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Entropy and uncertainty of squeezed quantum open systems

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We define the entropy S and uncertainty function of a squeezed system interacting with a thermal bath, and study how they change in time by following the evolution of the reduced density matrix in the influence functional formalism. As examples, we calculate the entropy of two exactly solvable squeezed systems: an inverted harmonic oscillator and a scalar field mode evolving in an inflationary universe. For the inverted oscillator with weak coupling to a bath at both high and low temperatures, $S \to r$, where r is the squeeze parameter. In the de Sitter case, at high temperatures, $S \to (1-c)r$ where $c=\gamma_0/H$, γ_0 being the coupling to the bath and H the Hubble constant. These three cases confirm previous results based on more *ad hoc* prescriptions for calculating entropy. But at low temperatures, the de Sitter entropy $S \to (1/2-c)r$ is noticeably different. This result, obtained from a more rigorous approach, shows that factors usually ignored by the conventional approaches, i.e., the nature of the environment and the coupling strength betwen the system and the environment, are important. [S0556-2821(97)06710-6]

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I. INTRODUCTION

In discussing the conceptual problems of entropy generation from cosmological particle creation [1,2] one of us was confronted in the early 1980s by the following apparent paradox: on the one hand, common sense suggests that entropy (S) is given by the number (N) of particles produced ($S \approx N^3$ for photons). On the other hand, theoretically, for a free field, particle pairs created in the vacuum will remain in a pure state and there should be no entropy generation. Inquiry into this paradox led to serious subsequent investigations into the statistical properties of particles and fields. In 1983, Hu [3] pointed out that the usual simplistic identification of entropy with the number of particles present is valid only in the thermodynamic-hydrodynamic regime, where interaction among particles and coarse graining can lead to entropy generation. This aspect was discussed later by Hu and Kandrup [4] using a statistical mechanics subdynamics analysis. The more intriguing case of entropy generation for free fields was addressed by Hu and Pavon [5]. They suggested that an intrinsic entropy of a (free) quantum field can be measured by the particle number (in a Fock-space representation) or by the variance (in the coherent-state representation). The entropy of a (free) quantum field is nonzero only if some information of the field is lost or excluded from consideration, either by choosing some special initial state and/or introducing some measure of coarse graining. For example, the predicted monotonic increase in the spontaneous creation of bosons is a consequence of adopting the Fock-space representation which amounts to a random phase initial condition implicitly assumed in most discussions of vacuum particle creation. (The difference of spontaneous and stimulated creation of bosons versus fermions was first pointed out by Parker [1], and discussed in squeezed state language by Hu, Kang and Matacz [6].) The relation of random phase and particle creation was further elaborated by Kandrup [7].

Following these early discussions of the theoretical meaning of entropy of quantum fields, a recent surge of interest in this issue was stimulated by the work of Brandenberger, Mukhanov, and Prokopec (BMP) [8], Gasperini and Giovanni (GG) [9], and others on the entropy content of primordial gravitons. The language of squeezed states for the description of cosmological particle creation was introduced by Grishchuk and Sidorov [10]. Though the physics is the same [6,11] as originally described by Parker [1] and Zel'dovich [2], the language brings closer the comparison with similar problems in quantum optics, which shares many interesting theoretical and practical issues [12]. BMP suggested a coarse graining of the field by integrating out the rotation angles in the probability functional, while GG considered a squeezed vacuum in terms of new variables which give the maximum and minimum fluctuations, and suggested a coarse graining by neglecting information about the subfluctuant variable. Keski-Vakkuri studied entropy generation from particle creation with many particle mixed initial states [13]. Matacz [14] considered a squeezed vacuum of a harmonic oscillator system with time-dependent frequency and, motivated by the special role of coherent states, modeled the effect of the environment by decohering the squeezed vacuum in the coherent-state representation. Kruczenski, Oxman, and Zal-

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darriaga [15] also used a procedure of setting off-diagonal elements in the density matrix to zero before calculating the entropy. Despite the variety of coarse-graining measures used, in the large squeezing limit (late times) these approaches all give an entropy of $S=2r$ per mode, where r is the squeezing parameter. This result which gives the number of particles created at late times agrees with that obtained in the original work of Hu and Pavon [5].

Noteworthy in this group of work is that the representation of the state of the quantum field and the coarse graining in the field are stipulated, not derived. What is implicitly assumed or grossed over in these approaches is the important process of decoherence, the diminution of the off-diagonal components of a reduced density matrix in a certain basis. It is a necessary condition for realizing the quantum to classical transition [16]. The deeper issues are to show explicitly how entropy of particle creation depends on the choice of specific initial state and/or particular ways of coarse graining, and to understand how natural or plausible these choices of the initial-state representation or the coarse-graining measure are in different realistic physical conditions [17].[1] To answer these questions, one needs to work with a more basic theoretical framework, that of statistical mechanics of quantum fields. In recent years we have approached the decoherence and entropy or uncertainty issues with the quantum open-system concept [18] and the influence functional formalism [19,20]. The purpose of this paper is to study the entropy and uncertainty of quantum fields using the statistical mechanics of squeezed quantum open systems as illustrated by quantum Brownian motion models.

In the quantum Brownian motion paradigmic depiction of quantum field theory studied in the series of papers by Hu, Paz, and Zhang [21] and Hu and Matacz [22], the system represented by the Brownian particle can act as a detector (as in the influence functional derivation of Unruh and Hawking radiation [22,23]), a particular mode of a quantum field (such as the homogeneous inflaton field), or the scale factor of the background spacetime (as in minisuperspace quantum cosmology), while the bath could be a set of coupled oscillators, a quantum field, or just the high-frequency sector of the field, as in stochastic inflation. The statistical properties of the system are depicted by the reduced density matrix (RDM) formed by integrating out the details of the bath. One can use the RDM or the associated Wigner function to calculate the statistical average of physical observables of the system, such as the uncertainty or the entropy functions. The von Neumann entropy of an open system is then

$$S \equiv -\mathrm{tr}\rho_{\mathrm{red}}\ln\rho_{\mathrm{red}}. \tag{1.1}$$

The uncertainty function measures the effects of vacuum and thermal fluctuations in the environment (at zero and finite temperature) on the observables of the system [24,25]. The increase of their variances because of these fluctuations gives rise to the uncertainty and entropy increase. The time dependence of the uncertainty function of an open system measures the varying relative importance of thermal and vacuum fluctuations and their roles in bringing about the decoherence of the system and the emergence of classical behavior [24,25].

The entropy function constructed from the reduced density matrix (or the Wigner function) of a particular state measures the information loss of the system in that state to the environment (or, in the phraseology of [26], the ''stability'' characterized by the loss of predictive power relative to the classical description). One can study the entropy increase for a specific state, or compare the entropy at each time for a variety of states characterized by the squeeze parameter. The time scale of entropy increase, when entropy arises from particle creation from the vacuum, should be comparable to the decoherence time which, for a high temperature bath, is very short. Interaction with the environment also changes its dynamics from strictly unitary to dissipative, the energy loss being measured by the viscosity function, which governs the relaxation of the system into equilibrium with the environment. The entropy function for such open systems can also be used [25,26] as a measure of how close different quantum states can lead to a classical dynamics. For example, the coherent state being the state of minimal uncertainty has the smallest entropy function [26] and a squeezed state in general has a greater uncertainty function [24]. One can thus use the uncertainty to measure how classical or ''nonclassical'' a quantum state is.

Using this first-principle approach for the calculation of the entropy function leads to more reliable results. With regard to the issue of entropy of quantum fields raised at the beginning, we can now ask what is the difference of our more vigorous definition and that defined earlier with more *ad hoc* prescriptions?

Foremost, the differences in design are obvious: the entropy of [5,8,9] and others refers to that of the field, and is obtained by coarse graining some information of the field itself, such as making a random phase approximation, adopting the number basis, or integrating over the rotation angles. The entropy of [24–26] refers to that of the open system and is obtained by coarse graining the environment. Why is it that for certain generic models in some common limit (late time, high squeezing), both groups of work obtain the same result? Under what conditions would they differ? Understanding this relation could provide a more solid theoretical foundation for the intuitively argued definitions of field entropy.

At the formal level, supposing we have some system which has been decomposed into two subsystems, it can be shown [27] that between the entropies S_1, S_2 of the two subsystems, and that of the total system, S_{12}, a triangle inequality holds:

$$|S_1 - S_2| \leqslant S_{12} \leqslant S_1 + S_2. \tag{1.2}$$

In particular, if the total system is closed and in a pure state, then it has zero entropy, so that the two subsystems

[1] This includes conditions when, for example, the quantum field is at a finite temperature or is in disequilibrium, interacting with other fields, or when its vacuum state is dictated by some natural choice, e.g., in the earlier quantum cosmology regime such as the Hartle-Hawking boundary condition leading to the Bunch-Davies vacuum in de Sitter spacetime.

necessarily have equal entropies.[2] Hence, asking for the entropy change of a system is equivalent to asking for the entropy change of the environment it couples to, if the overall closed system is in a pure state. Now, consider the case of the system as a detector (or a single mode of a field) and the environment as the field. The information lost in coarse graining the field which was used to define the field entropy in the above examples is precisely the information lost as registered in the particle detector, which shows up in the calculaton of entropy from the reduced density matrix. The bilinear coupling between the system and the bath as used in the simple quantum Brownian motion models also ensures that the information registered in both sectors is directly commutable. This explains the commonalities. However, not all coarse graining and coupling will lead to the same results, as we shall explicitly demonstrate in some examples.

Another important feature of the entropy function obtained in our present investigation, which is not at all clear in earlier studies, is that it depends nonlocally on the entire history of the squeezing parameter. This can be seen from the fact that the rate of particle creation varies in time and its effect is history dependent [32]. Existing methods of calculating the entropy generation give results which only depend on the squeezing parameter at the time when a particular coarse graining (or dropping the off-diagonal components of the density matrix) is implemented. These *ad hoc* choices of coarse graining and the time it is introduced affect the generality of the earlier results.

The plan of this paper is as follows. In Sec. II we give a brief summary of a squeezed quantum system, using a general oscillator Hamiltonian as an example. The notation is that of [6,22]. This is followed by a brief summary of open-quantum systems in terms of influence functionals [19], following the treatment of [21,22]. Readers familiar with these background material can go directly to Sec. III, which contains the central material for the derivation of entropy and uncertainty functions as well as fluctuations and coherence functions. In Sec. IV we apply these formulas to an oscillator system, recovering en route the earlier results of [24,25] for uncertainty at finite temperature, and of [26] on entropy of coherent states. In Sec. V we apply our result to the consideration of a scalar field in a de Sitter universe. We show the conditions where one recovers the $S=2r$ result of all previous work and, more significantly, the cases when they differ. We give a short discussion of our findings in Sec. VI. The appendices contain details of derivations.

[2]This could be the reason why the derivation of black hole entropy (see the recent review of Bekenstein [28]) can be obtained equivalently by computing the entropy of the radiation (e.g., [29]) emitted by the black hole, or by counting the internal states (if one knows how) of the black hole (e.g., [30]). Physically, one can view what happens to the particle as a probe into the state of the field. The application of open-system concepts to black hole entropy is a very fruitful avenue [31].

