



An equivariant Atiyah–Patodi–Singer index theorem for proper actions II: the K -theoretic index

Peter Hochs¹ · Bai-Ling Wang² · Hang Wang³

Received: 15 June 2020 / Accepted: 28 November 2021 / Published online: 17 January 2022
© The Author(s) 2022

Abstract

Consider a proper, isometric action by a unimodular locally compact group G on a Riemannian manifold M with boundary, such that M/G is compact. Then an equivariant Dirac-type operator D on M under a suitable boundary condition has an equivariant index $\text{index}_G(D)$ in the K -theory of the reduced group C^* -algebra C_r^*G of G . This is a common generalisation of the Baum–Connes analytic assembly map and the (equivariant) Atiyah–Patodi–Singer index. In part I of this series, a numerical index $\text{index}_g(D)$ was defined for an element $g \in G$, in terms of a parametrix of D and a trace associated to g . An Atiyah–Patodi–Singer type index formula was obtained for this index. In this paper, we show that, under certain conditions,

$$\tau_g(\text{index}_G(D)) = \text{index}_g(D),$$

for a trace τ_g defined by the orbital integral over the conjugacy class of g . This implies that the index theorem from part I yields information about the K -theoretic index $\text{index}_G(D)$. It also shows that $\text{index}_g(D)$ is a homotopy-invariant quantity.

Contents

1	Introduction	1334
	Outline of this paper	1336
2	Preliminaries and results	1336
2.1	The localised equivariant Roe algebra	1337
2.2	The localised equivariant coarse index	1338
2.3	Orbital integrals	1339
2.4	Manifolds with boundary	1340

✉ Peter Hochs
p.hochs@math.ru.nl

Bai-Ling Wang
bai-ling.wang@anu.edu.au

Hang Wang
wanghang@math.ecnu.edu.cn

¹ Radboud University, Nijmegen, The Netherlands
² Australian National University, Canberra, Australia
³ East China Normal University, Shanghai, China

2.5 The main result	1341
3 A parametrix and properties of the g -trace	1344
3.1 A parametrix	1344
3.2 Properties of S_0 and S_1	1345
3.3 Properties of the g -trace	1345
3.4 G -integrable kernels	1346
3.5 Basic estimates for heat operators	1348
4 S_0^2 and S_1^2 are g -trace class	1349
4.1 Convergence of an integral for small t	1349
4.2 Convergence of an integral for large t	1350
4.3 Proof of Proposition 4.1	1354
5 The trace of the index	1356
5.1 An explicit index	1356
5.2 The map $\overline{\text{TR}}$	1357
5.3 Two maps from Roe algebras to $C_r^*G \otimes \mathcal{K}$	1358
5.4 Proof of Proposition 5.6	1360
5.5 Proof of Proposition 5.1	1363
5.6 Proof of Theorem 2.7	1363
6 Non-invertible D_N	1364
6.1 A shifted Dirac operator	1364
6.2 A shifted parametrix	1365
References	1366

1 Introduction

This paper is about a K -theoretic index defined for Dirac operators on manifolds with boundary, equivariant with respect to proper, cocompact actions by locally compact groups. It is a companion paper to part I [23] of this series of two papers, in which numerical indices were defined for such operators, and an index formula was proved for those indices. The main result in this paper is Theorem 2.7, stating that, under certain conditions, numerical invariants extracted from the K -theoretic index via orbital integral traces equal the indices from [23]. In that way, the index formula from [23] applies to the K -theoretic index as well.

Consider a unimodular, locally compact group G acting properly and isometrically on a Riemannian manifold M , with boundary N , such that M/G is compact. Let D be a G -equivariant Dirac-type operator on a G -equivariant, \mathbb{Z}_2 -graded Hermitian vector bundle $E = E_+ \oplus E_- \rightarrow M$. Suppose that all structures have a product form near N . In particular, suppose that near N , the restriction of D to sections of E_+ equals

$$\sigma\left(-\frac{\partial}{\partial u} + D_N\right), \quad (1.1)$$

where $\sigma: E_+|_N \rightarrow E_-|_N$ is an equivariant vector bundle isomorphism, u is the coordinate in $(0, 1]$ in a neighbourhood of N equivariantly isometric to $N \times (0, 1]$, and D_N is a Dirac operator on $E_+|_N$.

We initially assume D_N to be invertible, and later show how to weaken this assumption to 0 being isolated in the spectrum of D_N . If D_N is invertible, then we use the construction of an index

$$\text{index}_G(D) \in K_0(C_r^*G) \quad (1.2)$$

from [18], where C_r^*G is the reduced group C^* -algebra of G . This index was defined in [18] in a more general setting, and applied to, for example, Callias-type operators and positive scalar curvature [19] and the quantisation commutes with reduction problem [20].

To extract relevant numbers from this K -theoretic index, we apply traces defined by *orbital integrals*. Let $g \in G$, let Z_g be its centraliser, and suppose that G/Z_g has a G -invariant measure $d(hZ_g)$. Then the orbital integral with respect to g of a function $f \in C_c(G)$ is the number

$$\tau_g(f) := \int_{G/Z_g} f(hgh^{-1}) d(hZ_g). \tag{1.3}$$

If the integral on the right hand side converges absolutely for all f in a dense subalgebra $\mathcal{A} \subset C_r^*G$, closed under holomorphic functional calculus, then this defines a trace τ_g on \mathcal{A} . That trace induces

$$\tau_g : K_0(C_r^*G) = K_0(\mathcal{A}) \rightarrow \mathbb{C}. \tag{1.4}$$

Orbital integrals for semisimple Lie groups are fundamental to Harish-Chandra’s development of harmonic analysis on such groups. They also play an important role in Bismut’s work on hypo-elliptic Laplacians [6]. The map (1.4) on K -theory is given by evaluating characters at g if G is compact. One can also use (1.4) to recover the values at elliptic elements g of characters of discrete series representation of semisimple groups [24]. This was used to link index theory to representation theory in [24]. Higher cyclic cocycles generalising orbital integrals and capturing all information about classes in $K_*(C_r^*G)$ were developed by Song and Tang [39].

For discrete groups, where they are sums over conjugacy classes, orbital integrals and the map (1.4) have found various applications to geometry and topology in recent years, see for example [26,40,41,43].

Applying (1.4) to (1.2) yields the number

$$\tau_g(\text{index}_G(D)), \tag{1.5}$$

which is the main object of interest in this paper. The index (1.2) and the number (1.5) generalise various earlier indices.

- If $N = \emptyset$, then (1.2) is the image of D under the Baum–Connes analytic assembly map [4], see Corollary 4.3 in [18]. That is the most natural and widely-used generalisation of the classical equivariant index to proper, cocompact actions. It has been applied to various problems in geometry and topology, such as questions about positive scalar curvature and the Novikov conjecture. In this context, the number (1.5) was shown to be relevant to representation theory, orbifold geometry and trace formulas [22,24,25,40].
- If M and G are compact, then (1.2) becomes the equivariant APS index used in [14], and (1.5) is the evaluation of that index at g . (See Lemma 2.9 in [23].) If G is trivial, then this index reduces to the usual APS index.
- In the case where M/G is a compact manifold with boundary, M is its universal cover, and G is its fundamental group, the number $\tau_e(\text{index}_G(D))$ is the index used by Ramachandran in [37], see Remark 2.14. In this setting, the index (1.2) was introduced in Section 3 of [42]. Indices with values in $K_*(C_r^*G)$ in this setting were also defined in [27–30], via operators on Hilbert C_r^*G -modules and in [34] in terms of Roe algebras. We expect these to be special cases of (1.2), because they generalise the case of manifolds without boundary [32], a special case of the Baum–Connes assembly map; see also for example Proposition 2.4 in [34]. (In this context, a K -theoretic index theorem involving Yu’s localisation algebras, and where M/G is not necessarily compact, was obtained by Zeidler. See Theorem 6.5 in [44].)

These special cases suggest that the index (1.2) and the number (1.5) are natural objects to study. They generalise to the case where 0 is isolated in the spectrum of D_N , as discussed in Sect. 6.

In [23], the notion of a *g-Fredholm operator* was introduced. Such operators have a numerical *g-index*, defined in terms of a parametrix of the operator and a trace related to τ_g . It was shown that for several classes of groups and actions, the Dirac operator D on the manifold with boundary M is *g-Fredholm*, and hence has a *g-index*, denoted by $\text{index}_g(D)$. An index formula was proved for this index. In the case where D is a twisted Spin^c -Dirac operator, this index formula takes the form

$$\text{index}_g(D) = \int_{M^g} \chi_g^2 \frac{\hat{A}(M^g) e^{c_1(L|_{M^g})/2} \text{tr}(g e^{-R_V|_{M^g}/2\pi i})}{\det(1 - g e^{-R_N/2\pi i})^{1/2}} - \frac{1}{2} \eta_g(D_N).$$

The first term in the right hand side is a direct generalisation of the right hand side of the Atiyah–Segal–Singer fixed point formula [2,3,5]. The number $\eta_g(D_N)$ is a *delocalised η -invariant*. These were first constructed by Lott [30,31].

The main result in this paper, Theorem 2.7, states that, under certain conditions,

$$\text{index}_g(D) = \tau_g(\text{index}_G(D)).$$

This links the index $\text{index}_g(D)$ to K -theory, and allows us to apply the index formula from [23] to the number (1.5). This generalises the index theorems in [1,14,37], for example. Furthermore, homotopy invariance of index_G implies homotopy invariance of index_g in these cases.

After this paper appeared, Piazza, Posthuma, Song and Tang [35] obtained an index theorem for $\tau_g(\text{index}_G(D))$ in the case where G is a semisimple Lie group. Their result applies even when D_N is not invertible, or 0 is not isolated in its spectrum. They used a construction of the index (1.2) in terms of *b*-calculus.

Outline of this paper

The index (1.2), and the Roe algebras needed to define it, are introduced in Sect. 2. There we also recall the definition of the index index_g from [23], and state the main result, Theorem 2.7.

We prepare for the proof of Theorem 2.7 in Sect. 3, by introducing a parametrix for the operator D , and discussing some properties of the *g*-trace and of heat kernels. Then we prove the two main steps in the proof of Theorem 2.7 in Sects. 4 and 5, Propositions 4.1 and 5.1. Combining these with a last extra step, Proposition 5.12, we obtain a proof of Theorem 2.7. In Sect. 6, we show how to weaken the assumption that the boundary Dirac operator D_N in (1.1) is invertible, to the assumption that 0 is isolated in its spectrum.

2 Preliminaries and results

For a proper, cocompact action by a general locally compact group G , the most widely-used equivariant index of equivariant elliptic operators is the Baum–Connes analytic assembly map [4]. (Here an action is called cocompact if its quotient is compact.) This is a generalisation of the usual equivariant index in the compact case, and takes values in $K_*(C_r^*G)$, the K -theory of the reduced group C^* -algebra of G . In [18], a generalisation of the assembly map was constructed and studied, which applies to possibly non-cocompact actions, as long as the operator it is applied to is invertible outside a cocompact set in the appropriate sense. This

index also generalises the Gromov–Lawson index [16], an equivariant index of Callias-type operators [17], the (equivariant) APS index on manifolds with boundary [1, 13], and the index used by Ramachandran for manifolds with boundary [37]. This index is an equivariant version of the localised coarse index of Roe [38], for actions by arbitrary locally compact groups. For actions by fundamental groups of manifolds on their universal covers, this index was constructed in [42].

We briefly review the construction of the index in [18] in Sect. 2.2, in the case we need here. This involves localised Roe algebras, which we discuss in Sect. 2.1. The index takes values in the K -theory of the reduced C^* -algebra of the group. Using traces on subalgebras of this algebra defined by orbital integrals, defined in Sect. 2.3, we extract numbers from that index. The main result in this paper is Theorem 2.7, which states that, under certain conditions, those numbers equal the numbers for which an index formula was proved in [23].

2.1 The localised equivariant Roe algebra

Let (X, d) be a metric space in which all closed balls are compact. Let G be a locally compact, unimodular group acting properly and isometrically on X . Let $Z \subset X$ be a nonempty, closed, G -invariant subset such that Z/G is compact. Fix a G -invariant Borel measure on X for which every open set has positive measure. Let $E \rightarrow X$ be a G -equivariant Hermitian vector bundle.

