname{tr}(P_i dP_i dP_i) - \alpha_n \sum_{i=1}^{n-1} \operatorname{tr}(P_i dP_i dP_i) \\
&\quad - \alpha_n \sum_{\substack{k,l=1 \\ k \neq l}}^{n-1} \operatorname{tr}(P_k dP_l dP_l) + \operatorname{tr}(P_l dP_l dP_k) \\
&= \sum_{i=1}^{n-1} (\alpha_i - \alpha_n) \operatorname{tr}(P_i dP_i dP_i),
\end{aligned}
$$

proving (4). □

Lemma A.2. *Consider* $\beta = -\frac{i}{4\pi} \sum_{\substack{i,k=1 \\ i \neq k}}^{n} p_i p_k^{-1} \operatorname{tr}(P_i dP_k dP_k)$. *Then*

(1) *there exist coefficients* β_{ij} *such that*

$$\beta = \sum_{i<j \leq n} \beta_{ij} \operatorname{tr}(P_j dP_i dP_i);$$

(2) *if* $n > 2$, *these* β_{ij} *satisfy*

$$\beta = \sum_{i<j<n} (\beta_{ij} - \beta_{in} + \beta_{jn}) \operatorname{tr}(P_j dP_i dP_i) - \sum_{i<n} \beta_{in} \operatorname{tr}(P_i dP_i dP_i);$$

(3) *if* $n > 2$, *these* β_{ij} *satisfy*

$$\sum_{i<j<n} (\beta_{ij} - \beta_{in} + \beta_{jn}) \operatorname{tr}(P_j dP_i dP_i) \neq 0.$$

Proof. It follows from Lemma A.1 (2) that $\beta = \sum_{i<j\leq n}(p_j p_i^{-1} - p_i p_j^{-1})\mathrm{tr}(P_j dP_i dP_i)$, so by setting $\beta_{ij} := p_j p_i^{-1} - p_i p_j^{-1}$ we obtain (1). It follows that we can write

$$\begin{aligned}
\beta &= \sum_{i<j<n} \beta_{ij}\mathrm{tr}(P_j dP_i dP_i) + \sum_{i<n} \beta_{in}\mathrm{tr}(P_n dP_i dP_i) \\
&= \sum_{i<j<n} \beta_{ij}\mathrm{tr}(P_j dP_i dP_i) - \sum_{i<n}\sum_{j=1}^{n-1} \beta_{in}\mathrm{tr}\left(P_j dP_i dP_i\right) \\
&= \sum_{i<j<n} \beta_{ij}\mathrm{tr}(P_j dP_i dP_i) - \sum_{i<j<n} \beta_{in}\mathrm{tr}\left(P_j dP_i dP_i\right) \\
&- \sum_{j<i<n} \beta_{in}\mathrm{tr}\left(P_j dP_i dP_i\right) - \sum_{i<n} \beta_{in}\mathrm{tr}\left(P_i dP_i dP_i\right) \\
&= \sum_{i<j<n} (\beta_{ij} - \beta_{in} + \beta_{jn})\mathrm{tr}(P_j dP_i dP_i) - \sum_{i<n} \beta_{in}\mathrm{tr}\left(P_i dP_i dP_i\right).
\end{aligned}$$

For (3), consider an element

$$\mathbf{S} := \begin{bmatrix} \mathbf{T} & 0 \\ 0 & \mathbf{I}_{n-2} \end{bmatrix} \in T$$

for $\mathbf{T} := \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$, $t \in U(1)$, and $\mathbf{I}_{n-2}$ the $(n-2)\times(n-2)$ identity matrix. Clearly $\beta_{ij} = \beta_{in}$ and $\beta_{jn} = 0$ if $j > 2$ evaluated at $\mathbf{S}$. Therefore the only non-zero coefficient in this summation evaluated at $\mathbf{S}$ is $\beta_{12} - \beta_{1n} + \beta_{2n} = 2t - 2t^{-1} + t^{-2} - t^2$. The result then follows by choosing t such that this coefficient is non-zero. $\square$

References

[1] M. F. Atiyah. On the K-theory of compact Lie groups. *Topology*, **4**, 95–99, 1965.

[2] D. Baraglia and P. Hekmati, A Fourier-Mukai approach to the K-theory of compact Lie groups, *Adv. Math.*, **269**, 335–345, 2015.

[3] K. E. Becker, *Bundle gerbes and the Weyl map*, MPhil Thesis, University of Adelaide, 2019. arXiv:1910.01776.

[4] I. N. Bernšteĭn, I. M. Gel′ fand, and S. I. Gel′ fand. Schubert cells, and the cohomology of the spaces G/P. *Uspehi Mat. Nauk*, **28**, 3 (171), 3–26, 1973. (*Russ. Math. Surv.* **28**, 1, 1–26, 1973.)

[5] J.-L. Brylinski. *Loop Spaces, Characteristic Classes and Geometric Quantization.* Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. Reprint of the 1993 edition.

[6] A. J. Harju and J. Mickelsson. Twisted K-theory constructions in the case of a decomposable Dixmier-Douady class, *J, K-theory* **14**, 247–272, 2014.

[7] S. Johnson. *Constructions with Bundle Gerbes.* PhD thesis, The University of Adelaide, 2002, arXiv:math/0312175.

[8] J. M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[9] V. Mathai and R. B. Melrose, Geometry of pseudodifferential algebra bundles and Fourier integral operators. *Duke Math. J.* **166**, 1859–1922, 2017.

[10] V. Mathai, R. B. Melrose and I. M. Singer, The index of projective families of elliptic operators: the decomposable case. *Astérisque* No. 328 (2009), 255–296 (2010).

[11] J. Mickelsson. Gerbes on quantum groups, 2003. arXiv:math/0308235.

[12] M. K. Murray. Bundle gerbes. *J. London Math. Soc. (2)*, **54**(2):403–416, 1996.

[13] M. K. Murray. An introduction to bundle gerbes. In *The Many Facets of Geometry*, pages 237–260. Oxford Univ. Press, Oxford, 2010.

[14] M. K. Murray, D. M. Roberts; D. Stevenson, R. F. Vozzo. Equivariant bundle gerbes. *Adv. Theor. Math. Phys.* **21**, no. 4, 921–975, (2017).

[15] M. K. Murray and D. Stevenson. Bundle gerbes: stable isomorphism and local theory. *J. London Math. Soc. (2)*, **62**(3):925–937, 2000.
[16] M. K. Murray and D. Stevenson. The basic bundle gerbe on unitary groups. *J. Geom. Phys.*, **58**(11):1571–1590, 2008.
[17] D. Stevenson. Bundle 2-gerbes. *Proc. London Math. Soc. (3)*, **88**(2):405–435, 2004.
[18] H. Weyl. Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen. I. *Math. Z.*, **23**(1):271–309, 1925.

(K. E. Becker) Wolfson College, Linton Road, Oxford, OX2 6UD, United Kingdom
E-mail address: `kimberly.becker@univ.ox.ac.uk`

(M. K. Murray) School of Mathematical Sciences, University of Adelaide, Adelaide, SA 5005, Australia
E-mail address: `michael.murray@adelaide.edu.au`

(D. Stevenson) School of Mathematical Sciences, University of Adelaide, Adelaide, SA 5005, Australia
E-mail address: `daniel.stevenson@adelaide.edu.au`