II. SQUEEZED OPEN SYSTEMS

A. Squeezed states and density matrices

Consider the general oscillator Hamiltonian

$$H(t)=f(t)\frac{a^2}{2}+f^*(t)\frac{a^{\dagger 2}}{2}+h(t)(a^\dagger a+1/2)+d(t)a+d^*(t)a^\dagger+g(t), \tag{2.1}$$

where d,f,g,h are arbitrary functions of time. The propagator for this has been calculated in [22] and is

$$U(t,t_i)=S(r,\phi)R(\theta)D(p)e^{w-|p|^2/2}, \tag{2.2}$$

where p,w are defined in terms of the coefficients appearing in H, and

$$D(p)=\exp(-p^*a-\text{H.c.}),$$

$$R(\theta)=\exp[-i\theta(a^\dagger a+1/2)],$$

$$S(r,\phi)=\exp(re^{-2i\phi}a^2/2-\text{H.c.}) \tag{2.3}$$

are the displacement, rotation, and squeeze operators [6], respectively. Suppose, we start with a simple harmonic oscillator with the Lagrangian

$$L=\frac{M}{2}(\dot{x}^2-\Omega^2x^2). \tag{2.4}$$

If we construct a Gaussian state in the position basis, with initially the same width σ_0 as that of the ground state of such an oscillator, displaced by some arbitrary amount and with a phase proportional to x, we find this to be an eigenstate of the lowering operator, and is called a coherent state. Suppose we locate the point $(\langle x\rangle,\langle p\rangle)$ in phase space and draw an ellipse about this point, the lengths of whose axes being the uncertainties $\Delta x^2,\Delta p^2$. Then, as the oscillator evolves this uncertainty ellipse revolves about the origin with angular speed Ω.

A squeezed state is again such a state, but with an arbitrary initial width σ. We find that as the oscillator evolves the uncertainty ellipse again revolves about the origin, but its axes change length and it can also rotate about its own center. It turns out that the squeeze parameter r is related to the width of such a state:

$$r=\ln\frac{\sigma_0}{\sigma},\quad \sigma_0\equiv\sqrt{\frac{\hbar}{2M\Omega}}. \tag{2.5}$$

Hence a coherent state has $r=0$, or zero squeezing. A Gaussian that initially has a width smaller than σ_0 will evolve to a squeezed state with some $r>0$. We can generate a squeezed state by applying $S(r,\phi)$ to the ground state of the simple oscillator. Consider the new operator

$$b=U^\dagger aU\equiv\alpha a+\beta^*a^\dagger, \tag{2.6}$$

where it can be shown that

$$\alpha = e^{-i\theta}\cosh r,$$

$$\beta = -e^{-i(\theta+2\phi)}\sinh r. \tag{2.7}$$

Going from a to b is then just a Bogoliubov transformation, and so α,β become Bogoliubov coefficients for our system. Their equations of motion are

$$\dot{\alpha} = -ih\alpha - if^*\beta,$$

$$\dot{\beta} = if\alpha + ih\beta,$$

$$\alpha(t_i) = 1, \quad \beta(t_i) = 0, \tag{2.8}$$

where f,h as defined in the Hamiltonian (2.1) are calculated from the general system Lagrangian. This Lagrangian has a time-dependent mass and frequency, and we will also allow it to have a time-dependent cross term denoted $2\mathcal{E}(t)$:

$$L = \frac{M(t)}{2}[\dot{x}^2 + 2\mathcal{E}(t)\dot{x}x - \Omega^2(t)x^2]. \tag{2.9}$$

Then f,h are given by [22]

$$f = \frac{1}{2}\left[\frac{M}{\kappa}(\Omega^2+\mathcal{E}^2) - \frac{\kappa}{M} + 2i\mathcal{E}\right],$$

$$h = \frac{1}{2}\left[\frac{M}{\kappa}(\Omega^2+\mathcal{E}^2) + \frac{\kappa}{M}\right], \tag{2.10}$$

and κ is an arbitrary positive constant that can be chosen to simplify the relevant equations.

Soon we shall find that the quantity of much importance to our work turns out to be the sum of the Bogoliubov coefficients $X \equiv \alpha + \beta$. It follows from Eq. (2.8) that X satisfies the classical equation of motion for the system:

$$\ddot{X} + \frac{\dot{M}}{M}\dot{X} + \left(\Omega^2 + \dot{\mathcal{E}} + \frac{\dot{M}\mathcal{E}}{M}\right)X = 0, \tag{2.11}$$

with initial conditions

$$X(t_i) = 1, \quad \dot{X}(t_i) = \frac{-i\kappa}{M(t_i)} - \mathcal{E}(t_i). \tag{2.12}$$

With this result, the usual task of finding the Bogoliubov coefficients α,β from two coupled first-order differential equations is reduced to that of solving one second-order equation for X.

B. Squeezing an inverted harmonic oscillator

For an inverted oscillator, i.e., one with $\Omega^2 < 0$, at late times r is expected to blow up. In that case we can calculate it from Eq. (2.7) as follows:

$$|\alpha| \to |\beta| \to e^r/2, \tag{2.13}$$

so that

$$r \to \ln(2|\alpha|). \tag{2.14}$$

Rather than use Eq. (2.8) to calculate α, once we have X we can extract α from it. This is done by writing, from Eq. (2.8),

$$X = \alpha + \beta,$$

$$\dot{X} = i(f-h)\alpha + i(h-f^*)\beta, \tag{2.15}$$

and solving for α,β using Eq. (2.10):

$$\left\{\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = \frac{1}{2}\left(1 \pm \frac{i\mathcal{E}M}{\kappa}\right)X \pm \frac{iM}{2\kappa}\dot{X}. \tag{2.16}$$

We can follow the behavior of r,ϕ,θ by writing Eq. (2.8) in terms of the squeeze parameter, with $f \equiv |f|e^{i\varepsilon}$:

$$\dot{r} = |f|\sin(2\phi+\varepsilon),$$

$$\dot{\phi} = -h + |f|\coth 2r\cos(2\phi+\varepsilon),$$

$$\dot{\theta} = h - |f|\tanh r\cos(2\phi+\varepsilon). \tag{2.17}$$

These equations are useful for numerical work. They also tell us of the existence of constant, and so possibly of attractor, solutions for ϕ,θ. If we set $r\to\infty$ then the equations for ϕ,θ become

$$\dot{\theta} = -\dot{\phi} = h - |f|\cos(2\phi+\varepsilon). \tag{2.18}$$

(1) Suppose there exist some θ and ϕ such that $\dot{\theta} = \dot{\phi} = 0$. Then, $h = |f|\cos(2\phi+\varepsilon)$, so that $|h| \leq |f|$. Thus, since h is real, we have $h^2 \leq |f|^2$, and from Eq. (2.10) this inequality is true if and only if $\Omega^2 \leq 0$.

(2) Conversely, suppose $\Omega^2 \leq 0$. Then by the previous argument, $|h| \leq |f|$, or $-1 \leq h/|f| \leq 1$. Thus, there must exist some ϕ such that $\cos(2\phi+\varepsilon) = h/|f|$. From Eq. (2.18) we see that for this value of ϕ, $\dot{\theta} = \dot{\phi} = 0$.

In other words, there will exist constant solutions for ϕ,θ if and only if $\Omega^2 \leq 0$ (the oscillator is ‘‘inverted’’). Of course, this does not reveal whether these constant solutions are attractors. Numerically, solving Eq. (2.17) with $\Omega^2 \leq 0$, for various $\mathcal{E}$, Ω, and κ, shows that ϕ,θ apparently do always quickly tend toward constants, always accompanied by one of $r \to \pm\infty$.

We note that it is common to eliminate the cross term in the action by adding a surface term:

$$L \to \frac{M}{2}(\dot{x}^2 + 2\mathcal{E}\dot{x}x - \Omega^2x^2) - \frac{1}{2}\frac{d}{dt}(M\mathcal{E}x^2)$$

$$= \frac{M}{2}\left[\dot{x}^2 - \left(\Omega^2 + \frac{\dot{M}\mathcal{E}}{M} + \dot{\mathcal{E}}\right)x^2\right]. \tag{2.19}$$

Although this leaves the classical equation of motion unchanged, it will change the squeeze parameters. In this paper we leave the cross term in our Lagrangians.

C. Open systems

We now let our system, a single harmonic oscillator in a squeezed state, interact with a bath of harmonic oscillators, resulting in an open squeezed system. The method most appropriately depicting such an open system under the influence of an environment we want to use is the influence functional (IF) formalism first introduced by Feynman and Vernon [19]. It was later applied by Caldeira and Leggett [19] to the high-temperature limit of a model where both system and environment are composed of oscillators with time-independent frequencies. A comprehensive review is given by Grabert *et al.* in [19].

In these earlier works, the influence functional for quantum Brownian motion has only been derived for Markovian processes corresponding to coupling to a high-temperature ohmic bath. An exact master equation for non-Markovian processes is recently derived by Hu, Paz, and Zhang [21] (see also [33,34]). Most work in this area since Feynman and Vernon has assumed a bilinear system-bath coupling, which together with a factorizable initial condition, yields an exact analytic form for the influence functional. Initial conditions with correlations have been considered by [35]. Weakly nonlinear couplings have also been considered using perturbation theory borrowed from field theory [21]. More relevant to our work here, Hu and Matacz [22] obtained the master equation for system and bath oscillators with time-dependent frequencies, a result readily generalizable to quantum fields (see, e.g., [36,37]).

In this paper we further develop the work of [22] by considering a squeezed system coupled bilinearly to a bath of oscillators with time-independent frequencies, but with a time-dependent coupling constant. We also lay out the groundwork for calculating such quantities as entropy and uncertainty as well as fluctuations and coherence, for the purpose of this paper, and a later one on the de Sitter universe [38].

Consider the quantum Brownian motion of an oscillator (system) with time-dependent mass, cross term, and natural frequency interacting bilinearly with an environment of n oscillators with the same time-dependent parameters. The total Lagrangian is

$$S[x,\mathbf{q}]=S[x]+S_E[\mathbf{q}]+S_{\rm int}[x,\mathbf{q}]$$
$$=\int_{t_i}^{t}ds\left\{\frac{M(s)}{2}[\dot{x}^2+2\mathcal{E}(s)x\dot{x}-\Omega^2(s)x^2]\right.$$
$$+\sum_n\left[\frac{m_n(s)}{2}[\dot{q}_n^2+2\varepsilon_n(s)q_n\dot{q}_n-\omega_n^2(s)q_n^2]\right]$$
$$\left.+\sum_n[-c(s)xq_n]\right\}, \tag{2.20}$$

where the particle and the bath oscillators have coordinates x and q_n, respectively.

We wish to start with some initial system density matrix $\rho_{sys}(x_i x_i' 0)$ assumed to be uncorrelated with the environment at time $t=0$ [Eq. (A4)]. The reduced density matrix ρ_r obtained by integrating out the environmental degrees of freedom [see Eq. (A5)] is evolved by its propagator J_r represented by the influence functional F, which contains in its exponent the dissipation and noise kernels μ and ν, respectively [Eq. (A11)]. These can be calculated from Eqs. (2.18) and (2.19) [22]. A summary of the influence functional formalism can be found in Appendix A.

For an environment of simple harmonic oscillators (that is, time-independent frequencies with no cross term), the dissipation and noise kernels take the form

$$\mu(s,s')=\int_0^\infty d\omega I(\omega,s,s')\ {\rm Im}[X(s)X^*(s')],$$

$$\nu(s,s')=\int_0^\infty d\omega I(\omega,s,s')\coth\frac{\omega}{2T}\,{\rm Re}[X(s)X^*(s')], \tag{2.21}$$

where by T we will always mean $k_BT/\hbar$, X is the sum of the Bogoliubov coefficients for the bath oscillators, and I is the "spectral density," a function defined by

$$I(\omega,s,s')=\frac{c(s)c(s')}{2\kappa}\sum_n\ \delta(\omega-\omega_n), \tag{2.22}$$

which encodes information of the action of the environment on the system. In general, the spectral density can be described by some function of ω^j, where j is set by the particular environment being modeled. The case of $j=1$, the so-called "ohmic" environment, is a borderline between the superohmic case ($j>1$), which models weak damping, and the subohmic case ($j<1$) modeling strong damping. We can in effect consider both damping extremes by taking an ohmic environment together with some strength γ_0 which can be altered from zero, for a free system, up to higher values.