The Hilbert space $L^2(E)$ of square-integrable sections of E has a natural unitary representation of G , and an action by $C_0(X)$ given by pointwise multiplication of sections by functions. In this sense, it is a G -equivariant $C_0(X)$ -module. We will not define the various types of such modules here, but always work with concrete examples. Apart from $L^2(E)$, we will also use the module $L^2(E) \otimes L^2(G)$, where G acts diagonally (acting on $L^2(G)$ via the left regular representation), and where $C_0(X)$ acts on the factor $L^2(E)$ via pointwise multiplication. If X/G is compact, then $L^2(E) \otimes L^2(G)$ is an *admissible* equivariant $C_0(X)$ -module, under the non-essential assumption that either X/G or G/K , for a maximal compact subgroup $K < G$, is infinite. See Theorem 2.7 in [18]. This type of $C_0(X)$ -module is central to the constructions in [18].

We denote the algebra of G -equivariant bounded operators on a Hilbert space H with a unitary representation of G by $\mathcal{B}(H)^G$.

Definition 2.1 Let $T \in \mathcal{B}(L^2(E) \otimes L^2(G))$. Then T is *locally compact* if the operators Tf and fT are compact for all $f \in C_0(X)$. The operator T has *finite propagation* if there is a number $r > 0$ such that for all $f_1, f_2 \in C_0(X)$ whose supports are further than r apart, we have $f_1 T f_2 = 0$. Finally, T is *supported near Z* if there is an $r' > 0$ such that for all $f \in C_0(X)$ whose support is further than r' away from Z , the operators Tf and fT are zero.

The *localised equivariant Roe algebra* of X is the closure in $\mathcal{B}(L^2(E) \otimes L^2(G))$ of the algebra of locally compact operators in $\mathcal{B}(L^2(E) \otimes L^2(G))^G$ with finite propagation, supported near Z . It is denoted by $C^*(X, Z)^G$.

The algebra $C^*(X, Z)^G$ is independent of the cocompact set Z . (It is denoted by $C^*(X)_{\text{loc}}^G$ in [18]). And, assuming either Z/G or G/K is an infinite set,

$$C^*(X, Z)^G \cong C_r^*G \otimes \mathcal{K}, \tag{2.1}$$

where C_r^*G is the reduced group C^* -algebra of G , and \mathcal{K} is the algebra of compact operators on a separable, infinite-dimensional Hilbert space. See (5) in [18]. (If Z/G and G/K are both finite, then (2.1) still holds with \mathcal{K} replaced by a matrix algebra.) This equality implies that $C^*(X, Z)^G$ is also independent of E . (In fact, it is independent of the choice of a more general kind of admissible module.)

Remark 2.2 There is no reason a priori to assume that Z/G is compact. The resulting localised Roe algebra will then depend on Z . We always assume that Z/G is compact, so that we have the isomorphism (2.1), and we can apply the traces of Sect. 2.3 to classes in the K -theory of $C^*(X, Z)^G$.

We will also use a version of the localised equivariant Roe algebra defined with respect to the $C_0(X)$ -module $L^2(E)$, instead of $L^2(E) \otimes L^2(G)$. This is defined exactly as in Definition 2.1, with $L^2(E) \otimes L^2(G)$ replaced by $L^2(E)$ everywhere. The resulting algebra is denoted by $C^*(X, Z; L^2(E))^G$. This algebra is less canonical than $C^*(X, Z)^G$, and is not stably isomorphic to C_r^*G in general. If X/G itself is compact, then we omit Z from the notation, since being supported near Z then becomes a vacuous condition.

2.2 The localised equivariant coarse index

Suppose, from now on, that $X = M$ is a complete Riemannian manifold, and E is a smooth, \mathbb{Z}_2 -graded, G -equivariant, Hermitian vector bundle. Let D be an elliptic, odd-graded, essentially self-adjoint, first order differential operator on E . Suppose that

$$D^2 \geq c \tag{2.2}$$

on $M \setminus Z$, for a positive constant c . Let $b \in C(\mathbb{R})$ be an odd function such that $b(x) = 1$ for all $x \geq c$. Lemma 2.3 in [38] states that $b(D)^2 - 1 \in C^*(X, Z; L^2(E))^G$. By Lemma 2.1 in [38], the operator $b(D)$ lies in the multiplier algebra $\mathcal{M}(C^*(X, Z; L^2(E))^G)$ of $C^*(X, Z; L^2(E))^G$. Hence the restriction of $b(D)$ to even-graded sections defines a class

$$[b(D)] \in K_1(\mathcal{M}(C^*(X, Z; L^2(E))^G)/C^*(X, Z; L^2(E))^G).$$

Let

$$\partial: K_1(\mathcal{M}(C^*(X, Z; L^2(E))^G)/C^*(X, Z; L^2(E))^G) \rightarrow K_0(C^*(X, Z; L^2(E))^G)$$

be the boundary map in the six-term exact sequence associated to the ideal $C^*(X, Z; L^2(E))^G$ of $\mathcal{M}(C^*(X, Z; L^2(E))^G)$. We set

$$\text{index}_G^{L^2(E)}(D) := \partial[b(D)] \in K_0(C^*(X, Z; L^2(E))^G). \tag{2.3}$$

To obtain an index in $K_0(C_r^*G)$, let $\chi \in C^\infty(M)$ be a cutoff function, in the sense that it is nonnegative, its support has compact intersections with all G -orbits, and that for all $m \in M$,

$$\int_G \chi(gm)^2 dg = 1. \tag{2.4}$$

The map

$$j: L^2(E) \rightarrow L^2(E) \otimes L^2(G), \tag{2.5}$$

given by

$$(j(s))(m, g) = \chi(g^{-1}m)s(m),$$

for $s \in L^2(E)$, $m \in M$ and $g \in G$, is a G -equivariant, isometric embedding. Let

$$\oplus 0: C^*(X, Z; L^2(E))^G \rightarrow C^*(X, Z)^G \tag{2.6}$$

be given by mapping operators on $L^2(E)$ to operators on $j(L^2(E))$ by conjugation with j , and extending them by zero on the orthogonal complement of $j(L^2(E))$. We denote the map on K -theory induced by $\oplus 0$ be the same symbol.

Definition 2.3 The *localised equivariant coarse index* of D is

$$\text{index}_G(D) := \text{index}_G^{L^2(E)}(D) \oplus 0 \in K_0(C_r^*G).$$

Remark 2.4 In [18], the localised equivariant coarse index is defined slightly differently from Definition 2.3, but also in terms of j . The two definitions agree by (13) in [18]. In that paper, a version for ungraded vector bundles, with values in odd K -theory, is also defined. An illustration of how (representation theoretic) information that may not be encoded by $\text{index}_G^{L^2(E)}(D)$ is recovered through the map $\oplus 0$ is Example 3.8 in [18].

The index of Definition 2.3 simultaneously generalises various other indices; some are mentioned in the introduction. For example, if M/G is compact, then it reduces to the analytic assembly map from the Baum–Connes conjecture [4]. See Sect. 3.5 in [18] for other special cases. In this paper, we apply the index to manifolds with boundary, to generalise the APS index and its generalisations in [1,13,37].

2.3 Orbital integrals

Fix an element $g \in G$. Let $Z_g < G$ be its centraliser. Suppose that G/Z_g has a G -invariant measure $d(hZ_g)$ such that for all $f \in C_c(G)$,

$$\int_G f(h) dh = \int_{G/Z_g} \int_{Z_g} f(hz) dz d(hZ_g),$$

for fixed Haar measures dh on G and dz on Z_g . (This is the case, for example, if G is discrete, or if G is real semisimple and g is a semisimple element).

The *orbital integral* of a function $f \in C_c(G)$ is

$$\tau_g(f) := \int_{G/Z_g} f(hgh^{-1}) d(hZ_g).$$

We assume that there is a dense subalgebra $\mathcal{A} \subset C_r^*G$, closed under holomorphic functional calculus, such that τ_g extends to a continuous linear functional on \mathcal{A} . Then it defines a trace on \mathcal{A} . Existence of \mathcal{A} is a nontrivial question. For semisimple Lie groups, such an algebra was constructed by Harish-Chandra, see Theorem 6 in [8]. For discrete groups, there are constructions by Connes–Moscovici for groups with polynomial growth (see Lemma 6.4 in [12]), and by Puschnigg [36] for word hyperbolic groups. See also [26] for conjugacy classes with polynomial growth in discrete groups.

The trace τ_g on \mathcal{A} defines a map

$$\tau_g : K_0(C_r^*G) = K_0(\mathcal{A}) \rightarrow \mathbb{C}.$$

Consider the setting of Sect. 2.2. Then we have the number

$$\tau_g(\text{index}_G(D)).$$

In part I [23], we used a trace related to τ_g to define the notion of a g -Fredholm operator, and the g -index of such operators. We briefly recall the definitions here.

Let $\chi \in C^\infty(M)$ be a cutoff function for the action, as in (2.4). Consider the bundle

$$\text{End}(E) := E \boxtimes E^* \rightarrow M \times M.$$

Definition 2.5 A section $\kappa \in \Gamma^\infty(\text{End}(E))^G$ is *g-trace class* if the integral

$$\int_{G/Z_g} \int_M \chi(hgh^{-1}m)^2 \text{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}m, m)) \, dm \, d(hZ_g) \tag{2.7}$$

converges absolutely. Then the value of this integral is the *g-trace* of κ , denoted by $\text{Tr}_g(\kappa)$. If T is a bounded, G -equivariant operator on $L^2(E)$, with a g -trace class Schwartz kernel κ , then we say that T is *g-trace class*, and define $\text{Tr}_g(T) := \text{Tr}_g(\kappa)$.

Definition 2.6 Let D be a G -equivariant, elliptic differential operator on E , odd with respect to a \mathbb{Z}_2 -grading on E . Let D_+ be its restriction to even-graded sections. Then D is *g-Fredholm* if D_+ has a parametrix R such that the operators

$$\begin{aligned} S_0 &:= 1 - RD_+; \\ S_1 &:= 1 - D_+R; \end{aligned} \tag{2.8}$$

are *g-trace class*.

The *g-index* of a g -Fredholm operator D is the number

$$\text{index}_g(D) := \text{Tr}_g(S_0) - \text{Tr}_g(S_1), \tag{2.9}$$

with S_0 and S_1 as in (2.8).

The g -index is independent of the parametrix R by Lemma 2.5 in [23].

2.4 Manifolds with boundary

We now specialise to the case we are interested in in this paper. The setting is the same as in Sect. 2.2 in [23].

Slightly changing notation from the previous subsections, we let M be a Riemannian manifold with boundary N . We still suppose that G acts properly and isometrically on M , preserving N , such that M/G is compact. We assume that a G -invariant neighbourhood U of N is G -equivariantly isometric to a product $N \times (0, \delta]$, for a $\delta > 0$. To simplify notation, we assume that $\delta = 1$; the case for general δ is entirely analogous.

As before, let $E = E_+ \oplus E_- \rightarrow M$ be a \mathbb{Z}_2 -graded G -equivariant, Hermitian vector bundle. We assume that E is a *Clifford module*, in the sense that there is a G -equivariant vector bundle homomorphism, the Clifford action, from the Clifford bundle of TM to the endomorphism bundle of E , mapping odd-graded elements of the Clifford bundle to odd-graded endomorphisms. We also assume that there is a G -equivariant isomorphism of Clifford modules $E|_U \cong E|_N \times (0, 1]$.

Let D be a Dirac-type operator on E ; i.e. the composition of a Clifford connection with the Clifford action. Let D_+ be the restriction of D to sections of E_+ . Suppose that

$$D_+|_U = \sigma \left(-\frac{\partial}{\partial u} + D_N \right), \tag{2.10}$$

where $\sigma : E_+|_N \rightarrow E_-|_N$ is a G -equivariant vector bundle isomorphism, u is the coordinate in the factor $(0, 1]$ in $U = N \times (0, 1]$, and D_N is an (ungraded) Dirac-type operator on $E_+|_N$. We initially assume that D_N is *invertible*, and show how to remove this assumption in Sect. 6.

Consider the cylinder $C := N \times [0, \infty)$, equipped with the product of the metric on M restricted to N , and the Euclidean metric. Because the metric, group action, Clifford module

and Dirac operator have a product form on U , all these structures extend to C . We form the complete manifold

$$\hat{M} := (M \sqcup C) / \sim,$$

where $m \sim (n, u)$ if $m = (n, u) \in U = N \times (0, 1]$. Let $\hat{E} \rightarrow \hat{M}$ and \hat{D} be the extensions of E and D to \hat{M} , respectively, obtained by gluing the relevant objects on M and C together along U .

Since D_N is invertible, there is a $c > 0$ such that

$$D_N^2 \geq c. \tag{2.11}$$

This implies that $\hat{D}^2 \geq c$ outside the cocompact set M , so that Definition 2.3 applies to \hat{D} . This gives us the localised coarse index

$$\text{index}_G(\hat{D}) \in K_0(C_r^*G), \tag{2.12}$$

which is the main object of study in this paper. Our goal is to give a topological expression for the number $\tau_g(\text{index}_G(\hat{D}))$.