Also, by considering the continuum limit of the coupling constant, it can be shown that this constant's independence of n also leads to an ohmic environment, so we will only consider spectral densities of the form

$$I(\omega,s,s')=\frac{2\gamma_0}{\pi}\omega c(s)c(s'). \tag{2.23}$$

For a general Lagrangian the sum of the Bogoliubov coefficients X will be complicated; however, we have simplified our calculations by taking the bath to be composed of unsqueezed (i.e., coherent) static oscillators with unit mass. For this type of bath the dissipation and noise can be calculated for an arbitrary bath temperature; we use the integral form of the noise as being easier to work with

$$\mu(s,s')=2\gamma_0 c(s)c(s')\delta'(s-s'),$$

$$\nu(s,s')=\frac{2\gamma_0}{\pi}c(s)c(s')\int_0^\infty \omega\coth\frac{\omega}{2T}\ \cos\omega(s-s')d\omega. \tag{2.24}$$

The dissipation is seen to be local for all temperatures, and the noise becomes white, that is, it tends toward a δ function in the high-temperature limit.

III. ENTROPY AND UNCERTAINTY, FLUCTUATIONS AND COHERENCE

A. Initial and final states

Assume the systems are initially in the vacuum state, so that their density matrix is Gaussian. So we start with an arbitrary Gaussian reduced density matrix

$$\rho_r(x_i x_i' t_i) \propto e^{-\xi x_i^2 + \chi x_i x_i' - \xi^* x_i'^2}, \tag{3.1}$$

and propagate it by using Eqs. (A5) and (A13) to give

$$\rho_r(x,x',t) = N e^{-A\Delta^2 - 2iB\Delta\Sigma - 4C\Sigma^2}, \tag{3.2}$$

where we have used the same A, B, and C notation of [14], and with ξ_r, ξ_i the real and imaginary parts of ξ:

$$N = 2\sqrt{C/\pi},$$

$$A = a_{22} + \frac{1}{D}\{[(2\xi_r + \chi)/4 + a_{11}]b_3^2 + (2\xi_i + b_4)a_{12}b_3 - (2\xi_r - \chi)a_{12}^2\},$$

$$B = -b_1/2 + \frac{1}{D}[(\xi_i + b_4/2)b_2 b_3 - (2\xi_r - \chi)a_{12}b_2],$$

$$C = \frac{1}{4D}(2\xi_r - \chi)b_2^2,$$

$$D = 4|\xi|^2 - \chi^2 + 4(2\xi_r - \chi)a_{11} + 4\xi_i b_4 + b_4^2. \tag{3.3}$$

These expressions form the basis of our later calculations. The quantity we are focusing on is the reduced density matrix, Eq. (3.2), using the expressions in Eq. (3.3). These in turn use Eq. (A32), which depends on our obtaining X, the sum of the Bogoliubov coefficients for the effective oscillator.

B. Entropy from the reduced density matrix

The entropy of a field mode has been calculated by Joos and Zeh [16]. It can be derived from the reduced density matrix at time t by using Eq. (1.1), and is given by

$$S = \frac{-1}{w}[w\ln w + (1-w)\ln(1-w)] \simeq 1 - \ln w \quad \text{if } w \to 0, \tag{3.4}$$

where

$$w \equiv \frac{2\sqrt{C/A}}{1+\sqrt{C/A}}. \tag{3.5}$$

The linear entropy is often more useful to work with owing to its simplicity:

$$S_{\text{lin}} \equiv -\text{tr}\rho^2 = -\sqrt{C/A}, \tag{3.6}$$

and $S = 0 \to \infty$ is equivalent to $S_{\text{lin}} = -1 \to 0$, both strictly increasing. Then if $S_{\text{lin}} \to 0$, we have

$$S \to -\ln|S_{\text{lin}}| + 1 - \ln 2, \quad \text{i.e., } S_{\text{lin}} \to -e^{1-S}/2. \tag{3.7}$$

As an example, suppose we have a system in an initially pure Gaussian state ($\chi = 0$), so that noise and dissipation are absent: $\gamma_0 = 0$. In this case, from Eqs. (2.24) and (A32) we have

$$a_{11} = a_{12} = a_{22} = 0, \tag{3.8}$$

so that Eq. (3.3) gives $C/A = 1$ and hence from Eq. (3.4) $S = 0$ as expected.

C. Fluctuations and coherence

A clearer picture of the dynamics of a closed and open system can be obtained if we rotate the phase-space axes so that the density matrix can be expressed in terms of the so-called super- and subfluctuant variables. (Alternatively, we are rotating the Wigner function in phase space so as to eliminate the cross term there.) Call these variables u,v, expressed as real linear combinations of q,p [they have nothing to do with the u,v of Eq. (A12)]. We fix the linear combinations such that one variable (u, the superfluctuant) grows exponentially while the other decays exponentially. In the case of no coupling to the environment we proceed by expressing $\langle u^2\rangle, \langle v^2\rangle$ in terms of $\langle q^2\rangle, \langle qp+pq\rangle, \langle p^2\rangle$, and then substituting for these the standard squeezed state results [14]. This enables us to write

$$\langle u^2\rangle = \frac{\kappa e^{2r}}{2}, \quad \langle v^2\rangle = \frac{e^{-2r}}{2\kappa}. \tag{3.9}$$

These relations fix u,v in terms of q,p, and we now use the same transformation for the case of nonzero dissipation:

$$u = -\kappa\sin\phi q + \cos\phi p,$$

$$v = \cos\phi q + \frac{\sin\phi}{\kappa}p. \tag{3.10}$$

What we wish to do is to take a density matrix in position, Eq. (3.2), and write it in the u,v basis. Consider first of all calculating $\rho(u,u')$:

$$\rho(u,u') = \int \langle u|q\rangle \rho(q,q') \langle q'|u'\rangle dq dq'. \tag{3.11}$$

We need $\langle u|q\rangle$. This can be found by solving the partial differential equation which follows by quantizing Eq. (3.10) and applying both sides to $\langle q|u\rangle$:

$$u\langle q|u\rangle = (-\kappa\sin\phi q - i\cos\phi\partial_q)\langle q|u\rangle, \tag{3.12}$$

which has solution

$$\langle q|u\rangle = f(u)\exp\left[\frac{i}{\cos\phi}\left(\frac{\kappa\sin\phi q^2}{2} + qu\right)\right] \tag{3.13}$$

for some function $f(u)$ to be determined [unrelated to Eq. (2.10)]. We determine $f(u)$ by redoing this calculation with the roles of q and u interchanged; since $[v,u] = i$, we have

$$q\langle u|q\rangle = \left(\frac{-\sin\phi u}{\kappa} + i\cos\phi\partial_u\right)\langle u|q\rangle. \tag{3.14}$$

Solving this determines $f(u)$ and allows us to finally write (up to a phase)

$$\langle q|u\rangle = \frac{1}{\sqrt{2\pi\cos\phi}}\exp\left[\frac{i}{\cos\phi}\left(\frac{\kappa\sin\phi q^2}{2} + qu + \frac{\sin\phi u^2}{2\kappa}\right)\right]. \tag{3.15}$$

Similarly, we find

$$\langle q|v\rangle = \sqrt{\frac{\kappa}{2\pi\sin\phi}}\exp\left[\frac{i\kappa}{\sin\phi}\left(\frac{-\cos\phi q^2}{2}+qv-\frac{\cos\phi v^2}{2}\right)\right]. \tag{3.16}$$

Now, suppose we start with a Gaussian density matrix as in Eq. (3.2). We can then easily change bases using Eqs. (3.11), (3.15), and (3.16) to get, with

$$\gamma \equiv \frac{\kappa}{2}\cot\phi, \quad \sigma \equiv \frac{\sin^2\phi}{\kappa^2}[4AC+(B-\gamma)^2], \tag{3.17}$$

$$\lambda \equiv \frac{4AC+(4\gamma\sigma+B-\gamma)^2}{4\sigma^2}, \tag{3.18}$$

$$\rho(u,u') = \sqrt{\frac{C}{\pi\sigma\lambda}}\exp\left[\left(\frac{-1}{4\sigma\lambda}\right)[A\Delta_u^2+2i(4\gamma\sigma+B-\gamma)\times\Delta_u\Sigma_u+4C\Sigma_u^2]\right],$$

$$\rho(v,v') = \sqrt{\frac{C}{\pi\sigma}}\exp\left[\left(\frac{-1}{4\sigma}\right)[A\Delta_v^2-2i(4\gamma\sigma+B-\gamma)\times\Delta_u\Sigma_u+4C\Sigma_v^2]\right], \tag{3.19}$$

where we have used the sum and difference variables, e.g., $\Sigma_u \equiv (u+u')/2, \Delta_u \equiv u-u'$, and γ has no relation to γ_0.

We can show that in the absence of a bath, these matrices reduce to the expected ones for a squeezed vacuum. First, in the q representation the density matrix of a squeezed vacuum is known to be [39]

$$\rho(q,q') \propto \frac{-\kappa}{2}\frac{1+e^{2i\phi}\tanh r}{1-e^{2i\phi}\tanh r}(q^2+q'^2). \tag{3.20}$$

If we write $\rho(q,q')$ in terms of sum and difference coordinates and compare with the definitions of A,B,C in Eq. (3.2), we find

$$A = C = \frac{\kappa}{4}\frac{1-\tanh^2 r}{1-2\cos 2\phi\tanh r+\tanh^2 r},$$

$$B = \frac{\kappa\sin 2\phi\tanh r}{1-2\cos 2\phi\tanh r+\tanh^2 r}. \tag{3.21}$$

Substituting these into Eq. (3.19) gives

$$\rho(u,u') = \frac{e^{-r}}{\sqrt{\pi\kappa}}\exp\left[\left(\frac{-e^{-2r}}{2\kappa}\right)(u^2+u'^2)\right],$$

$$\rho(v,v') = \sqrt{\frac{\kappa}{\pi}}e^r\exp\left[\left(\frac{-\kappa e^{2r}}{2}\right)(v^2+v'^2)\right]. \tag{3.22}$$

These are the expected results, as can be seen by the fact that with p,q replaced by u,v, respectively, they are produced when ϕ is set to zero in $\rho(p,p')$ and $\rho(q,q')$.

Measures of fluctuations and coherence. Returning to the general case of dissipation, the fluctuations in u and v are calculated from the density matrices:

$$\Delta u^2 = \langle u^2\rangle - \langle u\rangle^2 = \int u^2\rho(u,u)du - \left[\int u\rho(u,u)du\right]^2 = \frac{\sigma\lambda}{2C},$$

$$\Delta v^2 = \frac{\sigma}{2C}, \tag{3.23}$$

and both of these are just equal to 1/2 divided by the coefficient of $-\Sigma^2$ in their density matrix.

As a measure of coherence we note that a large coefficient of $-\Delta^2$ means that the density matrix is strongly peaked along its diagonal, i.e., there is very little coherence in the system. A measure of coherence was defined in [40] as a squared coherence length L^2, equal to 1/8 divided by the coefficient of $-\Delta^2$, so that a large L^2 means a high degree of coherence in the system. With this definition of L^2, Eq. (3.19) gives

$$L_u^2 = \frac{\sigma\lambda}{2A}, \quad L_v^2 = \frac{\sigma}{2A}. \tag{3.24}$$

We can also relate the coherence lengths and fluctuations to the entropy of the system (see Sec. III B for definitions). We can write

$$\frac{L_u^2}{\Delta u^2} = \frac{L_v^2}{\Delta v^2} = S_{\rm lin}^2 = \frac{C}{A}. \tag{3.25}$$

(A note of caution: linear entropy is negative by definition in order for it to increase with S. Then as $S_{\rm lin}$ increases, $S_{\rm lin}^2$ will decrease.) Also, the uncertainty relation for u,v becomes, from Eqs. (3.17), (3.18), and (3.23),

$$\Delta u^2\Delta v^2 = \frac{1}{S_{\rm lin}^2}\left[\frac{1}{4}+\frac{(4\gamma\sigma+B-\gamma)^2}{16AC}\right]. \tag{3.26}$$

For the free field the last term in the square brackets is zero while $S_{\rm lin} = -1$ (since $S=0$), so that $\Delta u\Delta v = 1/2$.

IV. ENTROPY AND UNCERTAINTY OF OSCILLATOR SYSTEM

We can now demonstrate how the previous results are used. In the simplest cases, such as a static oscillator coupled to a thermal bath of static oscillators, with a static ohmic coupling, the entropy is easily compared with known results in equilibrium statistical mechanics. From Sec. A 2, we know that this case has local dissipation [i.e., $\mu \propto \delta'(\Delta)$], and at $T\to\infty$ the noise becomes white [$\nu \propto \delta(\Delta)$].

For thermal equilibrium, the standard statistical mechanics result for the entropy at high temperature is

$$S \to 1+\ln\frac{T}{k}. \tag{4.1}$$

Obtaining this result with this formalism is a good example of its application. We will leave the details in Appendix B but show the numerical results in plots. Figure 1 shows a plot of S versus z for $\sigma=1$, $k=1$, $\gamma_0=0.1$, $T=10^5$. For these numbers, Eq. (4.1) gives $S\to 12.513$ as $z\to\infty$, as compared

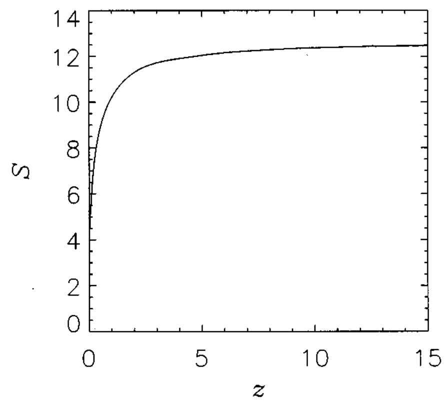

FIG. 1. Entropy growth over time.

with $S\rightarrow 12.514$ numerically at $z=100$, a result indicated by the figure. The relaxation time, defined to be

$$\frac{1}{2\gamma_0}=5, \tag{4.2}$$

is apparent in the figure as a characteristic time over which the entropy climbs to its final value, while the decoherence time scale [41]

$$\frac{1}{4M\gamma_0 T\sigma^2}=2.5\times 10^{-5} \tag{4.3}$$

is too small to be noticeable.

Coherent state as the state of least entropy. We now use our entropy expression to investigate the claim that for large times the state of least entropy for the static oscillator is the coherent one, at least for white noise and local dissipation. This was shown in [26] in the small γ_0 limit by using a Wigner function approach.

Using our expression for the entropy S, we can plot S versus the initial squeeze parameter r for various times in Fig. 2. We have chosen $k=10, \gamma_0=0.1$. The squeeze parameter r is related to σ, the width of the Gaussian wave function, by

$$r\equiv \ln\frac{\sigma_0}{\sigma}, \quad \sigma_0\equiv\sqrt{\frac{1}{2\kappa}} \tag{4.4}$$

or

$$\sigma=\frac{e^{-r}}{\sqrt{2\kappa}}. \tag{4.5}$$

Note that at early times (e.g., $z=0.001$), the entropy is minimized for high initial squeezing, as noted in [26], Fig. 1; this is not unreasonable since such a highly squeezed state will spread with time, becoming indistinguishable at later times from states which started out being less highly squeezed. At late times the entropy is minimized by starting with small or zero squeezing, i.e., an initially coherent state is the one which minimizes entropy at late times. Thus, our approach agrees with [26], and may be more useful in that it allows us to directly calculate the entropy at all times.

The static inverted oscillator is the simplest squeezed system. It also models the zero mode of the inflaton field in new inflation [42]. Its Lagrangian is

$$L(t)=\frac{1}{2}[\dot{x}^2+k^2x^2]. \tag{4.6}$$

Suppose this is coupled to the usual environment of harmonic oscillators in a thermal state, with coupling constant $c(s)=1$. Then the equivalent oscillator we consider has unit mass, no cross term, and frequency

$$\Omega_{\rm eff}^2=-k^2-\gamma_0^2\equiv-\kappa^2, \tag{4.7}$$

so that from Eq. (2.11) the sum of its Bogoliubov coefficients is (taking $t_i=0$)

$$X(t)=\cosh z-i\sinh z. \tag{4.8}$$

Hence, from Eq. (2.16) we have

$$\alpha=\cosh z, \quad \beta=-i\sinh z, \tag{4.9}$$

so that from Eq. (2.14) at late times ($z\rightarrow\infty$)

$$r\rightarrow z. \tag{4.10}$$

To investigate the dependence of the entropy on the various quantities in the propagator coefficients, we calculate these coefficients first for white noise analytically; we then calculate them numerically for zero temperature.

The b_i's are independent of the temperature, and using Eq. (A32) they are found to be (where here and elsewhere a carat will denote division by κ)

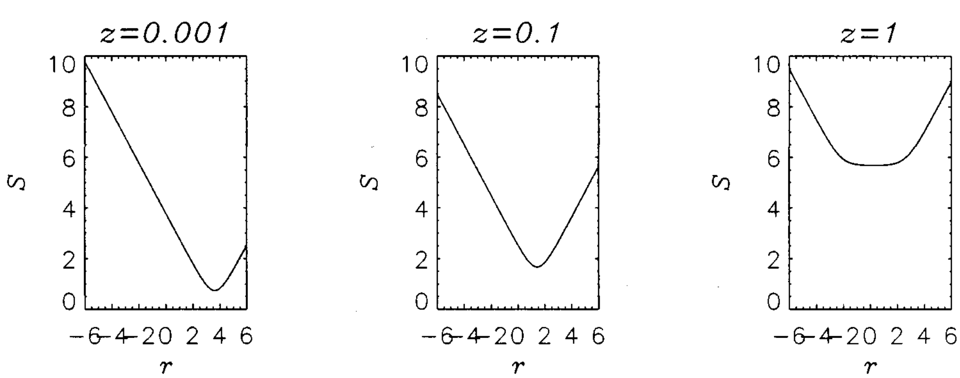

FIG. 2. Entropy at various times.

$$b_{\{{1\atop 4}\}}=\kappa(\pm\coth z-\hat{\gamma}_0),\quad b_{\{{2\atop 3}\}}=\frac{\pm\kappa e^{\pm\hat{\gamma}_0 z}}{\sinh z}. \tag{4.11}$$

High temperature. White noise is given by $\nu(s,s')=4\gamma_0 T\delta(s-s')$, or $\nu(\zeta,\zeta')=4\hat{\gamma}_0\kappa^2 T\delta(\zeta-\zeta')$; the relevant quantities are inserted into Eq. (A32) with the a_{ij}'s then becoming

$$a_{11}=\frac{T}{2\hat{k}^2\sinh^2 z}[\hat{k}^2+e^{2\hat{\gamma}_0 z}-\hat{\gamma}_0\sinh 2z-\hat{\gamma}_0^2\cosh 2z],$$

$$a_{12}=\frac{Te^{-\hat{\gamma}_0 z}}{\hat{k}^2\sinh^2 z}[(1-e^{2\hat{\gamma}_0 z})\cosh z+(1+e^{2\hat{\gamma}_0 z})\hat{\gamma}_0\sinh z],$$

$$a_{22}=\frac{Te^{-2\hat{\gamma}_0 z}}{2\hat{k}^2\sinh^2 z}[-\hat{k}^2e^{2\hat{\gamma}_0 z}-1+\hat{\gamma}_0 e^{2\hat{\gamma}_0 z}(\hat{\gamma}_0\cosh 2z-\sinh 2z)]. \tag{4.12}$$

Note that $\hat{\gamma}_0=\gamma_0/\kappa<1$; however, if we assume small dissipation ($\hat{\gamma}_0\ll 1$), we can write down large time limits of these quantities:

$$a_{11}\to\frac{T\hat{\gamma}_0}{1-\hat{\gamma}_0},\quad a_{12}\to\frac{2Te^{-(1-\hat{\gamma}_0)z}}{1+\hat{\gamma}_0},\quad a_{22}\to\frac{T\hat{\gamma}_0}{1+\hat{\gamma}_0},$$

$$b_{\{{1\atop 4}\}}\to\kappa(\pm 1-\hat{\gamma}_0),\quad b_{\{{2\atop 3}\}}\to\pm 2\kappa e^{-(1\mp\hat{\gamma}_0)z}. \tag{4.13}$$

We can now calculate large time limits of the density matrix coefficients from Eq. (3.3):

$$A\to a_{22},\quad B\to -b_1/2,\quad C\to\frac{b_2^2}{16a_{11}}. \tag{4.14}$$

These coefficients are independent of the initial conditions, which might be expected since the dissipation is acting to damp out any late time dependence on these initial conditions. So we have

$$S_{\rm lin}=-\sqrt{\frac{C}{A}}\to\frac{-\kappa^2 e^{-z}}{2\gamma_0 T}, \tag{4.15}$$

so that, from Eqs. (3.7) and (4.10),

$$S\to r+1+\ln\frac{T\gamma_0}{\kappa^2}. \tag{4.16}$$

Zero temperature. At $T=0$, the environment exerts only quantum effects through a_{ij}'s. If we write the noise in its primitive form as the usual integral over frequency then we can leave this frequency integration until last after the time integrations have been done. We will follow a more sophisticated approach in a later paper [38], but we show it here to investigate what value it might have.

So we refer to Eqs. (A32) and (2.24), swapping the limits of integration to write

$$a_{11}=\frac{\gamma_0}{\pi\sinh^2 z}\int_0^{\hat{\omega}_{\rm max}}d\hat{\omega}\,\hat{\omega}\coth\frac{\hat{\omega}\kappa}{2T}\int_0^z d\zeta\int_0^z d\zeta' e^{\hat{\gamma}_0(\zeta+\zeta')}\times\sinh(z-\zeta)\sinh(z-\zeta')\cos\hat{\omega}(\zeta-\zeta')$$
$$=\frac{\gamma_0}{2\pi\sinh^2 z}\int_0^{\hat{\omega}_{\rm max}}d\hat{\omega}\,\hat{\omega}\coth\frac{\hat{\omega}\kappa}{2T}I_{11}, \tag{4.17}$$

where

$$I_{11}\equiv\{\hat{k}^2-\hat{\omega}^2+2e^{2\hat{\gamma}_0 z}+(1+\hat{\gamma}_0^2+\hat{\omega}^2)\cosh 2z-4e^{\hat{\gamma}_0 z}[\cos\hat{\omega}z(\cosh z+\hat{\gamma}_0\sinh z)+\hat{\omega}\sin\hat{\omega}z\sinh z]+2\hat{\gamma}_0\sinh 2z\}/[\hat{k}^4+2\hat{\omega}^2(1+\hat{\gamma}_0^2)+\hat{\omega}^4]. \tag{4.18}$$