The index (2.12) and the number $\tau_g(\text{index}_G(\hat{D}))$ simultaneously generalise several widely-used indices, as mentioned in the introduction. The index generalises to a case where D_N is not invertible, as discussed in Sect. 6.

2.5 The main result

In the case where $G = \Gamma$ is discrete and finitely generated, let l be a word length function on Γ with respect to a fixed, finite, symmetric, generating set. Because Γ is finitely generated, there are $C, k > 0$ such that for all $n \in \mathbb{N}$,

$$\#\{\gamma \in \Gamma; l(\gamma) = n\} \leq C e^{kn}. \tag{2.13}$$

Fix $m_0 \in M$. By the Svarc–Milnor lemma, there are $a_1, a_2 > 0$ such that for all $\gamma \in \Gamma$,

$$d(\gamma m_0, m_0) \geq a_1 l(\gamma) - a_2. \tag{2.14}$$

Let c be as in (2.11).

Theorem 2.7 *Suppose that \hat{D} is g -Fredholm. Suppose that an algebra A as in Sect. 2.3 exists. If either*

- (a) G/Z_g is compact; or
- (b) $G = \Gamma$ is discrete and finitely generated, and (2.13) holds for a $k < \frac{2a_1\sqrt{c}}{3}$,

then

$$\tau_g(\text{index}_G(\hat{D})) = \text{index}_g(\hat{D}).$$

Conditions for \hat{D} to be g -Fredholm were given in Theorem 2.11 and Corollaries 2.16, 2.19 and 2.21 in [23].

Remark 2.8 The growth condition on Γ in part (b) of Theorem 2.7 holds in particular if Γ has slower than exponential growth. In general, the condition depends on D, Γ and the group action. The factor $2/3$ in the bound $\frac{2a_1\sqrt{c}}{3}$ may be increased to any number smaller than 1. This can be achieved if we replace the factors $1/3$ on the right hand sides of (4.6) by other factors smaller than $1/2$.

The first corollary of Theorem 2.7 is invariance of $\text{index}_g(\hat{D})$ under a suitable notion of homotopy. This follows by homotopy invariance of $\text{index}_G(\hat{D})$.

Corollary 2.9 *Suppose that D_0 and D_1 are Dirac operators on M like the operator D , and let \hat{D}_0 and \hat{D}_1 be their extensions to \hat{M} . Suppose that D_0 and D_1 both satisfy the conditions of Theorem 2.7. Suppose that these operators are homotopic, in the sense that the Kasparov (\mathbb{C}, C_r^*G) -cycles representing their indices in $K_0(C_r^*G)$ are homotopic. Then $\text{index}_g(\hat{D}_0) = \text{index}_g(\hat{D}_1)$.*

Proof In the setting of this corollary, $\text{index}_G(D_0) = \text{index}_G(D_1)$. So Theorem 2.7 implies that

$$\text{index}_g(\hat{D}_0) = \tau_g(\text{index}_G(D_0)) = \tau_g(\text{index}_G(D_1)) = \text{index}_g(\hat{D}_1).$$

□

Combining Theorem 2.7 with Corollaries 2.16 and 2.19 in [23], we obtain an index formula for $\tau_g(\text{index}_G(\hat{D}))$.

Corollary 2.10 *Let D be a twisted Spin^c -Dirac operator. Suppose that either*

- $g = e$, or
- $G = \Gamma$ is discrete and finitely generated, (2.13) holds for a $k < \frac{2a_1\sqrt{c}}{3}$, and (g) has polynomial growth.

Then

$$\tau_g(\text{index}_G(\hat{D})) = \int_{M^g} \chi_g^2 \frac{\hat{A}(M^g) e^{c_1(L|_{M^g})/2} \text{tr}(g e^{-R_V|_{M^g}/2\pi i})}{\det(1 - g e^{-R_N/2\pi i})^{1/2}} - \frac{1}{2} \eta_g(D_N). \tag{2.15}$$

Notation is as in [23]; the integrand on the right hand side is the Atiyah–Segal–Singer integrand [2,3,5] times a cutoff function χ_g^2 , and $\eta_g(D_N)$ is the delocalised η -invariant of D_N , as in [30,31] and Sect. 2.3 of [23].

Remark 2.11 If $G = \Gamma$ is discrete and finitely generated and has polynomial growth, then (g) has polynomial growth for all g , and for all $k > 0$ there is a $C > 0$ such that (2.13) holds. So Corollary 2.10 applies in this case.

If Γ does not necessarily have polynomial growth, then we assume a bound on the number k in the growth condition (2.13) on the whole group Γ , rather than just on the conjugacy class (g) , because our proof of Theorem 2.7 involves the notion of G -integrable or Γ -summable operators (see Definition 3.7). We use this notion, because it is well-behaved with respect to compositions, as in Lemma 3.8. A different proof of case (b) of Corollary 2.10, not involving Γ -summable operators, may be possible without the spectral gap assumption.

Theorem 2.7, combined with results from [10], also implies a version of Proposition 5.3 in [43] and Theorem 1.4 in [10] in the case of fundamental groups of compact manifolds with boundary acting on their universal covers.

Corollary 2.12 *Suppose that X is a compact Riemannian Spin^c -manifold with boundary, with a product structure near the boundary. Let M be the universal cover of X , and let $N = \partial M$ as before. Let $G = \Gamma = \pi_1(X)$. Let D be the lift to M of a twisted Spin^c -Dirac operator on X . Let $g \in \Gamma$ be different from the identity element. Suppose that either that (g) has polynomial growth, or that Γ satisfies (the surjectivity part of) the Baum–Connes conjecture. If the constant c such that $D_N^2 \geq c$ is large enough, then*

$$\tau_g(\text{index}_G(\hat{D})) = -\frac{1}{2} \eta_g(D_N).$$

Proof If the constant c is large enough, then the delocalised η -invariant $\eta_g(D_N)$ converges by Theorem 1.1 in [10]. Furthermore, condition (b) in Theorem 2.7 also holds if c is large enough.

If (g) has polynomial growth, then the index formula for $\text{index}_g(\hat{D})$ in Corollary 2.19 in [23] applies. The interior contribution in this index formula now equals zero, because a nontrivial group element has no fixed points because the action is free. Together with Theorem 2.7, this implies the claim.

If (g) does not necessarily have polynomial growth, but the Baum–Connes assembly map for Γ is surjective, then $\text{index}_\Gamma(\hat{D}) \in K_0(C_r^*\Gamma)$ equals the index of an operator on a cocompact Γ -space. The latter index may be replaced by an index in $K_0(l^1(\Gamma))$; see for example Remark A.2 in [43]. The trace τ_g converges on $l^1(\Gamma)$ without growth conditions on the conjugacy class of g . So again, the claim follows from Corollary 2.19 in [23] and Theorem 2.7. □

Remark 2.13 The index theorem in [23] also applies to semisimple Lie groups. So it is a natural question if a version of Theorem 2.7 applies in that setting. We expect the techniques needed to prove this (particularly Proposition 4.1) to be very different from the discrete case. We have not looked into the details so far. In any case, the resulting version of Corollary 2.10 was obtained in [35].

Remark 2.14 The case of Theorem 2.7 where $g = e$, combined with Lemma 2.7 in [23], shows that $\tau_e(\text{index}_G(\hat{D}))$ generalises the index used by Ramachandran in [37], and that Corollary 2.10 generalises Ramachandran’s index theorem for manifolds with boundary.

Remark 2.15 Consider the setting of Corollary 2.12. Let D_X be the twisted Spin^c -Dirac operator on X that lifts to the operator D on M . As a consequence of Theorem 3.9 in [20], where reduced group C^* -algebras and Roe algebras are replaced by maximal ones (one can also use $l^1(\Gamma)$ if Γ satisfies the Baum–Connes conjecture), we have

$$\sum_{(g)} \tau_g(\text{index}_\Gamma(\hat{D})) = \text{index}(D_X), \tag{2.16}$$

where the sum runs over all conjugacy classes (g) in Γ , and the index on the right hand side is the APS index of D_X . Since Γ acts freely on M , Corollaries 2.10 and 2.12 imply that the left hand side of (2.16) equals

$$\int_M \chi_e^2 \hat{A}(\tilde{M}) e^{c_1(L)/2} \text{tr}(g e^{-R_{\tilde{V}}/2\pi i}) - \frac{1}{2} \sum_{(g)} \eta_g(D_N).$$

The first term is exactly the interior contribution to the topological side of the APS index of D_X . We conclude that

$$\eta(D_Y) = \sum_{(g)} \eta_g(D_N),$$

where D_Y is the Dirac operator on the boundary $Y = N/\Gamma$ of X corresponding to D_N . In other words, the delocalised η -invariants of D_N are refinements of the η -invariant of D_Y . This remains true in a case where D_N is not invertible, but there is a large enough gap in the spectrum of D_N around zero. See Sect. 6. (See (I.6) in [13] for the case where G is finite.)

We expect Corollary 2.10, and its extension to non-invertible D_N , to refine Farsi’s orbifold APS index theorem (Theorem 4.1 in [15]) in a similar way.

Remark 2.16 In [10], Chen, Wang, Xie and Yu proved convergence of delocalised and higher η -invariants for Dirac operators with large enough spectral gaps around zero. The assumption on k in case (b) of Theorem 2.7 is closely related to the spectral gap assumption in [10]. Furthermore, there are analogous convergence results for certain integrals, compare for example Propositions 3.12 and 3.13 in [10] with Lemma 4.3 and Proposition 4.4 in this paper. In [10], these results are used to prove convergence of delocalised η -invariants; in the current paper they are used to prove that the squares of certain g -trace class operators are again g -trace class.

3 A parametrix and properties of the g -trace

We prepare for the proof of Theorem 2.7 by introducing a specific parametrix for \hat{D} , and discussing some properties of the g -trace and of heat operators. We will use these things in Sect. 4 to prove that for the parametrix chosen, the squares of the remainder terms S_j as in (2.8) are g -trace class in the setting of Theorem 2.7.

3.1 A parametrix

We will use a parametrix of \hat{D} introduced in Sect. 5.1 of [23]. Consider the setting of Sect. 2.4. As before, let \tilde{M} be the double of M , and let $\tilde{E} = \tilde{E}_+ \oplus \tilde{E}_-$ and \tilde{D} be the extensions of E and D to \tilde{M} , respectively. More explicitly, as on page 55 of [1], \tilde{M} is obtained from M by gluing together a copy of M and a copy of M with reversed orientation, while \tilde{E} is obtained by gluing together a copy of E and a copy of E with reversed grading. To glue these copies of E together along N , we use the isomorphism σ . Let \tilde{D}_\pm be the restrictions of \tilde{D} to the sections of \tilde{E}_\pm .

Let $\psi_1 : (0, \infty) \rightarrow [0, 1]$ be a smooth function such that ψ_1 equals 1 on $(0, \varepsilon)$ and 0 on $(1 - \varepsilon, \infty)$, for some $\varepsilon \in (0, 1/2)$. Set $\psi_2 := 1 - \psi_1$. Let $\varphi_1, \varphi_2 : (0, \infty) \rightarrow [0, 1]$ be smooth functions such that φ_1 equals 1 on $(0, 1 - \varepsilon/2)$ and 0 on $(1, \infty)$, while φ_2 equals 0 on $(0, \varepsilon/4)$ and 1 on $(\varepsilon/2, \infty)$. Then $\varphi_j \psi_j = \psi_j$ for $j = 1, 2$, and φ'_j and ψ_j have disjoint supports.

We pull back the functions φ_j and ψ_j to C along the projection onto $(0, \infty)$, and extend these functions smoothly to \hat{M} by setting ψ_1 and φ_1 equal to 1 on $M \setminus U$, and ψ_2 and φ_2 equal to 0 on $M \setminus U$. We denote the resulting functions by the same symbols ψ_j and φ_j . (No confusion is possible in what follows, because we will always use these symbols to denote the functions on \hat{M} .) We denote the derivatives of these functions in the $(0, \infty)$ directions by φ'_j and ψ'_j , respectively. These derivatives are only defined and used on $N \times (0, \infty) \subset \hat{M}$.

Fix $t > 0$, and consider the parametrix

$$\tilde{Q} := \frac{1 - e^{-t\tilde{D}_- \tilde{D}_+}}{\tilde{D}_- \tilde{D}_+} \tilde{D}_- \tag{3.1}$$

of \tilde{D}_+ . (The part without the last factor \tilde{D}_- is formed via functional calculus, by an application of the function $x \mapsto \frac{1 - e^{-tx}}{x}$ to $\tilde{D}_- \tilde{D}_+$; this does not require invertibility of $\tilde{D}_- \tilde{D}_+$.)