Similarly,

$$a_{12}=\frac{\gamma_0 e^{-\hat{\gamma}_0 z}}{\pi\sinh^2 z}\int_0^{\hat{\omega}_{\rm max}}d\hat{\omega}\,\hat{\omega}\coth\frac{\hat{\omega}\kappa}{2T}I_{12}, \tag{4.19}$$

where

$$I_{12}\equiv\{-2\cosh z(1+e^{2\hat{\gamma}_0 z})-2\hat{\gamma}_0\sinh z(1-e^{2\hat{\gamma}_0 z})+e^{\hat{\gamma}_0 z}\cos\hat{\omega}z[3+\hat{\gamma}_0^2+\hat{\omega}^2+(\hat{k}^2-\hat{\omega}^2)\cosh 2z]+2\hat{\omega}e^{\hat{\gamma}_0 z}\sin\hat{\omega}z\sinh 2z\}/[\hat{k}^4+2\hat{\omega}^2(1+\hat{\gamma}_0^2)+\hat{\omega}^4], \tag{4.20}$$

and

$$a_{22}=\frac{\gamma_0 e^{-2\hat{\gamma}_0 z}}{2\pi\sinh^2 z}\int_0^{\hat{\omega}_{\rm max}}d\hat{\omega}\,\hat{\omega}\coth\frac{\hat{\omega}\kappa}{2T}I_{22}, \tag{4.21}$$

where

$$I_{22}\equiv\{2+e^{2\hat{\gamma}_0 z}[\hat{k}^2-\hat{\omega}^2+(1+\hat{\gamma}_0^2+\hat{\omega}^2)\cosh 2z-2\hat{\gamma}_0\sinh 2z]+4e^{\hat{\gamma}_0 z}[\cos\hat{\omega}z(-\cosh z+\hat{\gamma}_0\sinh z)-\hat{\omega}\sin\hat{\omega}z\sinh z]\}/[\hat{k}^4+2\hat{\omega}^2(1+\hat{\gamma}_0^2)+\hat{\omega}^4]. \tag{4.22}$$

With $T=0$ the coth term is set to one. Then in all cases a_{ij} starts at zero at $z=0$; for low dissipation a_{11},a_{22} quickly climb to similar constant values while a_{12} climbs briefly but then rapidly decreases to zero. This behavior quantitatively matches the large time limits of the white noise a_{ij}'s in Eq. (4.13), even though the two calculations were done quite differently. The asymptotic value of a_{11} increases in even steps as we increase $\hat{\omega}_{\rm max}$ exponentially. So we can make a_{11} arbitrarily large by taking a large enough cutoff, so that it will always dominate D.

In that case, with $\hat{\gamma}_0\ll 1$ we have, at late times, using the b_i's in Eq. (4.13),

$$A\to a_{22},\quad B\to -b_1/2,\quad C\to\frac{b_2^2}{16a_{11}}. \tag{4.23}$$

Again, the coefficients are independent of the initial conditions. Since b_2 is unchanged from the high-temperature case and a_{11}, a_{22} tend toward constants, we now can say

$$S_{\rm lin} \to \frac{-\kappa e^{-z}}{2\sqrt{a_{11}a_{22}}} \tag{4.24}$$

and so again from Eqs. (3.7) and (4.10)

$$S \to r + 1 + \ln \frac{\sqrt{a_{11}a_{22}}}{\kappa}. \tag{4.25}$$

V. SCALAR FIELD IN DE SITTER SPACETIME

We now turn to an example in cosmology, that of an inflationary universe [42]. We want to calculate the entropy of a massless scalar field minimally coupled to gravity in a de Sitter spacetime by examining the evolution of the density matrix. As we shall see, it is a generally solvable squeezed system.

Consider a scalar field Φ of mass m, described by the Lagrangian density

$$\mathcal{L} = \frac{\sqrt{-g}}{2}[g^{\mu\nu}\Phi,_\mu \Phi,_\nu - (m^2 + \xi R)\Phi^2], \tag{5.1}$$

coupled by ξ to the curvature $R = 6(\dot{a}^2/a^2 + \ddot{a}/a)$ of a spatially flat Friedmann-Robertson-Walker (FRW) universe with metric

$$ds^2 = dt^2 - a^2(t)\sum_i (dx^i)^2. \tag{5.2}$$

In conformal time $\eta = \int dt/a$, the conformally related field $\chi = a\Phi$ is described by a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left[\chi'^2 - \sum \chi,_i^2 - 2\frac{a'}{a}\chi\chi' - \chi^2\left(m^2a^2 - \frac{a'^2}{a^2} + 6\xi\frac{a''}{a}\right)\right]. \tag{5.3}$$

Decomposing the field into normal modes k with amplitudes q_k, the Lagrangian can be expressed as

$$L(\eta) = \sum \frac{1}{2}\left[q'^2 - 2\frac{a'}{a}qq' - q^2\left(k^2 + m^2a^2 - \frac{a'^2}{a^2} + 6\xi\frac{a''}{a}\right)\right]. \tag{5.4}$$

Inside the square brackets if we add a surface term of $6\xi(q^2a'/a)'$ to eliminate the a'' term (for justification, see [22]), we get a new Lagrangian:

$$L_{\rm new}(\eta) = \sum \frac{1}{2}\left[q'^2 + 2(6\xi - 1)\frac{a'}{a}qq' - q^2\left(k^2 + m^2a^2 + (6\xi - 1)\frac{a'^2}{a^2}\right)\right]. \tag{5.5}$$

For a massless minimally coupled scalar field in de Sitter space,

$$L_{\rm new}(\eta) = \sum \frac{1}{2}\left[q'^2 + \frac{2}{\eta}qq' - q^2\left(k^2 - \frac{1}{\eta^2}\right)\right]. \tag{5.6}$$

We also use a spectral density of the form

$$I(\omega, \eta, \eta') = \frac{2\gamma_0}{\pi H}\frac{\omega}{\sqrt{\eta\eta'}}, \tag{5.7}$$

so that $c(\eta) = 1/\sqrt{-H\eta}$. This form of spectral density will be justified in a later paper [38], although for now we note that it does not make the equation of motion for X any harder to solve than if we had used a static coupling. Since γ_0/H is dimensionless, we rewrite it as c [not to be confused with $c(\eta)$]. Incorporating the bath gives the equivalent oscillator with $M = 1, \mathcal{E} = 1/\eta$, and frequency, from Eq. (A24),

$$\Omega_{\rm eff}^2 = k^2 - \frac{1 + c^2}{\eta^2}. \tag{5.8}$$

Also, we choose $\kappa = k$ to simplify the equation of motion. With $z = k\eta$ we can write this together with its initial conditions from Eqs. (2.9), (2.11), and (2.12) as

$$X''(z) + \left(1 - \frac{2 + c^2}{z^2}\right)X = 0,$$

$$X(z_i) = 1, \quad X'(z_i) = -i - 1/z_i, \tag{5.9}$$

where $z < 0$. The solution of this equation can be constructed using Bessel functions whose index is a function of c; however, since we are interested in small c we take the solution to be approximately that of the same equation but with c set to zero. This simplifies things greatly:

$$X(z) = \left(1 + \frac{i}{2z_i}\right)f(z) + \frac{i}{z_i}f^*(z), \tag{5.10}$$

where

$$f(z) \equiv \left(1 - \frac{i}{z}\right)e^{i(z_i - z)}. \tag{5.11}$$

We can further simplify X by using a very early initial time, setting $z_i \to -\infty$. We also disregard the phase in the resulting expression for X, since this is not expected to make any difference to physical quantities. In this case we obtain a new function which we rename X:

$$X(z) \rightsquigarrow \left(1 - \frac{i}{z}\right)e^{-iz}. \tag{5.12}$$

The Bogoliubov coefficients can now be found from Eq. (2.16):

$$\alpha = \left(1 - \frac{i}{2z}\right)e^{-iz}, \quad \beta = \frac{-i}{2z}e^{-iz}, \tag{5.13}$$

and so from Eq. (2.14) at late times

$$r \to -\ln|z|. \tag{5.14}$$

This result was also obtained in [14] using a different formalism.

First, we calculate the b_i's. Since we are only interested in late times we can work to leading order in z (although with hindsight we include some next higher order terms which will be needed later). Using Eq. (A32) we find

$$b_1 = ck/z + kz + O(z^3),$$

$$b_{\{^2_3\}} = \mp k|z|^{1\mp c}|z_i|^{\pm c},$$

$$b_4 = (c+1)k/z_i + kz^3/3 + O(z^5), \tag{5.15}$$

and for the a_{ij}'s we need the following expressions, calculated from Eq. (5.12):

$$\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)} \simeq \frac{(1-z/\zeta)\cos(\zeta - z) - (z+1/\zeta)\sin(\zeta - z)}{\cos z + z\sin z},$$

$$\frac{\mathrm{Im}\,[X(\zeta)]}{\mathrm{Im}X(z)} \simeq \frac{\dfrac{\cos\zeta}{\zeta} + \sin\zeta}{\dfrac{\cos z}{z} + \sin z}, \tag{5.16}$$

$$\exp\left(\hat{\gamma}_0 \int_{z_i}^{\zeta} \frac{c^2(\zeta'')}{M} d\zeta''\right) = (\zeta/z_i)^{-c},$$

$$\exp\left(-\hat{\gamma}_0 \int_{\zeta}^{z} \frac{c^2(\zeta'')}{M} d\zeta''\right) = (z/\zeta)^{c}. \tag{5.17}$$

High temperature. We begin by writing

$$\nu = 4cc^2(s)T\delta(s-s') = \frac{-4ck^2T}{\zeta}\delta(\zeta - \zeta'). \tag{5.18}$$

We calculate a_{11} here and leave the details of a_{12}, a_{22} to Appendix C. First, Eq. (A32) gives

$$a_{11} = \frac{1}{2k^2}\int_{z_i}^{z} d\zeta \int_{z_i}^{z} d\zeta' \left(\frac{\zeta}{z_i}\right)^{-c} \frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)} \frac{4ck^2T}{-\zeta}$$
$$\times \delta(\zeta - \zeta')\left(\frac{\zeta'}{z_i}\right)^{-c} \frac{\mathrm{Im}\,[X(z)X^*(\zeta')]}{\mathrm{Im}X(z)}$$
$$= 2cT\int_{z_i}^{z} d\zeta \left(\frac{\zeta}{z_i}\right)^{-2c}\left(\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)}\right)^2 \frac{1}{-\zeta}. \tag{5.19}$$

We wish to investigate the dependence of the a_{ij}'s on z as $z \to 0$, and so we now separate each integral into a sum of two parts. The first is obtained by integrating in to some constant λ close to z, while the second integral contains the z upper limit:

$$a_{11} = 2cT\left[\int_{z_i}^{\lambda} + \int_{\lambda}^{z}\right] d\zeta \left(\frac{\zeta}{z_i}\right)^{-2c}\left(\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)}\right)^2 \frac{1}{-\zeta}. \tag{5.20}$$

It is only necessary to work to leading order in z. We need the following expressions. When only $z \approx 0$, we have the z dependence in the integrands as

$$\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)} = \cos\zeta - \sin\zeta/\zeta + O(z^2) \equiv f_1(\zeta) + O(z^2),$$

$$\frac{\mathrm{Im}\,[X(\zeta)]}{\mathrm{Im}X(z)} \simeq z(\cos\zeta/\zeta + \sin\zeta) \equiv z f_2(\zeta), \tag{5.21}$$

while if both $z, \zeta \approx 0$, then to leading order

$$\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)} \simeq (-\zeta^2 + z^3/\zeta)/3, \quad \frac{\mathrm{Im}\,[X(\zeta)]}{\mathrm{Im}X(z)} \simeq z/\zeta. \tag{5.22}$$

We are now in a position to write

$$a_{11} \propto cT\left[\int_{z_i}^{\lambda} d\zeta |\zeta|^{-2c-1} f_1^2(\zeta) + \int_{\lambda}^{z} d\zeta |\zeta|^{-2c-1}(-\zeta^2 + z^3/\zeta)^2/9\right]$$
$$= cT(O(1) + O|z|^{-2c+5}) = cTO(1), \tag{5.23}$$

since we have taken c to be small. A similar approach gives the following results for a_{12}, a_{22} (details can be found in Appendix C):

$$a_{12} = cTO|z|^{c+1}, \quad a_{22} = cTO(1). \tag{5.24}$$

Since T is large, a_{11} dominates D while a_{22} dominates A; so we have

$$A \to a_{22}, \quad B \to -b_1/2, \quad C \to \frac{b_2^2}{16a_{11}}. \tag{5.25}$$