Let D_C be the Dirac operator on $N \times \mathbb{R}$ given by (2.10). This operator is essentially self-adjoint and positive. Hence its self-adjoint closure is invertible. Let Q_C be the restriction to sections of E_- of the inverse of that closure. We define

$$R := \varphi_1 \tilde{Q} \psi_1 + \varphi_2 Q_C \psi_2.$$

Note that the operator \tilde{Q} is well-defined on the supports of φ_1 and ψ_1 , and that Q_C is well-defined on the supports of φ_2 and ψ_2 . The following two operators play key roles in this paper.

$$\begin{aligned} S_0 &:= 1 - R\hat{D}_+; \\ S_1 &:= 1 - \hat{D}_+R. \end{aligned} \tag{3.2}$$

3.2 Properties of S_0 and S_1

Consider the setting of Sect. 2.5. In addition to the parametrix R and the remainder terms S_0 and S_1 , we will also use the remainders

$$\begin{aligned} \tilde{S}_0 &:= 1 - \tilde{Q}\tilde{D}_+ = e^{-t\tilde{D}_-\tilde{D}_+}; \\ \tilde{S}_1 &:= 1 - \tilde{D}_+\tilde{Q} = e^{-t\tilde{D}_+\tilde{D}_-}. \end{aligned} \tag{3.3}$$

We recall Lemmas 5.1 and 5.2 from [23].

Lemma 3.1 *We have*

$$\begin{aligned} S_0 &= \varphi_1\tilde{S}_0\psi_1 + \varphi_1\tilde{Q}\sigma\psi'_1 + \varphi_2Q_C\sigma\psi'_2; \\ S_1 &= \varphi_1\tilde{S}_1\psi_1 - \varphi'_1\sigma\tilde{Q}\psi_1 - \varphi'_2\sigma Q_C\psi_2. \end{aligned} \tag{3.4}$$

Lemma 3.2 *The operators S_0 and S_1 have smooth kernels.*

Lemma 3.3 *The operators S_0 and S_1 lie in $C^*(\hat{M}, M; L^2(\hat{E}))^G$.*

Proof The operators S_0 and S_1 have smooth kernels by Lemma 3.2. This implies that these operators are locally compact.

The operator Q_C equals $b(D_C)$, where $b \in C_0(\mathbb{R})$ satisfies $b(x) = 1/x$ for all $x \in \text{spec}(D_N) \not\cong 0$. Hence, by Lemma 2.1 in [38], Q_C is a norm-limit of a sequence $(Q_{C,j})_{j=1}^\infty$ operators with finite propagation. Similarly, \tilde{Q} is a norm-limit of operators with finite propagation. So S_0 and S_1 are norm-limits of operators with finite propagation.

Since φ'_2 and ψ'_2 are supported near M and $Q_{C,j}$ has finite propagation, the operators $\varphi'_2\sigma Q_{C,j}\psi_2$ and $\varphi_2Q_{C,j}\sigma\psi'_2$ are supported near M . Hence $\varphi'_2\sigma Q_C\psi_2$ and $\varphi_2Q_C\sigma\psi'_2$ are norm-limits of operators that are supported near M . The other terms on the right hand sides of (3.4) are supported near M because φ_1 and ψ_1 are. So S_0 and S_1 are norm-limits of operators that are supported near M . □

3.3 Properties of the g -trace

We consider a general setting, where $E \rightarrow M$ is an equivariant, Hermitian vector bundle over a complete Riemannian metric with a proper, isometric action by G . In Sect. 4.3, we return to the setting of Sect. 2.4.

This trace property is Lemma 3.2 in [23].

Lemma 3.4 *Let S and T are G -equivariant operators on $\Gamma^\infty(E)$. Suppose that S has a distributional kernel supported on the diagonal, and T has a smooth kernel in $\Gamma^\infty(\text{End}(E))^G$. If ST and TS are g -trace class, then they have the same g -trace.*

Lemma 3.5 *A section $\kappa \in \Gamma^\infty(\text{End}(E))^G$ is g -trace class if and only if the integral*

$$\int_{G/Z_g} \int_M \chi(m)^2 |\text{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}m, m))| dm d(hZ_g) \tag{3.5}$$

converges.

Proof In (2.7), substituting $m' = hgh^{-1}m$, using G -invariance of κ and the trace property shows that (2.7) equals

$$\begin{aligned} & \int_{G/Z_g} \int_M \chi(m')^2 |\text{tr}(hgh^{-1}\kappa(hg^{-2}h^{-1}m', hg^{-1}h^{-1}m'))| dm' d(hZ_g) \\ &= \int_{G/Z_g} \int_M \chi(m')^2 |\text{tr}(\kappa(hg^{-1}h^{-1}m', m')hgh^{-1})| dm' d(hZ_g) \\ &= \int_{G/Z_g} \int_M \chi(m')^2 |\text{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}m', m'))| dm' d(hZ_g). \end{aligned}$$

□

Lemma 3.6 *Let $\kappa \in \Gamma^\infty(\text{End}(E))^G$ be such that there exists a cocompactly supported $\varphi \in C^\infty(M)^G$ such that either $\kappa = (\varphi \otimes 1)\kappa$ or $\kappa = (1 \otimes \varphi)\kappa$. Suppose that G/Z_g is compact. Then κ is g -trace class.*

Proof We prove the case where $\kappa = (\varphi \otimes 1)\kappa$, the other case is analogous. The integral (3.5) then equals

$$\int_{G/Z_g} \int_M \chi(m)^2 \varphi(m) |\text{tr}(hgh^{-1}\kappa(hg^{-1}h^{-1}m, m))| dm d(hZ_g).$$

Because G/Z_g and the support of $\chi^2\varphi$ are compact, this integral converges. □

In the setting of Lemma 3.6, if κ^2 is well-defined, then it has the same property as κ , so that it is also g -trace class.

3.4 G -integrable kernels

The composition of two g -trace class operators need not be g -trace class. The notion of G -integrability (or Γ -summability for discrete groups Γ) can be used to prove that such compositions are g -trace class under certain conditions.

Definition 3.7 A section $\kappa \in \Gamma^\infty(\text{End}(E))^G$ is G -integrable if for all $\varphi, \psi \in C_c^\infty(M)$, the integral

$$\int_G \left(\int_{M \times M} \varphi(m)\psi(m') \|\kappa(x^{-1}m, m')\|^2 dm dm' \right)^{1/2} dx$$

converges.

Lemma 3.8 *Let $\kappa, \lambda \in \Gamma^\infty(\text{End}(E))^G$ be G -integrable, and such that there exist cocompactly supported $\varphi, \psi \in C^\infty(M)^G$ such that either $\kappa = (\varphi \otimes 1)\kappa$ and $\lambda = (\psi \otimes 1)\lambda$ or $\kappa = (1 \otimes \varphi)\kappa$ and $\lambda = (1 \otimes \psi)\lambda$. Suppose that the composition $\kappa\lambda$ is a well-defined element of $\Gamma^\infty(\text{End}(E))^G$. Then the integral*

$$\int_G \int_M \chi(m)^2 |\text{tr}(x(\kappa\lambda)(x^{-1}m, m))| dm dx \tag{3.6}$$

converges.

Proof We prove the case where $\kappa = (\varphi \otimes 1)\kappa$ and $\lambda = (\psi \otimes 1)\lambda$, the other case is analogous. In this situation, the integral (3.6) equals

$$\int_G \int_M \chi(m)^2 \left| \int_M \varphi(m)\psi(m') \operatorname{tr}(x\kappa(x^{-1}m, m')\lambda(m', m)) \, dm' \right| dm \, dx.$$

Inserting a factor $1 = \int_G \chi(y m')^2 \, dy$ and substituting $m'' = y m'$, we find that this integral equals at most

$$\begin{aligned} & \int_G \int_M \int_M \int_G \chi(m)^2 \chi(y m')^2 |\varphi(m)\psi(m') \operatorname{tr}(x\kappa(x^{-1}m, m')\lambda(m', m))| \, dy \, dm' \, dm \, dx \\ &= \int_G \int_M \int_M \int_G \chi(m)^2 \chi(m'')^2 |\varphi(m)\psi(m'') \operatorname{tr}(x\kappa(x^{-1}m, y^{-1}m'')\lambda(y^{-1}m'', m))| \, dy \, dm'' \, dm \, dx. \end{aligned}$$

By Fubini’s theorem, the integral on the right converges if and only if

$$\int_G \int_G \int_M \int_M \chi(m)^2 \chi(m'')^2 |\varphi(m)\psi(m'') \operatorname{tr}(x\kappa(x^{-1}m, y^{-1}m'')\lambda(y^{-1}m'', m))| \, dm'' \, dm \, dx \, dy$$

converges. It is enough to consider the case where χ, φ and ψ are nonnegative. Then the latter integral is at most equal to

$$\int_G \int_G \int_M \int_M \chi(m)^2 \varphi(m) \chi(m'')^2 \psi(m'') \|x\kappa(x^{-1}m, y^{-1}m'')\lambda(y^{-1}m'', m)\| \, dm'' \, dm \, dx \, dy.$$

Using G -invariance of κ , substituting $z = x y^{-1}$ for x and applying the Cauchy–Schwartz inequality, we see that this integral equals

$$\begin{aligned} & \int_M \int_M \chi(m)^2 \varphi(m) \chi(m'')^2 \psi(m'') \int_G \int_G \|x y^{-1} \kappa(y x^{-1} m, m'') y \lambda(y^{-1} m'', m)\| \, dx \, dy \, dm'' \, dm \\ & \leq \int_G \left(\int_{M \times M} \chi(m)^2 \varphi(m) \chi(m'')^2 \psi(m'') \|z \kappa(z^{-1} m, m'')\|^2 \right)^{1/2} dz \\ & \cdot \int_G \left(\int_{M \times M} \chi(m)^2 \varphi(m) \chi(m'')^2 \psi(m'') \|y \lambda(y^{-1} m, m'')\|^2 \right)^{1/2} dy. \end{aligned} \tag{3.7}$$

The right hand side converges by G -integrability of κ and λ . □

If $G = \Gamma$ is discrete, then we will call a G -integrable smooth kernel Γ -summable.

Lemma 3.9 *Suppose $G = \Gamma$ is discrete. Let $\kappa, \lambda \in \Gamma^\infty(\operatorname{End}(E))^\Gamma$ be Γ -summable, and such that there exist cocompactly supported $\varphi, \psi \in C^\infty(M)^\Gamma$ such that either $\kappa = (\varphi \otimes 1)\kappa$ and $\lambda = (\psi \otimes 1)\lambda$ or $\kappa = (1 \otimes \varphi)\kappa$ and $\lambda = (1 \otimes \psi)\lambda$. Suppose that the composition $\kappa\lambda$ is a well-defined element of $\Gamma^\infty(\operatorname{End}(E))^\Gamma$. Then $\kappa\lambda$ is γ -trace class for all $\gamma \in \Gamma$.*

Proof By Lemma 3.8,

$$\sum_{\gamma' \in \Gamma} \int_M \chi(m)^2 |\operatorname{tr}(\gamma'(\kappa\lambda)(\gamma'^{-1}m, m))| \, dm$$

converges. So the sum over the conjugacy class of γ also converges, which is (3.5) in this case. □

3.5 Basic estimates for heat operators

Let D be a Dirac operator on $E \rightarrow M$.

Lemma 3.10 *Let $f \in \mathcal{S}(\mathbb{R})$. Let $r \geq 0$. Consider bounded endomorphisms Φ and Ψ of E whose supports are at least a distance r apart. Then*

$$\|\Phi f(D)\Psi\|_{\mathcal{B}(L^2(E))} \leq \frac{1}{2\pi} \|\Phi\| \|\Psi\| \int_{\mathbb{R} \setminus [-r,r]} |\hat{f}(\xi)| d\xi.$$

Proof For $D = \sqrt{-\Delta}$, with Δ the scalar Laplacian and f even, this is Proposition 1.1 in [9]. The arguments apply directly to D : the claim follows from the decomposition

$$f(D) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\lambda) e^{i\lambda D} d\lambda$$

and the fact that $e^{i\lambda D}$ has propagation at most $|\lambda|$. See Propositions 10.3.5 and 10.3.1 in [21], respectively. □

Corollary 3.11 *In the setting of Lemma 3.10, for all $t > 0$,*

$$\begin{aligned} \|\Phi e^{-tD^2} \Psi\|_{\mathcal{B}(L^2(E))} &\leq \frac{2}{\sqrt{\pi}} \|\Phi\| \|\Psi\| e^{-\frac{r^2}{4t}} \\ \|\Phi D e^{-tD^2} \Psi\|_{\mathcal{B}(L^2(E))} &\leq \frac{1}{\sqrt{\pi t}} \|\Phi\| \|\Psi\| e^{-\frac{r^2}{4t}}. \end{aligned}$$

Proof Applying Lemma 3.10 with $f(x) = e^{-tx^2}$, we obtain

$$\begin{aligned} \|\Phi e^{-tD^2} \Psi\|_{\mathcal{B}(L^2(E))} &\leq \frac{1}{\sqrt{\pi t}} \|\Phi\| \|\Psi\| \int_r^\infty e^{-\frac{\lambda^2}{4t}} d\lambda \\ &= \frac{2}{\sqrt{\pi}} \|\Phi\| \|\Psi\| \operatorname{erfc}\left(\frac{r}{2\sqrt{t}}\right). \end{aligned}$$

The first inequality now follows from the inequality $\operatorname{erfc}(x) \leq e^{-x^2}$ for all $x > 0$.

For the second inequality, we take $f(x) = xe^{-tx^2}$. Then Lemma 3.10 yields

$$\begin{aligned} \|\Phi D e^{-tD^2} \Psi\|_{\mathcal{B}(L^2(E))} &\leq \frac{1}{2\sqrt{\pi t^{3/2}}} \|\Phi\| \|\Psi\| \int_r^\infty \lambda e^{-\frac{\lambda^2}{4t}} d\lambda \\ &= \frac{1}{\sqrt{\pi t}} \|\Phi\| \|\Psi\| e^{-\frac{r^2}{4t}}. \end{aligned}$$

□

If M has bounded geometry, then the Schwartz kernel κ_t of e^{-tD^2} or $D e^{-tD^2}$ has Gaussian off-diagonal decay behaviour. More explicitly, for all $t_0 > 0$, there are $b_1, b_2, b_3 > 0$ such that for all $t \in (0, t_0]$ and all $m, m' \in M$,

$$\|\kappa_t(m, m')\| \leq b_1 t^{-b_2} e^{-b_3 d(m, m')^2/t}, \tag{3.8}$$

where d is the Riemannian distance. Estimates of this type were proved in many places. A classical result is the one by Chen–Li–Yau [11] for the scalar Laplacian. A general result, which applies in our current setting, is Proposition 4.2 in [7].

Lemma 3.12 *If M has bounded geometry, then the operators $e^{-tD^2}\varphi$ and $e^{-tD^2}D\varphi$ are Hilbert–Schmidt operators for all $t > 0$ and $\varphi \in C_c^\infty(M)$.*

Proof Let κ be the Schwartz kernel of either $e^{-tD^2}D$ or e^{-tD^2} . The bound (3.8) means that $\kappa\varphi$ can be bounded by a Gaussian function. Since M has bounded geometry, volumes of balls in M are bounded by an exponential function of their radii. This implies that a Gaussian function is square-integrable. \square

4 S_0^2 and S_1^2 are g -trace class

Let S_0 and S_1 be as in (3.2). Our main goal in this section is to prove the following proposition.

Proposition 4.1 *Under the conditions in Theorem 2.7, the operators S_0^2 and S_1^2 are g -trace class.*

In [23], it is shown that S_0 and S_1 are g -trace class in a general setting. An important subtlety is that this is true for the notion of g -trace class operators in Definition 2.5, which is relatively weak. For example, it does not reduce to the usual notion of trace class operators if G is trivial, and it is not preserved by composition with bounded, or even other g -trace class operators. For this reason, Proposition 4.1 does not follow directly from the fact that S_0 and S_1 are g -trace class, and the arguments in this section are needed to prove it.

4.1 Convergence of an integral for small t

In this subsection and the next, we consider a general setting, where $E \rightarrow M$ is an equivariant, Hermitian vector bundle over a complete Riemannian manifold with bounded geometry, and a proper, isometric action by G .

Let D be a Dirac operator on E , assuming a Clifford action is given. Because M has bounded geometry, the kernels of e^{-tD^2} and De^{-tD^2} satisfy bounds of the type (3.8). We choose $t_1 > 0$ such that these bounds hold for $t \in (0, t_1]$.

We will use some calculus.

Lemma 4.2 *Let $a, b > 0$, and $t_0 \in (0, b/a]$. Then*

$$\int_0^{t_1} t^{-a} e^{-b/t} dt \leq t_1 \min(t_0, t_1)^{-a} e^{-b/t_1}. \tag{4.1}$$

Proof The function $t \mapsto t^{-a} e^{-b/t}$ is increasing on $(0, b/a]$, hence on $(0, t_0]$. So

$$\int_0^{t_0} t^{-a} e^{-b/t} dt \leq t_0 t_0^{-a} e^{-b/t_0} \leq t_0^{1-a} e^{-b/t_1}, \tag{4.2}$$

and a similar estimate holds for the integral from 0 to t_1 if $t_1 \leq t_0$. If $t_1 \geq t_0$, then

$$\int_{t_0}^{t_1} t^{-a} e^{-b/s} ds \leq (t_1 - t_0) t_0^{-a} e^{-b/t_1}. \tag{4.3}$$

The claim (4.1) follows from a combination of (4.2) and (4.3). \square

Lemma 4.3 *Let κ_t be the Schwartz kernel of either e^{-tD^2} or $e^{-tD^2}D$. Let $\varphi, \psi \in C^\infty(M)^\Gamma$ have supports separated by a positive distance ε , and let $\tilde{\varphi}, \tilde{\psi} \in C_c^\infty(M)$. The integral*

$$\sum_{\gamma \in \Gamma} \left(\int_{M \times M} \tilde{\varphi}(m)\varphi(m)\tilde{\psi}(m')\psi(m') \int_0^{t_1} \|\gamma\kappa_t(\gamma^{-1}m, m')\|^2 dt dm dm' \right)^{1/2} \tag{4.4}$$

converges.

Proof For $\gamma \in \Gamma$, set

$$r(\gamma) := d(\gamma \operatorname{supp}(\tilde{\varphi}\varphi), \operatorname{supp}(\tilde{\psi}\psi)).$$

The Gaussian bound (3.8) on κ_t implies that for all $\gamma \in \Gamma$ and $t \in (0, t_1]$,

$$\begin{aligned} & \int_{M \times M} \tilde{\varphi}(m)\varphi(m)\tilde{\psi}(m')\psi(m') \|\gamma\kappa_t(\gamma^{-1}m, m')\|^2 dm dm' \\ &= \int_{M \times M} \tilde{\varphi}(\gamma m)\varphi(\gamma m)\tilde{\psi}(m')\psi(m') \|\kappa_t(m, m')\|^2 dm dm' \\ &\leq b_1^2 t^{-2b_2} e^{-2b_3 r(\gamma)^2/t} \|\tilde{\varphi}\varphi\|_{L^1} \|\tilde{\psi}\psi\|_{L^1}. \end{aligned}$$

So

$$\begin{aligned} & \left(\int_{M \times M} \tilde{\varphi}(m)\varphi(m)\tilde{\psi}(m')\psi(m') \int_0^{t_1} \|\gamma\kappa_t(\gamma^{-1}m, m')\|^2 dt dm dm' \right)^{1/2} \\ &\leq b_1 \|\tilde{\varphi}\varphi\|_{L^1}^{1/2} \|\tilde{\psi}\psi\|_{L^1}^{1/2} \left(\int_0^{t_1} t^{-2b_2} e^{-2b_3 r(\gamma)^2/t} \right)^{1/2}. \end{aligned}$$

The assumptions on φ and ψ imply that $r(\gamma) \geq \varepsilon$ for all $\gamma \in \Gamma$. Set $t_0 := b_3\varepsilon^2/b_2$. Then by Lemma 4.2,

$$\left(\int_0^{t_1} t^{-2b_2} e^{-2b_3 r(\gamma)^2/t} \right)^{1/2} \leq t_1^{1/2} \min(t_0, t_1)^{-b_2} e^{-b_3 r(\gamma)^2/t_1}.$$

The Svarc–Milnor lemma and compactness of the supports of $\tilde{\varphi}$ and $\tilde{\psi}$ imply that there are $a, b > 0$ such that for all $\gamma \in \Gamma$, $r(\gamma) \geq al(\gamma) - b$, where l denotes the word length with respect to a fixed, finite, symmetric, generating set. So there are $\alpha, \beta > 0$ such that for all $\gamma \in \Gamma$,

$$e^{-b_3 r(\gamma)^2/t_1} \leq e^{-b_3(al(\gamma)-b)^2/t_1} \leq \alpha e^{-\beta l(\gamma)^2/t_1}.$$

The sum of the right hand side over $\gamma \in \Gamma$ converges, because of (2.13). □

4.2 Convergence of an integral for large t

We still consider a Dirac operator D , and now assume that $D^2 \geq c > 0$.

As before, let l be a word length function on Γ with respect to a fixed, finite, symmetric, generating set. Because Γ is finitely generated, there are $C, k > 0$ such that (2.13) holds for all $n \in \mathbb{N}$. Let $\varphi, \psi \in C_c^\infty(M)$, and fix $m_0 \in \operatorname{supp}(\psi)$. Let a_1 and a_2 be as in (2.14).

Proposition 4.4 *Suppose that M has bounded geometry. Suppose that (2.13) holds for a $k < \frac{2a_1\sqrt{c}}{3}$. Then for all $t_1 > 0$, the expression*

$$\sum_{\gamma \in \Gamma} \left(\int_{M \times M} \int_{t_1}^{\infty} \varphi(m)\psi(m') \|\gamma e^{-sD^2} D(\gamma^{-1}m, m')\|^2 ds dm dm' \right)^{1/2} \tag{4.5}$$

converges.

By Lemma 3.12, the operators $e^{-tD^2} \varphi$ and $e^{-tD^2} D\varphi$ are Hilbert–Schmidt for all $t > 0$, because M has bounded geometry.

Lemma 4.5 *For all $\varphi \in C_c^\infty(M)$, and all $t_1 > 0$, there exists an $a > 0$ such that for all $t > t_1$,*

$$\begin{aligned} \|e^{-tD^2} \varphi\|_{\text{HS}} &\leq a e^{-ct}; \\ \|e^{-tD^2} D\varphi\|_{\text{HS}} &\leq a e^{-ct}. \end{aligned}$$

Proof For $t > 0$, let A_t be either the operator e^{-tD^2} or $e^{-tD^2} D$. Then for all $t > t_1 > 0$, and all $s \in L^2(E)$,

$$\begin{aligned} \|A_t \varphi s\|^2 &= \|e^{-(t-t_1)D^2} A_{t_1} \varphi s\|^2 \\ &= (e^{-2(t-t_1)D^2} A_{t_1} \varphi s, A_{t_1} \varphi s) \\ &\leq e^{-2c(t-t_1)} \|A_{t_1} \varphi s\|^2. \end{aligned}$$

Let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis of $L^2(E)$. Then by the above estimate,

$$\|A_t \varphi\|_{\text{HS}}^2 = \sum_{j=1}^\infty \|A_t \varphi e_j\|^2 \leq e^{-2c(t-t_1)} \|A_{t_1} \varphi\|_{\text{HS}}^2.$$

□

Let $\varphi, \psi \in C_c^\infty(M)$, and suppose for simplicity that these functions take values in $[0, 1]$. For $\gamma \in \Gamma$, set

$$r(\gamma) := d(\gamma \text{ supp}(\varphi), \text{ supp}(\psi)).$$

(Here we note that $r(\gamma)$ may be zero.) Fix $\gamma \in \Gamma$ and $t > 0$. Let $\zeta \in C_c^\infty(M)$ be a function with values in $[0, 1]$ such that

$$\begin{aligned} d(\text{supp}(\psi), \text{supp}(1 - \zeta)) &\geq r(\gamma)/3; \\ d(\gamma \text{ supp}(\varphi), \zeta) &\geq r(\gamma)/3. \end{aligned} \tag{4.6}$$

Write

$$(\gamma \cdot \varphi) e^{-tD^2} D\psi = A(\gamma) + B(\gamma),$$

where

$$\begin{aligned} A(\gamma) &:= (\gamma \cdot \varphi) e^{-tD^2/2} \zeta e^{-tD^2/2} D\psi; \\ B(\gamma) &:= (\gamma \cdot \varphi) e^{-tD^2/2} (1 - \zeta) e^{-tD^2/2} D\psi. \end{aligned}$$