These, of course, have the same form as for the static oscillator case, although it is by no means clear whether such a fact could have been deduced from the general expressions for the a_{ij}'s. We now have

$$S_{\mathrm{lin}} \to \frac{-|b_2|}{4\sqrt{a_{11}a_{22}}} = O|z|^{1-c}, \tag{5.26}$$

and using Eqs. (3.7) and (5.14) we can write

$$S \to (1-c)r + \mathrm{const.} \tag{5.27}$$

Finite temperature. Here we leave the frequency integration until last as was done for the static oscillator. The integrals can then be done in the same way as in the last section, although some subtleties are present in this case (see Appendix C). We finally obtain

$$a_{11} = ckO(1), \quad a_{12} = ckO|z|^{1/2}, \quad a_{22} = ckO(z). \tag{5.28}$$

Again, since we integrate over $\hat{\omega}$, a_{11} will be large and so dominate D, leading to

$$A \to a_{22} - \frac{a_{12}^2}{4a_{11}}, \quad B \to -b_1/2, \quad C \to \frac{b_2^2}{16a_{11}}, \tag{5.29}$$

and so

$$S_{\mathrm{lin}} \to O|z|^{1/2-c}. \tag{5.30}$$

Then, with Eqs. (3.7) and (5.14) we have

$$S \rightarrow (1/2 - c) r + \text{const.} \tag{5.31}$$

VI. DISCUSSION

In the last two sections we calculated the entropy of two physical and exactly solvable squeezed systems: an inverted harmonic oscillator and a scalar field mode evolving in a de Sitter inflationary universe. Our aim was to compare these results, based on our rigorous quantum open-system framework, with that of the previous more *ad hoc* approaches described in the introduction. We must bear in mind that these previous results referred to a field mode that could be split into two independent sine and cosine (standing wave) components. We should, therefore, expect a result of $S = r$ (rather than $2r$) if we are to compare with previous work.

For the inverted oscillator, in both temperature regimes with low coupling, we obtained $S \rightarrow r + \text{const}$. In the de Sitter case, the high-temperature result is $S \rightarrow (1-c) r + \text{const}$. These three examples certainly do confirm the *ad hoc* approaches to calculating entropy that have been used by others. However, at lower temperatures the de Sitter entropy is $S \rightarrow (1/2 - c) r + \text{const}$. This last result requires us to look more closely at A and C which together give the entropy.

From Eqs. (3.6) and (3.7), and neglecting the added constants which are always implied, we find that in the high squeezing limit the entropy behaves as

$$S \rightarrow \frac{1}{2} \ln A - \frac{1}{2} \ln C.$$

When the system-environment coupling is small, all of the above cases give $-1/2 \ln C \rightarrow r$, which is the expected result. The dominant contribution to C always comes from b_2 in the high squeezing limit. This parameter is determined by the squeezing of the system and is essentially independent of the nature of the environment and its coupling to the system. We can, therefore, conclude that the $\ln C$ contribution to the entropy represents entropy intrinsic to the squeezed system itself. This is in agreement with the previous results and should also be true quite generally for squeezed systems. However, these results cannot but fail to take into account the contributions to the entropy from the $\ln A$ term. This contribution is determined by the a_{ij} factors which strongly depend on the nature of the environment and its coupling to the system. There is, *a priori*, no reason to expect this contribution to be small, a point illustrated by our finite-temperature de Sitter example for which we found $1/2 \ln A \rightarrow -r/2$. This highlights the danger in using the previous *ad hoc* approaches to entropy of squeezed systems. The critical point is that the entropy of a system depends not only on the system itself but also on the nature of the environment it is coupled to.

In conclusion, approaching the problem of entropy and uncertainty from the open-system viewpoint as we have demonstrated improves on the earlier work in that it makes explicit how their dependence on the coarse graining of the environment and on the system-environment couplings. It also clarifies the relation between quantum and classical descriptions, it is through decoherence that the quantum field becomes classical [37,44]. These issues are important as they rest at the foundation of statistical and quantum mechanics. (For a discussion of the deeper meaning of the dependence of persistent structures on coarse graining, see [17].)

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APPENDIX A: INFLUENCE FUNCTIONAL THEORY

1. Propagator for the density matrix

The primary object we wish to consider is the evolution of the reduced density matrix of our system via the Feynman-Vernon influence functional method. Since this has been discussed at length in [22] we will just state the main results without showing much derivation.

Consider our system described by x which interacts with its environment q through some interaction. The combined action is

$$S[x,q] = S[x] + S_E[q] + S_{\text{int}}[x,q]. \tag{A1}$$

We require the reduced density matrix of the system at time t. This is found by tracing out the environment:

$$\rho_r(x x' t) = \int_{-\infty}^{\infty} dq \, \rho(xq, x'q, t). \tag{A2}$$

The full density matrix $\rho(xq, x'q, t)$ evolves unitarily. Suppose, we expand it using completeness relations and then path integrals:

$$\begin{aligned}
\rho(xq,x'q,t) &= \langle xq,t|\rho|x'q,t\rangle \\
&= \int dx_i dq_i \int dx_i' dq_i' \langle xq,t|x_i q_i,\ 0\rangle \\
&\quad \times \langle x_i q_i\ 0|\rho|x_i' q_i' 0\rangle \langle x_i' q_i',0|x'q,t\rangle \\
&= \int dx_i dq_i \int dx_i' dq_i' \int_{x_i}^{x} Dx \int_{q_i}^{q} Dq\, e^{iS[x,q]} \\
&\quad \times \rho(x_i q_i, x_i' q_i', 0) \int_{x_i'}^{x'} Dx' \int_{q_i'}^{q} Dq'\, e^{-iS[x',q']} \\
&\equiv \int dx_i dq_i \int dx_i' dq_i' J(xq,x'q,t|x_i q_i, x_i' q_i',0) \\
&\quad \times \rho(x_i q_i, x_i' q_i', 0),
\end{aligned} \tag{A3}$$

where J is seen to be an evolution operator for the entire system plus bath. We make the assumption that the system and bath are initially uncorrelated, i.e.,

$$\rho(x_i q_i, x_i' q_i', 0) = \rho_{\text{sys}}(x_i x_i', 0) \rho_E(q_i q_i', 0). \tag{A4}$$

In this case we are able to rearrange the order of integration to write the reduced density matrix in the following way:

$$\rho_r(xx',t)=\int dx_i dx_i' J_r(xx',t|x_i x_i',0)\rho_{\rm sys}(x_i x_i',0), \tag{A5}$$

where the evolution operator for the reduced density matrix is defined by

$$J_r(xx't|x_i x_i' 0)\equiv\int_{x_i}^{x}Dx\int_{x_i'}^{x'}Dx' e^{iS[x]-iS[x']}F[x,x'], \tag{A6}$$

and $F[x,x']$ is the so-called influence functional:

$$F[x,x']=\int dq\,dq_i\,dq_i'\,\rho_E(q_i q_i',0)\int_{q_i}^{q}Dq\,e^{iS_E[q]+iS_{\rm int}[x,q]}\times\int_{q_i'}^{q}Dq'\,e^{-iS_E[q']-iS_{\rm int}[x',q']}. \tag{A7}$$

We can also write the influence functional in a basis-independent form as follows. First, we write the path integrals as propagators

$$F[x,x']=\int dq\,dq_i\,dq_i'\,\rho_E(q_i q_i',0)\langle q|U(t)|q_i\rangle\times\langle q_i'|U'^{\dagger}(t)|q\rangle, \tag{A8}$$

where $U(t),U'(t)$ are the propagators for $S_E[q]+S_{\rm int}[x,q]$ and $S_E[q]+S_{\rm int}[x',q]$, respectively. Then upon integrating over q,q_i and writing the remaining integral as a trace, we obtain

$$F[x,x']=\ {\rm tr}U(t)\rho_E(0)U'^{\dagger}(t). \tag{A9}$$

Using this form to calculate the influence functional was done earlier in [22]. Here we just list the result: if we use the sum and difference coordinates defined by

$$\Sigma\equiv(x+x')/2,\quad \Delta\equiv x-x', \tag{A10}$$

then the influence functional can be written in terms of two new quantities, the "dissipation" $\mu(s,s')$ and "noise" $\nu(s,s')$:

$$F[x,x']=\exp\frac{-1}{\hbar}\int_0^t ds\int_0^s ds'\,\Delta(s)[\nu(s,s')\Delta(s')+i\mu(s,s')2\Sigma(s')]. \tag{A11}$$

Thus the influence of the environment is completely invested in the dissipation and noise.

Using the sum and difference coordinates defined in Eq. (A10), the classical paths followed by the system, $\Sigma_{\rm cl},\Delta_{\rm cl}$, can be written in terms of more elementary functions u,v:

$$\Sigma_{\rm cl}(s)=\Sigma_{\rm cl}(t_i)u_1(s)+\Sigma_{\rm cl}(t)u_2(s),$$
$$\Delta_{\rm cl}(s)=\Delta_{\rm cl}(t_i)v_1(s)+\Delta_{\rm cl}(t)v_2(s). \tag{A12}$$

It can be shown that the superpropagator J_r is equal to

$$J_r(x,x',t|x_i,x_i',t_i)=\frac{|b_2|}{2\pi\hbar}\exp\left[\frac{i}{\hbar}(b_1\Sigma\Delta-b_2\Sigma\Delta_i+b_3\Sigma_i\Delta-b_4\Sigma_i\Delta_i)-\frac{1}{\hbar}(a_{11}\Delta_i^2+a_{12}\Delta_i\Delta+a_{22}\Delta^2)\right]. \tag{A13}$$

The functions $b_1\rightarrow b_4$ can be expressed as

$$b_1(t,t_i)=M(t)\dot{u}_2(t)+M(t)\mathcal{E}(t),$$
$$b_2(t,t_i)=M(t_i)\dot{u}_2(t_i),$$
$$b_3(t,t_i)=M(t)\dot{u}_1(t),$$
$$b_4(t,t_i)=M(t_i)\dot{u}_1(t_i)+M(t_i)\mathcal{E}(t_i), \tag{A14}$$

while the functions a_{ij} are defined by

$$a_{ij}(t,t_i)=\frac{1}{1+\delta_{ij}}\int_{t_i}^{t}ds\int_{t_i}^{t}ds'\,v_i(s)\,\nu(s,s')v_j(s'). \tag{A15}$$

The functions $u_1\rightarrow v_2$ are solutions to the following equations (dropping subscripts on u,v):

$$\ddot{u}(s)+\frac{\dot{M}}{M}\dot{u}+\left(\Omega^2+\dot{\mathcal{E}}+\frac{\dot{M}}{M}\mathcal{E}\right)u+\frac{2}{M(s)}\int_{t_i}^{s}ds'\,\mu(s,s')u(s')=0, \tag{A16}$$

$$\ddot{v}(s)+\frac{\dot{M}}{M}\dot{v}+\left(\Omega^2+\dot{\mathcal{E}}+\frac{\dot{M}}{M}\mathcal{E}\right)v-\frac{2}{M(s)}\int_{s}^{t}ds'\,\mu(s,s')v(s')=0, \tag{A17}$$

subject to the boundary conditions

$$u_1(t_i)=v_1(t_i)=1,\quad u_1(t)=v_1(t)=0,$$
$$u_2(t_i)=v_2(t_i)=0,\quad u_2(t)=v_2(t)=1. \tag{A18}$$

2. Propagator J_r for an Ohmic environment

To proceed further we need explicit expressions for $a_{11}\rightarrow b_4$. These are expressed in terms of $u_1\rightarrow v_2$, which in turn come from solving Eqs. (A16) and (A17). To solve these equations we need to know the dissipation μ of the environment, which is determined by the coupling and the spectral density function of the environment. We consider an Ohmic bath with a spectral function of the form (2.23) in the following derivation.

a. Calculating $u_1 \rightarrow v_2$

First, consider Eq. (A16). We treat the integral of a δ function and its derivative in the following way. Use a smooth step function [i.e., $\theta(0)\equiv 1/2$] to write ($x_1 > x_0$)

$$\int_{x_0}^{x_1} f(x)\,\delta(x-a)\,dx \equiv f(a)\,\theta(x_1-a)\,\theta(a-x_0), \quad \text{(A19)}$$

$$\int_{x_0}^{x_1} f(x)\,\delta'(x-a)\,dx \equiv -f'(a)\,\theta(x_1-a)\,\theta(a-x_0). \quad \text{(A20)}$$

These relations can easily be proved by checking the five cases individually, of $a<x_0$, $a=x_0$, $x_0<a<x_1$, etc. Note that treating the δ function in this ''smoothed'' way eliminates the need for the frequency renormalization in [41]. This smoothing essentially just defines $\int_0^\infty \delta(x)dx = 1/2$ (see, e.g., [43] for a discussion of this).

Hence Eq. (A16) together with Eq. (2.24) becomes (with u being either u_1 or u_2)

$$\ddot{u}(s) + \left(\frac{\dot{M}}{M} + \frac{2\gamma_0 c^2}{M}\right)\dot{u} + \left(\Omega^2 + \frac{\dot{M}\mathcal{E}}{M} + \dot{\mathcal{E}} + \frac{2\gamma_0 c\dot{c}}{M}\right)u = 0. \quad \text{(A21)}$$

Now define $\tilde{u}$ by

$$\tilde{u} \equiv u \exp\left[\gamma_0 \int_{t_i}^{s} \frac{c^2(s')}{M(s')} ds'\right], \quad \text{(A22)}$$

in which case it follows that

$$\ddot{\tilde{u}} + \frac{\dot{M}}{M}\dot{\tilde{u}} + \left(\Omega^2 + \frac{\dot{M}\mathcal{E}}{M} + \dot{\mathcal{E}} - \frac{\gamma_0^2 c^4}{M^2}\right)\tilde{u} = 0. \quad \text{(A23)}$$

Comparing with Eq. (2.11), we recognize this as just the equation of motion of an oscillator with mass M, cross term $\mathcal{E}$, and an effective frequency

$$\Omega_{\text{eff}}^2 \equiv \Omega^2 - \frac{\gamma_0^2 c^4}{M^2}. \quad \text{(A24)}$$

So, we are in a position to describe our system in terms of an equivalent system. Hence, we know a solution for $\tilde{u}(s)$, it is the sum X of the Bogoliubov coefficients for this new system. So we write (with g_1, g_2 constants to be determined)

$$u(s) = \exp\left[-\gamma_0 \int_{t_i}^{s} \frac{c^2}{M} ds'\right][g_1 X(s) + g_2 X^*(s)]. \quad \text{(A25)}$$

By including the boundary conditions for u_1 and u_2, we obtain

$$u_1(s) = \exp\left[-\gamma_0 \int_{t_i}^{s} \frac{c^2}{M} ds'\right] \frac{\text{Im}\,[X(t)X^*(s)]}{\text{Im}X(t)},$$

$$u_2(s) = \exp\left[\gamma_0 \int_{s}^{t} \frac{c^2}{M} ds'\right] \frac{\text{Im}X(s)}{\text{Im}X(t)}. \quad \text{(A26)}$$

This tying in of the propagator formalism to the language of squeezed states (such as Bogoliubov coefficients) will be very useful for relating the entropy of a field mode to its squeeze parameter r.

In the same way that we solved Eq. (A16), Eq. (A17) becomes

$$\ddot{v}(s) + \left(\frac{\dot{M}}{M} - \frac{2\gamma_0 c^2}{M}\right)\dot{v} + \left(\Omega^2 + \frac{\dot{M}\mathcal{E}}{M} + \dot{\mathcal{E}} - \frac{2\gamma_0 c\dot{c}}{M}\right)v = 0. \quad \text{(A27)}$$

Now write

$$\tilde{v} \equiv v \exp\left[-\gamma_0 \int_{t_i}^{s} \frac{c^2}{M} ds'\right], \quad \text{(A28)}$$

and, just as for the case of u, we have

$$\ddot{\tilde{v}} + \frac{\dot{M}}{M}\dot{\tilde{v}} + \left(\Omega^2 + \frac{\dot{M}\mathcal{E}}{M} + \dot{\mathcal{E}} - \frac{\gamma_0^2 c^4}{M^2}\right)\tilde{v} = 0. \quad \text{(A29)}$$

So, now v_1 and v_2 can also be written as combinations of X and X^*. Including the boundary conditions, we eventually obtain

$$v_1(s) = \exp\left[\gamma_0 \int_{t_i}^{s} \frac{c^2}{M} ds'\right] \frac{\text{Im}\,[X(t)X^*(s)]}{\text{Im}X(t)},$$

$$v_2(s) = \exp\left[-\gamma_0 \int_{s}^{t} \frac{c^2}{M} ds'\right] \frac{\text{Im}X(s)}{\text{Im}X(t)}. \quad \text{(A30)}$$

b. Calculating $a_{11} \rightarrow b_4$

To facilitate our calculations we introduce dimensionless parameters for time

$$z \equiv \kappa t, \quad \zeta \equiv \kappa s,$$

$$X(z) \equiv X(t), \quad \text{etc.} \quad \text{(A31)}$$

and a carat will denote division by κ, e.g., $\hat{\gamma}_0 = \gamma_0/\kappa$. Note that t is the Lagrangian time, which is not necessarily cosmic.

Now, we are able to calculate the propagator. Making use of Eqs. (A15) and (A14), we obtain

$$a_{11}(z,z_i)=\frac{1}{2\kappa^2}\int_{z_i}^{z}d\zeta\int_{z_i}^{z}d\zeta'\exp\left(\hat{\gamma}_0\int_{z_i}^{\zeta}\frac{c^2}{M}d\zeta''\right)\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)}\,\nu(\zeta,\zeta')\exp\left(\hat{\gamma}_0\int_{z_i}^{\zeta'}\frac{c^2}{M}d\zeta''\right)\frac{\mathrm{Im}[X(z)X^*(\zeta')]}{\mathrm{Im}X(z)},$$

$$a_{12}=\frac{1}{\kappa^2}\int_{z_i}^{z}d\zeta\int_{z_i}^{z}d\zeta'\exp\left(\hat{\gamma}_0\int_{z_i}^{\zeta}\frac{c^2}{M}d\zeta''\right)\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)}\,\nu(\zeta,\zeta')\exp\left(-\hat{\gamma}_0\int_{\zeta'}^{z}\frac{c^2}{M}d\zeta''\right)\frac{\mathrm{Im}X(\zeta')}{\mathrm{Im}X(z)},$$

$$a_{22}=\frac{1}{2\kappa^2}\int_{z_i}^{z}d\zeta\int_{z_i}^{z}d\zeta'\exp\left(-\hat{\gamma}_0\int_{\zeta}^{z}\frac{c^2}{M}d\zeta''\right)\frac{\mathrm{Im}X(\zeta)}{\mathrm{Im}X(z)}\,\nu(\zeta,\zeta')\exp\left(-\hat{\gamma}_0\int_{\zeta'}^{z}\frac{c^2}{M}d\zeta''\right)\frac{\mathrm{Im}X(\zeta')}{\mathrm{Im}X(z)},$$

$$b_1(z,z_i)=-\hat{\gamma}_0\kappa c^2(z)+\kappa M(z)\frac{\mathrm{Im}X'(z)}{\mathrm{Im}X(z)}+M(z)\mathcal{E}(z),$$

$$b_{\{^2_3\}}=\frac{\mp\kappa\exp\left(\pm\hat{\gamma}_0\int_{z_i}^{z}\frac{c^2}{M}d\zeta\right)}{\mathrm{Im}X(z)},$$

$$b_4=-\hat{\gamma}_0\kappa c^2(z_i)+\kappa\frac{\mathrm{Re}\,X(z)}{\mathrm{Im}X(z)}+M(z_i)\mathcal{E}(z_i). \tag{A32}$$

APPENDIX B: ENTROPY OF A STATIC OSCILLATOR IN A THERMAL BATH

The Lagrangian for the static oscillator with unit mass is given by

$$L=\frac{1}{2}(\dot{x}^2-k^2x^2). \tag{B1}$$

From Eq. (A24) with $M=c=1$, the effective frequency is

$$\Omega_{\mathrm{eff}}^2=k^2-\gamma_0^2\equiv\kappa^2. \tag{B2}$$

Then the equation of motion for X is, from Eq. (2.11) with $\Omega\rightarrow\Omega_{\mathrm{eff}}$,

$$\ddot{X}+\kappa^2X=0, \tag{B3}$$

$$X(0)=1,\quad \dot{X}(0)=-i\kappa, \tag{B4}$$

which leads to

$$X(z)=e^{-iz}, \tag{B5}$$

with $z=\kappa t$. Then,

$$\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)}=\frac{\sin(z-\zeta)}{\sin z},\quad \frac{\mathrm{Im}X(\zeta)}{\mathrm{Im}X(z)}=\frac{\sin\zeta}{\sin z}, \tag{B6}$$

$$\exp\left(\hat{\gamma}_0\int_{z_i}^{\zeta}\frac{c^2}{M}d\zeta''\right)=e^{\hat{\gamma}_0\zeta},$$

$$\exp\left(-\hat{\gamma}_0\int_{\zeta}^{z}\frac{c^2}{M}d\zeta''\right)=e^{-\hat{\gamma}_0(z-\zeta)}, \tag{B7}$$

with noise for $T\rightarrow\infty$ being white:

$$\nu(\zeta,\zeta')=4\kappa\gamma_0T\delta(\zeta-\zeta'). \tag{B8}$$

Then, $a_{11}\rightarrow b_4$ follow:

$$a_{11}=\frac{T}{\sin^2z}\,\frac{e^{2\hat{\gamma}_0z}-1-\hat{\gamma}_0\sin2z-\hat{\gamma}_0^2(1-\cos2z)}{2(1+\hat{\gamma}_0^2)},$$

$$a_{12}=\frac{2T}{\sin^2z}\,\frac{-\cos z\sinh\hat{\gamma}_0z+\hat{\gamma}_0\sin z\cosh\hat{\gamma}_0z}{1+\hat{\gamma}_0^2},$$

$$a_{22}=\frac{T}{\sin^2z}\,\frac{-e^{-2\hat{\gamma}_0z}+1-\hat{\gamma}_0\sin2z+\hat{\gamma}_0^2(1-\cos2z)}{2(1+\hat{\gamma}_0^2)},$$

$$b_{\{^1_4\}}=\kappa(-\hat{\gamma}_0\pm\cot z),\quad b_{\{^2_3\}}=\frac{\pm\kappa e^{\pm\hat{\gamma}_0z}}{\sin z}. \tag{B9}$$

To evaluate S, we need A and C; in turn for these we need $a_{11}\rightarrow b_4$. These are calculated from Eq. (A32).[3]

The oscillator is assumed to be initially in its ground state

$$\psi(x,0)\propto\exp\left(\frac{-x^2}{4\sigma^2}\right), \tag{B10}$$

so that its density matrix is

$$\rho(xx',0)\propto\exp\left(\frac{-x^2-x'^2}{4\sigma^2}\right), \tag{B11}$$

and in Eq. (3.1), we have

---

[3]Various notations exist describing these results; see, for example, [14,26,24]. To compare with [24], Eq. (2.2.7) is a matter of carefully transcribing the notation; key things to note are that $X\equiv\Sigma$, $Y\equiv-\Delta$; here we have taken $x_0=p_0=0$; [24], Eq. (2.2.6c) should have an a_{11} in place of the a_{22}; the b_i's in [22] are written explicitly in [24] via [22], Eq. (3.11); [22], a_{12} equals [24], $a_{12}+a_{21}$; [22], γ_0 equals [24], $\gamma_0/2$.

$$\xi = \frac{1}{4\sigma^2}, \quad \chi = 0. \tag{B12}$$

The reduced density matrix evolves into Eq. (3.2), with

$$A = a_{22} + \frac{1}{D}\left\{\left[\frac{1}{8\sigma^2} + a_{11}\right] b_3^2 + a_{12} b_3 b_4 - \frac{a_{12}^2}{2\sigma^2}\right\},$$

$$B = \frac{-b_1}{2} + \frac{b_2}{2D}\left\{b_3 b_4 - \frac{a_{12}}{\sigma^2}\right\},$$

$$C = \frac{b_2^2}{8D\sigma^2},$$

$$D = \frac{1}{4\sigma^4} + \frac{2a_{11}}{\sigma^2} + b_4^2. \tag{B13}$$

It is by no means trivial to show that the entropy calculated using these expressions does indeed tend toward Eq. (4.1), and in particular the cscz terms in the a_{ij}'s and b_i's mean their values can diverge depending on the time. But this divergence cancels out when physical quantities are measured, as we can see by verifying numerically that our entropy really does tend toward the usual asymptotic value at late times (Fig. 1).