Lemma 4.6 *The operator $A(\gamma)$ is Hilbert–Schmidt, and there is a $b > 0$, independent of γ , such that for all $t \geq t_1$,*

$$\|A(\gamma)\|_{\text{HS}} \leq be^{-r(\gamma)^2/9t-ct}.$$

Proof For all $s \in L^2(E)$ and $\gamma \in \Gamma$,

$$\|A(\gamma)s\| \leq \|(\gamma \cdot \varphi)e^{-tD^2/2}\zeta^{1/2}\|_{\mathcal{B}(L^2(E))} \|\zeta^{1/2}e^{-tD^2/2}D\psi s\|.$$

By Corollary 3.11 and the second inequality in (4.6),

$$\|(\gamma \cdot \varphi)e^{-tD^2/2}\zeta^{1/2}\|_{\mathcal{B}(L^2(E))} \leq \frac{2}{\sqrt{\pi}}e^{-\frac{r(\gamma)^2}{18r}}.$$

So, if $\{e_j\}_{j=1}^\infty$ is an orthonormal basis of $L^2(E)$,

$$\|A(\gamma)\|_{\text{HS}}^2 \leq \frac{4}{\pi}e^{-\frac{r(\gamma)^2}{9r}} \sum_{j=1}^\infty \|\zeta^{1/2}e^{-tD^2/2}D\psi e_j\|^2 \leq \frac{4}{\pi}e^{-\frac{r(\gamma)^2}{9r}} \|e^{-tD^2/2}D\psi\|_{\text{HS}}^2.$$

The claim now follows by Lemma 4.5. □

Lemma 4.7 *The operator $B(\gamma)$ is Hilbert–Schmidt, and there is a $b > 0$, independent of γ , such that for all $t \geq t_1$,*

$$\|B(\gamma)\|_{\text{HS}} \leq be^{-r(\gamma)^2/9t-ct}.$$

Proof The operator $B(\gamma)$ is Hilbert–Schmidt if and only its adjoint is, and then these operators have the same Hilbert–Schmidt norm. Now

$$\begin{aligned} B(\gamma)^* &= \psi e^{-tD^2/2}D(1-\zeta)e^{-tD^2/2}(\gamma \cdot \varphi) \\ &= \psi e^{-tD^2/2}(1-\zeta)De^{-tD^2/2}(\gamma \cdot \varphi) - \varphi e^{-tD^2/2}c(d\zeta)e^{-tD^2/2}(\gamma \cdot \varphi). \end{aligned}$$

The distance between the supports of φ and $1-\zeta$ is at least $r(\gamma)/3$. The support of $d\zeta$ lies inside the support of $1-\zeta$, so the distance between the supports of φ and $d\zeta$ is at least $r(\gamma)/3$ as well. So Corollary 3.11 implies that

$$\begin{aligned} \|\psi e^{-tD^2/2}(1-\zeta)\| &\leq \frac{2}{\sqrt{\pi}}e^{-\frac{r(\gamma)^2}{18r}}; \\ \|\psi e^{-tD^2/2}c(d\zeta)\| &\leq \|d\zeta\|_\infty \frac{2}{\sqrt{\pi}}e^{-\frac{r(\gamma)^2}{18r}}. \end{aligned}$$

And the Hilbert–Schmidt norms of

$$e^{-tD^2/2}(\gamma \cdot \varphi) = \gamma e^{-tD^2/2}\varphi\gamma^{-1}$$

and

$$De^{-tD^2/2}(\gamma \cdot \varphi) = \gamma De^{-tD^2/2}\varphi\gamma^{-1}$$

are independent of γ . So a similar argument to the proof of Lemma 4.6 applies to show that there is a $b > 0$ such that for all $t \geq t_1$,

$$\|B(\gamma)\|_{\text{HS}} = \|B(\gamma)^*\|_{\text{HS}} \leq be^{-r(\gamma)^2/9t-ct}.$$

□

Lemma 4.8 *Let $C, k, \alpha_1, \alpha_2, \alpha_3, t_1 > 0$, and suppose that (2.13) holds for all $n \in \mathbb{N}$. Suppose that $k^2 < 4\alpha_1\alpha_3$. Then*

$$\sum_{\gamma \in \Gamma} \int_{t_1}^{\infty} e^{-\alpha_1 \frac{(l(\gamma) - \alpha_2)^2}{s} - \alpha_3 s} ds \tag{4.7}$$

converges.

Proof The sum (4.7) equals

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{\gamma \in \Gamma; l(\gamma)=n} \int_{t_1}^{\infty} e^{-\alpha_1 \frac{(n - \alpha_2)^2}{s} - \alpha_3 s} ds \leq C \sum_{n=0}^{\infty} \int_{t_1}^{\infty} e^{-\alpha_1 \frac{(n - \alpha_2)^2}{s} - \alpha_3 s + kn} ds \\ & = C e^{k\alpha_2} \int_{t_1}^{\infty} e^{\left(\frac{k^2}{4\alpha_1} - \alpha_3\right)s} \left(\sum_{n=0}^{\infty} e^{-\frac{\alpha_1}{s} \left(n - \alpha_2 - \frac{ks}{2\alpha_1}\right)^2} \right) ds. \end{aligned} \tag{4.8}$$

(Because all terms and integrands are positive, convergence does not depend on the order of summation and integration.) Convergence of the right hand side of (4.8) is equivalent to convergence of the double integral

$$\int_{t_1}^{\infty} e^{\left(\frac{k^2}{4\alpha_1} - \alpha_3\right)s} \left(\int_0^{\infty} e^{-\frac{\alpha_1}{s} \left(x - \alpha_2 - \frac{ks}{2\alpha_1}\right)^2} dx \right) ds. \tag{4.9}$$

And for all $s > 0$,

$$\int_0^{\infty} e^{-\frac{\alpha_1}{s} \left(x - \alpha_2 - \frac{ks}{2\alpha_1}\right)^2} dx \leq \int_{\mathbb{R}} e^{-\frac{\alpha_1}{s} \left(x - \alpha_2 - \frac{ks}{2\alpha_1}\right)^2} dx = \sqrt{\frac{\pi s}{\alpha_1}}.$$

We find that a sufficient condition for the convergence of (4.9) is convergence of

$$\int_{t_1}^{\infty} e^{\left(\frac{k^2}{4\alpha_1} - \alpha_3\right)s} \sqrt{\frac{\pi s}{\alpha_1}} ds.$$

This is equivalent to the condition $k^2 < 4\alpha_1\alpha_3$. □

Proof of Proposition 4.4 The integral (4.5) equals

$$\sum_{\gamma \in \Gamma} \left\| \int_{t_1}^{\infty} \varphi \circ \gamma \circ e^{-sD^2} D \circ \psi ds \right\|_{\text{HS}} \leq \sum_{\gamma \in \Gamma} \int_{t_1}^{\infty} \|\varphi \circ \gamma \circ e^{-sD^2} D \circ \psi\|_{\text{HS}} ds. \tag{4.10}$$

By Lemmas 4.6 and 4.7, there is a $b > 0$ such that for all $t \geq t_1$ and all $\gamma \in \Gamma$,

$$\|\varphi \circ \gamma \circ e^{-sD^2} D \circ \psi\|_{\text{HS}} = \|(\gamma \cdot \varphi) \circ e^{-sD^2} D \circ \psi\|_{\text{HS}} \leq b e^{-r(\gamma)^2/9s - cs}.$$

The condition (2.14) and compactness of $\text{supp}(\varphi)$ and $\text{supp}(\psi)$ imply that there is $a_3 > 0$ such that for all $\gamma \in \Gamma$, $r(\gamma) \geq a_1 l(\gamma) - a_3$. So

$$\|\varphi \circ \gamma \circ e^{-sD^2} D \circ \psi\|_{\text{HS}} \leq b e^{-\frac{(a_1 l(\gamma) - a_3)^2}{9s} - cs}.$$

So the right hand side of (4.10) is at most equal to

$$b \sum_{\gamma \in \Gamma} \int_{t_1}^{\infty} e^{-\frac{(a_1 l(\gamma) - a_3)^2}{9s} - cs} ds.$$

By Lemma 4.8, this converges if Γ satisfies (2.13) for some $C, k > 0$ with $k^2 < \frac{4a_1^2 c}{9}$. □

4.3 Proof of Proposition 4.1

We return to the setting of Sect. 2.4, where M is a manifold with boundary N , on which G acts cocompactly, and \hat{M} is obtained from M by attaching a cylinder $N \times [0, \infty)$.

The operators \tilde{Q} and Q_C as in Sect. 3.1 do not have smooth kernels, but if $\varphi, \psi \in C^\infty(M)$ have disjoint supports, then $\varphi\tilde{Q}\psi$ and $\varphi Q_C\psi$ do.

Lemma 4.9 *If $\varphi, \psi \in C^\infty(M)^\Gamma$ have supports separated by a positive distance, then $\varphi\tilde{Q}\psi$ is Γ -summable.*

Proof We have

$$\tilde{Q} = - \int_0^t e^{-s\tilde{D}_+\tilde{D}_-} \tilde{D}_- ds.$$

Because \tilde{M} has bounded geometry, the claim follows from Lemma 4.3. □

Proposition 4.10 *Consider the setting of Theorem 2.7(b). If $\varphi, \psi \in C^\infty(M)^\Gamma$ have supports separated by a positive distance, then $\varphi Q_C\psi$ is Γ -summable.*

Proof We have

$$Q_C = \int_0^\infty e^{-s(D_C)_+(D_C)_-} (D_C)_- ds.$$

The operator

$$\varphi \int_0^1 e^{-s(D_C)_+(D_C)_-} (D_C)_- ds \psi$$

is Γ -summable by Lemma 4.3, because \hat{M} has bounded geometry. The operator

$$\varphi \int_1^\infty e^{-s(D_C)_+(D_C)_-} (D_C)_- ds \psi$$

is Γ -summable by Proposition 4.4. The coefficient that appears a_1 in (2.14) and in the growth condition on Γ is independent of the choice of $m_0 \in \text{supp}(\psi)$ by compactness of M/Γ and Γ -invariance of the distance on M . □

Let the functions φ_j and ψ_j , and the operator S_0 be as in Sect. 3.1, and let \tilde{S}_0 be as in (3.3).

Proposition 4.11 *Consider the setting of Theorem 2.7(b). The operator $(\varphi_1\tilde{Q} - \varphi_2Q_C)\psi'_1$ has a smooth kernel, and is Γ -summable.*

Proof The operator S_0 has a smooth kernel by Lemma 3.2, and $\varphi_1\tilde{S}_0\psi_1$ has a smooth kernel as well. Hence so does

$$(\varphi_1\tilde{Q} - \varphi_2Q_C)\psi'_1 = S_0 - \varphi_1\tilde{S}_0\psi_1.$$

As in the proof of Proposition 5.8 in [23],

$$\begin{aligned}
 (\varphi_1 \tilde{Q} - \varphi_2 Q_C) \psi'_1 &= (\varphi_1 \tilde{Q} - \varphi_2 Q'_C) \psi'_1 - \varphi_2 (Q_C - Q'_C) \psi'_1 \\
 &= (\varphi_1 \tilde{Q} - \varphi_2 Q'_C) \psi'_1 - \varphi_2 e^{-D_C + D_{C,-}} Q_C \psi'_1 \\
 &\quad - \int_0^t (\varphi_1 e^{-s \tilde{D}_+ \tilde{D}_-} - \varphi_2 e^{-s D_C + D_{C,-}}) D_- ds \sigma \psi'_1 \\
 &\quad - \varphi_2 \int_t^\infty e^{-s D_C - D_{C,+} D_{C,-}} ds \sigma \psi'_1
 \end{aligned}
 \tag{4.11}$$

The second term on the right hand side is Γ -summable by Proposition 4.4, in which it is not assumed that the functions φ and ψ have disjoint supports. Here we again use the fact that the coefficient a_1 that appears in (2.14) and in the growth condition on Γ is independent of the choice of $m_0 \in \text{supp}(\psi)$ by compactness of M/Γ and Γ -invariance of the distance on M .

We now focus on the first term on the right hand side of (4.11). As in the proof of Lemma 5.5 in [23], let $\varphi \in C^\infty(\tilde{M})$ be such that for $j = 1, 2$, φ equals 1 on the support of ψ'_j , and zero outside the support of $1 - \varphi_j$. Since $(1 - \varphi)$ and ψ'_1 have supports separated by a positive distance, Lemma 4.3 implies that

$$\int_0^t (1 - \varphi) (\varphi_1 e^{-s \tilde{D}_+ \tilde{D}_-} - \varphi_2 e^{-s (D_C + D_{C,-})}) D_- \sigma \psi'_1 ds$$

is Γ -summable.