APPENDIX C: CALCULATION OF a_{ij}'S IN SEC. V

1. de Sitter with high temperature

Here we evaluate the a_{ij}'s leading to Eq. (5.24). We are using the following small z,ζ approximations:

$$\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)} \xrightarrow{z\to 0} \cos\zeta - \sin\zeta/\zeta + O(z^2) \equiv f_1(\zeta) + O(z^2) \xrightarrow{z,\zeta\to 0} (-\zeta^2 + z^3/\zeta)/3,$$

$$\frac{\mathrm{Im}X(\zeta)}{\mathrm{Im}X(z)} \xrightarrow{z\to 0} z(\cos\zeta/\zeta + \sin\zeta) \equiv z f_2(\zeta) \xrightarrow{z,\zeta\to 0} z/\zeta. \tag{C1}$$

First,

$$a_{12} = \frac{1}{k^2}\int_{z_i}^{z} d\zeta \int_{z_i}^{z} d\zeta' \left(\frac{\zeta}{z_i}\right)^{-c} \frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)} \frac{4ck^2T}{-\zeta}\,\delta(\zeta-\zeta')\left(\frac{z}{\zeta'}\right)^{c} \frac{\mathrm{Im}X(\zeta')}{\mathrm{Im}X(z)}$$
$$= 4cT\int_{z_i}^{z} d\zeta \left(\frac{\zeta}{z_i}\right)^{-c} \frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)} \frac{1}{-\zeta}\left(\frac{z}{\zeta}\right)^{c} \frac{\mathrm{Im}X(\zeta)}{\mathrm{Im}X(z)}$$
$$\propto cT|z|^c\left[\int_{z_i}^{\lambda} d\zeta |\zeta|^{-2c-1} f_1(\zeta) z f_2(\zeta) + \int_{\lambda}^{z} d\zeta |\zeta|^{-2c-1}(-\zeta^2 + z^3/\zeta)\frac{z}{3\zeta}\right] = cTO|z|^{c+1}, \tag{C2}$$

provided $c<1/2$. Finally,

$$a_{22} = \frac{1}{2k^2}\int_{z_i}^{z} d\zeta \int_{z_i}^{z} d\zeta' \left(\frac{z}{\zeta}\right)^{c} \frac{\mathrm{Im}X(\zeta)}{\mathrm{Im}X(z)} \frac{4ck^2T}{-\zeta}\,\delta(\zeta-\zeta')\left(\frac{z}{\zeta'}\right)^{c} \frac{\mathrm{Im}X(\zeta')}{\mathrm{Im}X(z)} = 2cT\int_{z_i}^{z} d\zeta \left(\frac{z}{\zeta}\right)^{2c}\left(\frac{\mathrm{Im}X(\zeta)}{\mathrm{Im}X(z)}\right)^2 \frac{1}{-\zeta}$$
$$\propto cT|z|^{2c}\left[\int_{z_i}^{\lambda} d\zeta |\zeta|^{-2c-1} f_2^2(\zeta) z^2 + \int_{\lambda}^{z} d\zeta |\zeta|^{-2c-1} z^2/\zeta^2\right] = cTO(1). \tag{C3}$$

2. de Sitter with finite temperature

We leave the frequency integration until last:

$$\nu = \frac{2c}{\pi}\frac{1}{\sqrt{ss'}}\int_0^\infty \omega\coth\frac{\omega}{2T}\cos\omega(s-s')\,d\omega = \frac{2c}{\pi}\frac{k^3}{\sqrt{\zeta\zeta'}}\int_0^\infty d\hat{\omega}\,\hat{\omega}\coth\frac{\hat{\omega}}{2T}\cos\hat{\omega}(\zeta-\zeta'). \tag{C4}$$

The a_{ij}'s are

$$a_{11}=z_i^{2c}\frac{ck}{\pi}\int_0^\infty d\hat{\omega}\,\hat{\omega}\coth\frac{\hat{\omega}k}{2T}\underbrace{\int_{z_i}^{z}d\zeta\int_{z_i}^{z}d\zeta'|\zeta|^{-c-1/2}\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)}\cos\hat{\omega}(\zeta-\zeta')|\zeta'|^{-c-1/2}\frac{\mathrm{Im}\,[X(z)X^*(\zeta')]}{\mathrm{Im}X(z)}}_{\equiv I_{11}},$$

$$a_{12}=(z_iz)^c\frac{2ck}{\pi}\int_0^\infty d\hat{\omega}\,\hat{\omega}\coth\frac{\hat{\omega}k}{2T}\underbrace{\int_{z_i}^{z}d\zeta\int_{z_i}^{z}d\zeta'|\zeta|^{-c-1/2}\frac{\mathrm{Im}\,[X(z)X^*(\zeta)]}{\mathrm{Im}X(z)}\cos\hat{\omega}(\zeta-\zeta')|\zeta'|^{-c-1/2}\frac{\mathrm{Im}X(\zeta')}{\mathrm{Im}X(z)}}_{\equiv I_{12}},$$

$$a_{22}=z^{2c}\frac{ck}{\pi}\int_0^\infty d\hat{\omega}\,\hat{\omega}\coth\frac{\hat{\omega}k}{2T}\underbrace{\int_{z_i}^{z}d\zeta\int_{z_i}^{z}d\zeta'|\zeta|^{-c-1/2}\frac{\mathrm{Im}X(\zeta)}{\mathrm{Im}X(z)}\cos\hat{\omega}(\zeta-\zeta')|\zeta'|^{-c-1/2}\frac{\mathrm{Im}X(\zeta')}{\mathrm{Im}X(z)}}_{\equiv I_{22}}. \tag{C5}$$

Using the expressions from Eqs. (C1) and (5.17), the first of the inner integrals becomes

$$\begin{aligned}I_{11}=&\int_{z_i}^{\lambda}d\zeta|\zeta|^{-c-1/2}f_1(\zeta)\Bigg[\int_{z_i}^{\lambda}d\zeta'\cos\hat{\omega}(\zeta-\zeta')\\&\times|\zeta'|^{-c-1/2}f_1(\zeta')+\int_{\lambda}^{z}d\zeta'\cos\hat{\omega}\zeta|\zeta'|^{-c-1/2}\\&\times(-\zeta'^2+z^3/\zeta')/3\Bigg]+\int_{\lambda}^{z}d\zeta|\zeta|^{-c-1/2}(-\zeta^2+z^3/\zeta)/3\\&\times\Bigg[\int_{z_i}^{\lambda}d\zeta'\cos\hat{\omega}\zeta'|\zeta'|^{-c-1/2}f_1(\zeta')\\&+\int_{\lambda}^{z}d\zeta'\cos\hat{\omega}(\zeta-\zeta')|\zeta'|^{-c-1/2}(-\zeta'^2+z^3/\zeta'/3)\Bigg].\end{aligned} \tag{C6}$$

We now have a difficulty. In order to get a reasonably useful analytic result, it will be an advantage to replace the $\cos\hat{\omega}(\zeta-\zeta')$ term in the fourth integral above by something simpler. We will have competition between $\hat{\omega}$ increasing in the frequency integral versus z decreasing in time. Suppose then, we use a frequency cutoff $\omega_{\max}$. In that case we can approximate $\cos\hat{\omega}(\zeta-\zeta')$ for $\zeta,\zeta'\approx 0$ by choosing $\hat{\omega}_{\max}$ such that $\cos\hat{\omega}(\zeta-\zeta')\approx 1$ in the fourth integral. This will be true provided

$$\hat{\omega}_{\max}\ll -1/\lambda. \tag{C7}$$

However, now we do not expect our result to necessarily agree with the high T result found in Eq. (5.23), since there we had taken $\hat{\omega}_{\max}\to\infty$, which was made possible by the use of the δ function.

At this point we refer to the discussion of the high-temperature limit in [45]. There it is shown that the high-temperature (δ function) regime is that for which $\omega_{\max}\ll T$ *and* $\omega_{\max}\to\infty$. This absence of a cutoff in the high-temperature limit is usually not stressed, but it forms the most relevant fact here. In general, we must impose a cutoff for all finite T values, otherwise the frequency integral is not well defined, unless $T\to\infty$. So, we conclude that the regime for which our analysis is valid here is $T\gtrsim\omega_{\max}$.

With the last cosine set equal to 1 as before, these integrals are all of order 1 and, therefore, so is a_{11}. Next,

$$\begin{aligned}I_{12}=&\int_{z_i}^{\lambda}d\zeta|\zeta|^{-c-1/2}f_1(\zeta)\left[\int_{z_i}^{\lambda}d\zeta'\cos\hat{\omega}(\zeta-\zeta')|\zeta'|^{-c-1/2}f_2(\zeta')z+\int_{\lambda}^{z}d\zeta'\cos\hat{\omega}\zeta|\zeta'|^{-c-1/2}z/\zeta'\right]\\&+\int_{\lambda}^{z}d\zeta|\zeta|^{-c-1/2}(-\zeta^2+z^3/\zeta)/3\left[\int_{z_i}^{\lambda}d\zeta'\cos\hat{\omega}\zeta'|\zeta'|^{-c-1/2}f_2(\zeta')z+\int_{\lambda}^{z}d\zeta'\cos\hat{\omega}(\zeta-\zeta')|\zeta'|^{-c-1/2}z/\zeta'\right].\end{aligned} \tag{C8}$$

Evaluating these integrals gives $I_{12}=O|z|^{-c+1/2}$ so that $a_{12}=O|z|^{1/2}$. Last,

$$I_{22}=\int_{z_i}^{\lambda}d\zeta|\zeta|^{-c-1/2}f_2(\zeta)z\left[\int_{z_i}^{\lambda}d\zeta'\cos\hat{\omega}(\zeta-\zeta')|\zeta'|^{-c-1/2}f_2(\zeta')z+\int_{\lambda}^{z}d\zeta'\cos\hat{\omega}\zeta|\zeta'|^{-c-1/2}z/\zeta'\right]$$
$$+\int_{\lambda}^{z}d\zeta|\zeta|^{-c-1/2}z/\zeta\left[\int_{z_i}^{\lambda}d\zeta'\cos\hat{\omega}\zeta'|\zeta'|^{-c-1/2}f_2(\zeta')z+\int_{\lambda}^{z}d\zeta'\cos\hat{\omega}(\zeta-\zeta')|\zeta'|^{-c-1/2}z/\zeta'\right]=O|z|^{-2c+1}, \qquad (C9)$$

so that $a_{22}=O(z)$.

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