Let $\tilde{\varphi}, \tilde{\psi} \in C_c^\infty(M)$. Then as in Lemma 5.4 in [23], for all $m, m' \in M$

$$\begin{aligned}
 &(\tilde{\varphi} \varphi (\varphi_1 e^{-s \tilde{D}_+ \tilde{D}_-} - \varphi_2 e^{-s (D_C + D_{C,-})}) D_- \sigma \psi'_1 \tilde{\psi})(m, m') \\
 &= \frac{1}{(2\pi s)^{\dim(M)/2}} e^{-d(m, m')^2/4s} F(s, m, m'),
 \end{aligned}$$

where $F(s, m, m')$ vanishes to all orders in s as $s \downarrow 0$, uniformly in m, m' in compact sets. This implies that $\tilde{\varphi} \varphi (\varphi_1 e^{-s \tilde{D}_+ \tilde{D}_-} - \varphi_2 e^{-s (D_C + D_{C,-})}) D_- \sigma \psi'_1$ is Γ -summable via a simpler version of the proof of Lemma 4.3. □

Proof of Proposition 4.1 First suppose that G/Z_g is compact. Because the functions φ_1 and φ'_2 are cocompactly supported, Lemma 3.1 implies that there is a cocompactly supported function $\varphi \in C^\infty(M)^G$ such that $\varphi S_1 = S_1$. So S_1^2 is g -trace class by Lemma 3.6 and the comment below it. And because ψ_1 and ψ'_2 are cocompactly supported, Lemma 3.1 implies that there is a cocompactly supported function $\varphi \in C^\infty(M)^G$ such that $S_0 \varphi = S_0$. So S_0^2 is g -trace class, again by Lemma 3.6. Part (a) follows.

For case (b) in Theorem 2.7, suppose that $G = \Gamma$ is discrete. The operator \tilde{S}_1 is Γ -summable, so Lemma 4.9 and Proposition 4.10 imply that the three terms in the expression for S_1 in Lemma 3.1 are all Γ -summable. As in the proof of part (a), there is a cocompactly supported function $\varphi \in C^\infty(M)^G$ such that $\varphi S_1 = S_1$. So S_1^2 is g -trace class by Lemma 3.9.

The operator \tilde{S}_0 is Γ -summable, so Proposition 4.11 and Lemma 3.1 imply that

$$S_0 = \varphi_1 \tilde{S}_0 \psi_1 + (\varphi_1 \tilde{Q} - \varphi_2 Q_C) \psi'_1$$

is Γ -summable as well. As in the proof of part (a), there is a cocompactly supported function $\varphi \in C^\infty(M)^G$ such that $S_0 \varphi = S_0$. So S_0^2 is g -trace class by Lemma 3.9. □

5 The trace of the index

Our main goal in this section is to prove the following part of Theorem 2.7.

Proposition 5.1 *If S_0^2 and S_1^2 are g -trace class, then*

$$\tau_g(\text{index}_G(\hat{D})) = \text{Tr}_g(S_0^2) - \text{Tr}_g(S_1^2). \tag{5.1}$$

Together with Proposition 4.1, this is the main part of the proof of Theorem 2.7.

5.1 An explicit index

Let $C^\infty(\hat{M}, M; L^2(\hat{E}))^G$ be the subalgebra of elements of $C^*(\hat{M}, M; L^2(\hat{E}))^G$ with smooth kernels. Because \hat{D} is a multiplier of $C^\infty(\hat{M}, M; L^2(\hat{E}))^G$, Lemmas 3.2 and 3.3 imply that

$$e := \begin{pmatrix} S_0^2 & S_0(1 + S_0)R \\ S_1\hat{D}_+ & 1 - S_1^2 \end{pmatrix} \tag{5.2}$$

is an idempotent in $C^\infty(\hat{M}, M; L^2(\hat{E}))^G$. (The 2×2 matrix notation is with respect to the decomposition $\hat{E} = \hat{E}_+ \oplus \hat{E}_-$.) See also page 353 of [12]. We write

$$p_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let

$$\iota: C^\infty(\hat{M}, M; L^2(\hat{E}))^G \rightarrow C^*(\hat{M}, M; L^2(\hat{E}))^G$$

be the inclusion map. Let

$$\text{index}_G^{L^2(\hat{E})}(\hat{D}) \in K_0(C^*(\hat{M}, M; L^2(\hat{E}))^G)$$

be defined as in (2.3).

Lemma 5.2 *We have*

$$\text{index}_G^{L^2(\hat{E})}(\hat{D}) = \iota_*([e] - [p_2]). \tag{5.3}$$

Proof The right hand side of (5.3) equals $\partial[\hat{D}]$, where

$$\partial: K_1(\mathcal{M}(C^\infty(\hat{M}, M; L^2(\hat{E}))^G / C^\infty(\hat{M}, M; L^2(\hat{E}))^G) \rightarrow K_0(C^\infty(\hat{M}, M; L^2(\hat{E}))^G)$$

is the boundary map in the six-term exact sequence. The image of $\partial[\hat{D}]$ in $K_0(C^*(\hat{M}; Z; L^2(\hat{E}))^G)$ equals $[\bar{e}] - [p_2]$, where \bar{e} is the idempotent defined as the right hand side of (5.2), with R replaced by \bar{R} , and S_j by \bar{S}_j , for any multiplier \bar{R} of $C^\infty(\hat{M}; Z; L^2(\hat{E}))^G$ such that $\bar{S}_0 := 1 - \bar{R}\hat{D}_+$ and $\bar{S}_1 := 1 - \hat{D}_+\bar{R}$ are in $C^\infty(\hat{M}, M; L^2(\hat{E}))^G$. In other words, for any such \bar{R} ,

$$[e] - [p_2] = [\bar{e}] - [p_2]. \tag{5.4}$$

Let b be the function used in Sect. 2.2. We now choose b such that $b(x) = O(x)$ as $x \rightarrow 0$, so that the function $x \mapsto b(x)/x$ has a continuous extension to \mathbb{R} . The function b is odd, and the function $x \mapsto b(x)/x$ is even. So the operator $\frac{b(\hat{D})}{\hat{D}}$ is even with respect to the grading on

E , whereas $b(\hat{D})$ is odd. We denote restrictions of operators to sections of E_{\pm} be subscripts \pm , respectively. We choose

$$\bar{R} := b(\hat{D})_- \left(\frac{b(\hat{D})}{\hat{D}} \right)_-$$

Then we obtain operators \bar{S}_0 and \bar{S}_1 which equal the restrictions of $1 - b(\hat{D})^2$ to even and odd graded sections of E , respectively. We claim that \bar{S}_0 and \bar{S}_1 lie in $C^\infty(\hat{M}; Z; L^2(\hat{E}))^G$. Indeed, by Lemma 2.3 in [38], these operators lie in $C^*(\hat{M}; Z; L^2(\hat{E}))^G$. And $1 - b^2$ is compactly supported, so $\hat{D}^j(1 - b(\hat{D})^2)$ is a bounded operator on $L^2(\hat{E})$ for all $j \in \mathbb{N}$. Hence, by elliptic regularity, $1 - b(\hat{D})^2$ maps $L^2(\hat{E})$, and any Sobolev space defined in terms of \hat{D} , continuously into $\Gamma^\infty(E)$. So this operator has a smooth kernel.

For this choice of \bar{R} , we have

$$\bar{e} = \begin{pmatrix} \bar{S}_0^2 & \bar{S}_0(1 + \bar{S}_0)b(\hat{D})_- \left(\frac{b(\hat{D})}{\hat{D}} \right)_- \\ \bar{S}_1 \hat{D}_+ & 1 - \bar{S}_1^2 \end{pmatrix}$$

For $s \in [0, 1]$, we write

$$A_s := \begin{pmatrix} \left(\frac{b(\hat{D})}{\hat{D}} \right)_+^{-s/2} & 0 \\ 0 & \left(\frac{b(\hat{D})}{\hat{D}} \right)_-^{s/2} \end{pmatrix},$$

and consider the idempotent

$$e_s := A_s \bar{e} A_s^{-1} = \begin{pmatrix} \bar{S}_0^2 & \bar{S}_0(1 + \bar{S}_0)b(\hat{D})_- \left(\frac{b(\hat{D})}{\hat{D}} \right)_-^{1-s} \\ \bar{S}_1 \hat{D}_+ \left(\frac{b(\hat{D})}{\hat{D}} \right)_+^s & 1 - \bar{S}_1^2 \end{pmatrix}$$

in $M_2(C^*(\hat{M}; Z; L^2(\hat{E}))^G)$. Via this continuous path of idempotents, we conclude from (5.4) that

$$[e] - [p_2] = [\bar{e}] - [p_2] = [e_1] - [p_2].$$

By the definition (2.3) of $\text{index}_G^{L^2(\hat{E})}(\hat{D})$, this index equals $[e_1] - [p_2]$. The map ι_* may be inserted here because the entries of e_1 have smooth kernels. \square

5.2 The map $\widetilde{\text{TR}}$

In this subsection, we temporarily return to the general setting of Subsection 2.1. Because Z/G is compact, the equivariant Roe algebra $C^*(Z; L^2(E|_Z))^G$ equals the closure in $\mathcal{B}(L^2(E|_Z))$ of the algebra of bounded operators on $L^2(E|_Z)$ with finite propagation, and G -invariant, continuous kernels

$$\kappa \in \Gamma(Z \times Z, \text{End}(E|_Z)). \tag{5.5}$$

This can be proved analogously to the arguments in Section 5.4 in [18]. We will not need this fact, however, since the operators in $C^*(Z; L^2(E|_Z))^G$ we work with always have continuous kernels.

Let $\chi \in C(X)$ be a cutoff function for the action by G , as in (2.4). Define the map

$$\widetilde{\text{TR}}: C^*(Z; L^2(E|_Z))^G \rightarrow C_r^*G \otimes \mathcal{K}(L^2(E|_Z))$$

by

$$\widetilde{\text{TR}}(\kappa)(h) = T_{(\chi \otimes \chi)h \cdot \kappa},$$

for $h \in G$ and κ as in (5.5). Here $T_{(\chi \otimes \chi)h \cdot \kappa}$ is the operator whose Schwartz kernel is given by

$$((\chi \otimes \chi)h \cdot \kappa)(z, z') = \chi(z)\chi(z')h\kappa(h^{-1}z, z'),$$

for all $h \in G$ and $z, z' \in Z$. (The map $\widetilde{\text{TR}}$ is not a trace, the notation is motivated by Lemma 5.4 below.)

Lemma 5.3 *The map $\widetilde{\text{TR}}$ is an injective $*$ -homomorphism.*

Proof The fact that $\widetilde{\text{TR}}$ is a $*$ -homomorphism follows from direct computations involving G -invariance of κ . It follows from G -invariance of κ that $\kappa = 0$ if $(\chi \otimes \chi)h \cdot \kappa = 0$ for all $h \in G$. □

Let $C_{\text{Tr}}^*(Z; L^2(E|_Z))^G \subset C^*(Z; L^2(E|_Z))^G$ be the subalgebra of operators with kernels κ such that $\widetilde{\text{TR}}(\kappa) \in C_r^*G \otimes \mathcal{L}^1(L^2(E|_Z))$, where \mathcal{L}^1 stands for the space of trace-class operators.

Analogously to Sect. 3.3 of [23], we define

$$\text{TR}(\kappa)(x) := \int_Z \chi(xm)^2 \text{tr}(x\kappa(x^{-1}m, m)) \, dm,$$

for $\kappa \in \Gamma^\infty(\text{End}(E|_Z))^G$ and $x \in G$ for which the integral converges.

Lemma 5.4 *For all $\kappa \in C_{\text{Tr}}^*(Z; L^2(E|_Z))^G$ and $x \in G$,*

$$\text{TR}(\kappa)(x) = \text{Tr}(\widetilde{\text{TR}}(\kappa)(x)).$$

Proof For any G -equivariant operator T on $L^2(E|_Z)$ with smooth kernel $\kappa \in C_{\text{Tr}}^*(Z; L^2(E|_Z))^G$,

and any $x \in G$, the trace property of the operator trace Tr and G -equivariance of T imply that

$$\text{TR}(T)(x) = \text{Tr}(x\chi^2T) = \text{Tr}(\chi xT\chi) = \text{Tr}(\widetilde{\text{TR}}(\kappa)(x)).$$

□

Lemma 5.5 *For all $\kappa \in C_{\text{Tr}}^*(Z; L^2(E|_Z))^G$ such that $\text{Tr} \circ \widetilde{\text{TR}}(\kappa) \in \mathcal{A}$,*

$$\tau_g \circ \text{Tr} \circ \widetilde{\text{TR}}(\kappa) = \text{Tr}_g(\kappa).$$

Proof It is immediate from the definitions that $\text{Tr}_g = \tau_g \circ \text{TR}$. So the claim follows from Lemma 5.4. □

5.3 Two maps from Roe algebras to $C_r^*G \otimes \mathcal{K}$

To apply τ_g to the localised coarse index of an operator, one needs a specific isomorphism (2.1). The key step in the proof of Proposition 5.1 is the fact that two maps from localised Roe algebras to group C^* -algebras tensored with the algebra of compact operators lead to the same result when one applies τ_g . See Proposition 5.6. One of these maps is the one applied in [18] to map the localised equivariant coarse index into the K -theory of a group C^* -algebra.

The other is defined in terms of the map $\widetilde{\text{TR}}$ from Sect. 5.2, and is suitable for computing g -traces.

Let X be a proper, isometric, Riemannian G -manifold, and let $Z \subset X$ be a cocompact subset. Suppose that $Z = G \times_K Y$ for a slice $Y \subset Z$ and a compact subgroup $K < G$. (We comment on how to remove this assumption in Remark 5.7.) Fix a Borel section $\phi : K \backslash G \rightarrow G$. The map

$$\psi : Z \times G \rightarrow G \times K \backslash G \times Y \tag{5.6}$$

given by

$$\psi(gy, h) = (h\phi(Kg^{-1}h)^{-1}, Kg^{-1}h, \phi(Kg^{-1}h)h^{-1}gy)$$

for $g, h \in G$ and $y \in Y$, is G -equivariant and bijective, with respect to the diagonal action by G on $Z \times G$ and the action by G on the factor G on the right hand side of (5.6). We always use the action by G on itself by left multiplication. The map ψ relates the measures $dz dg$ and $dg d(Kg) dy$ to each other, as shown in Lemma 5.2 in [18].

Let $E \rightarrow X$ be a G -equivariant, Hermitian vector bundle. Write

$$H := L^2(K \backslash G) \otimes L^2(E|_Y).$$

Then pulling back along ψ defines a G -equivariant, unitary isomorphism

$$\psi^* : L^2(G) \otimes H \rightarrow L^2(E|_Z) \otimes L^2(G). \tag{5.7}$$

Let ψ_1 and ψ_2 be the projections of ψ onto G and $K \backslash G \times Y$, respectively. Define the map

$$\eta : Z \rightarrow K \backslash G \times Y$$

by

$$\eta(z) = \psi_2(z, e).$$

This induces a unitary isomorphism

$$\eta^* : H \rightarrow L^2(E|_Z).$$

Let $C_{\text{ker}}^*(Z)^G$ be the algebra as in Definition 5.10 in [18], of continuous kernels

$$\kappa_G : G \times G \rightarrow \mathcal{K}(H)$$

with finite propagation, and the invariance property that for all $g, g', h \in G$,

$$\kappa_G(hg, hg') = \kappa_G(g, g'). \tag{5.8}$$

Such a kernel defines an operator on $L^2(G) \otimes H$, which corresponds to an operator on $L^2(E|_Z) \otimes L^2(G)$ via (5.7). This gives a map

$$a : C_{\text{ker}}^*(Z)^G \rightarrow C^*(Z)^G$$

with dense image; see Proposition 5.11 in [18]. We also have an injective $*$ -homomorphism

$$W : C_{\text{ker}}^*(Z)^G \rightarrow C_r^*G \otimes \mathcal{K}(H)$$

with dense image, given by

$$W(\kappa_G)(g) = \kappa_G(g^{-1}, e),$$

for $\kappa_G \in C_{\text{ker}}^*(Z)^G$ and $g \in G$.

There are natural maps

$$\begin{aligned} \varphi &: C^*(Z)^G \rightarrow C^*(X, Z)^G; \\ \varphi_E &: C^*(Z; L^2(E|_Z))^G \rightarrow C^*(X, Z; L^2(E))^G, \end{aligned} \tag{5.9}$$

defined by extending operators by zero outside Z , that induce isomorphisms on K -theory; see Section 7.2 in [18]. Consider the map $\oplus 0$ from (2.6).

Proposition 5.6 *The diagram*

$$\begin{array}{ccc} C^*(X, Z; L^2(E))^G & \xrightarrow{\oplus 0} & C^*(X, Z)^G \\ \varphi_E \uparrow & & \uparrow \varphi \\ C^*(Z; L^2(E|_Z))^G & & C^*(Z)^G \\ \widetilde{\text{TR}} \downarrow & & \downarrow a \\ C_c(G) \otimes \mathcal{K}(L^2(E|_Z)) & & C_{\text{ker}}^*(Z)^G \\ \tau_g \otimes 1 \downarrow & & \downarrow W \\ C_c(G) \otimes \mathcal{K}(L^2(E|_Z)) & & C_c(G) \otimes \mathcal{K}(H) \\ \eta^* \downarrow & & \downarrow \tau_g \otimes 1 \\ \mathcal{K}(L^2(E|_Z)) & \xrightarrow{\eta^*} & \mathcal{K}(H). \end{array} \tag{5.10}$$

commutes in the following sense: the maps a , φ_E and φ are injective, with dense images, and the diagram commutes on the relevant dense subalgebras for the inverses of these maps. More explicitly, if $\kappa \in C^*(Z; L^2(E|_Z))^G$, $\kappa_G \in C_{\text{ker}}^*(Z)^G$ and $\varphi_E(\kappa) \oplus 0 = \varphi \circ a(\kappa_G)$, then

$$\eta^* \circ (\tau_g \otimes 1) \circ \widetilde{\text{TR}}(\kappa) = (\tau_g \otimes 1) \circ W(\kappa_G).$$

Remark 5.7 In general, Z is a finite disjoint union of subsets of the form $Z_j = G \times_{K_j} Y_j$; see [33]. We can generalise Proposition 5.6 to that setting, by viewing operators on $L^2(E|_Z)$ as finite matrices of operators between the spaces $L^2(E|_{Z_j})$, and comparing them with analogous matrices of operators between the spaces $H_j := L^2(K_j \setminus G) \otimes L^2(E|_{Y_j})$.

5.4 Proof of Proposition 5.6

For simplicity, we will prove Proposition 5.6 in the case where E is the trivial line bundle. The general case can be proved analogously.

By definition of the maps (5.9), as in [18], the diagram

$$\begin{array}{ccc} C^*(X, Z; L^2(E))^G & \xrightarrow{\oplus 0} & C^*(X, Z)^G \\ \varphi_E \uparrow & & \uparrow \varphi \\ C^*(Z; L^2(E|_Z))^G & \xrightarrow{\oplus 0} & C^*(Z)^G \end{array}$$

commutes. (This is in fact the only property of these maps that we use here.) For this reason, we disregard the top line in (5.10), and only work with Roe algebras on Z .

Let an element of $C^*(Z; L^2(Z))^G$ be given by a continuous kernel $\kappa : Z \times Z \rightarrow \mathbb{C}$ with finite propagation.

Lemma 5.8 For all $\zeta \in L^2(Z) \otimes L^2(G)$, $g \in G$ and $z \in Z$,

$$((\varphi_E(\kappa) \oplus 0)\zeta)(z, g) = \left(\int_G \widetilde{\text{TR}}(\kappa)(h)(h^{-1}g^{-1} \cdot \zeta(-, gh)) dh \right)(g^{-1}z).$$

In this lemma, $\zeta(-, gh) \in L^2(Z)$, on which G acts via its action on Z .

Proof Consider the map (2.5) in this setting,

$$j : L^2(Z) \rightarrow L^2(Z) \otimes L^2(G).$$

Then $\oplus 0$ is given by mapping operators on $L^2(Z)$ to the corresponding operators on $j(L^2(Z))$ by conjugation with j , and extending them by zero on the orthogonal complement of $j(L^2(Z))$. Let $p : L^2(Z) \otimes L^2(G) \rightarrow j(L^2(Z))$ be the orthogonal projection. Then

$$\varphi_E(\kappa) \oplus 0 = j \circ \varphi_E(\kappa) \circ j^{-1} \circ p. \tag{5.11}$$

One checks directly that for all $\zeta \in L^2(Z) \otimes L^2(G)$ and $z \in Z$,

$$(j^{-1} \circ p)(\zeta)(z) = \int_G \chi(g^{-1}z)\zeta(z, g) dg. \tag{5.12}$$

The lemma can now be proved via a straightforward computation involving (5.11), (5.12), G -invariance of κ , and left invariance of the Haar measure on G . \square

Next, fix $\kappa_G \in C^*_{\text{ker}}(Z)^G$.

Lemma 5.9 For all $\zeta \in L^2(Z) \otimes L^2(G)$, $g \in G$ and $z \in Z$,

$$\begin{aligned} & ((\varphi \circ a)(\kappa_G)\zeta)(z, g) \\ &= \left(\int_G W(\kappa_G)(\psi_1(z, g)^{-1}h\psi_1(z, g))\zeta(\psi^{-1}(h\psi_1(z, g), -)) dh \right)(\psi_2(z, g)). \end{aligned}$$

Proof This is a straightforward computation involving G -invariance of κ_G and right invariance of the Haar measure on G . \square

Lemma 5.10 Let $\eta : X_1 \rightarrow X_2$ be a measurable bijection between measure spaces (X_1, μ_1) and (X_2, μ_2) , such that $\eta^*\mu_2 = \mu_1$. Let $\sigma : X_1 \rightarrow G$ be any map. Define

$$\Psi : C_c(G) \otimes \mathcal{K}(L^2(X_2)) \rightarrow C_c(G) \otimes \mathcal{K}(L^2(X_1))$$

by

$$((\Psi(f)(g))u)(x) = ((\eta^* \circ f(\sigma(x)^{-1}g\sigma(x)) \circ (\eta^{-1})^*)u)(x)$$

for all $f \in C_c(G) \otimes \mathcal{K}(L^2(X_2))$, $g \in G$, $u \in L^2(X_1)$ and $x \in X_1$. Then the following diagram commutes:

$$\begin{array}{ccc} C_c(G) \otimes \mathcal{K}(L^2(X_1)) & \xleftarrow{\Psi} & C_c(G) \otimes \mathcal{K}(L^2(X_2)) \\ \tau_g \otimes 1 \downarrow & & \downarrow \tau_g \otimes 1 \\ \mathcal{K}(L^2(X_1)) & \xrightarrow{\eta^*} & \mathcal{K}(L^2(X_2)). \end{array}$$

Proof This is a straightforward computation, involving G -invariance of the measure $d(hZ_g)$ on G/Z_g . □

Remark 5.11 The map Ψ in Lemma 5.10 is not a homomorphism in general, unless σ is constant.

Applying Lemma 5.10 with $X_1 = Z, X_2 = K \backslash G \times Y$ and $\sigma(z) = \psi_1(z, e)$, we obtain a commutative diagram

$$\begin{CD}
 C_c(G) \otimes \mathcal{K}(L^2(Z)) @<\Psi<< C_c(G) \otimes \mathcal{K}(H) \\
 @V{\tau_g \otimes 1}VV @VV{\tau_g \otimes 1}V \\
 \mathcal{K}(L^2(Z)) @>\eta^*>> \mathcal{K}(H).
 \end{CD}
 \tag{5.13}$$

Proof of Proposition 5.6 As before, fix an element of $C^*(Z; L^2(Z))^G$ given by a continuous kernel $\kappa : Z \times Z \rightarrow \mathbb{C}$ with finite propagation, and $\kappa_G \in C_{\text{ker}}^*(Z)^G$. Suppose that

$$(\varphi \circ a)(\kappa_G) = (\varphi_E(\kappa) \oplus 0) \in C^*(X, Z)^G.
 \tag{5.14}$$

Then Lemmas 5.8 and 5.9, applied with $g = e$, imply that for all $\zeta \in L^2(Z) \otimes L^2(G)$ and $z \in Z$,

$$\begin{aligned}
 & \left(\int_G \widetilde{\text{TR}}(\kappa)(h)(h^{-1} \cdot \zeta(-, h)) dh \right)(z) \\
 &= \left(\int_G \eta^* \circ W(\kappa_G)(\psi_1(z, e)^{-1} h \psi_1(z, e)) \zeta(\psi^{-1}(h \psi_1(z, e), -)) dh \right)(z).
 \end{aligned}
 \tag{5.15}$$

One has for all $z \in Z$ and $h \in G$,

$$\psi(hz, h) = (h \psi_1(z, e), \eta(z)).$$

(Recall that ψ_1 is the projection of ψ onto G .) Hence the right hand side of (5.15) equals

$$\left(\int_G \eta^* \circ W(\kappa_G)(\psi_1(z, e)^{-1} h \psi_1(z, e)) \circ (\eta^{-1})^*(h^{-1} \cdot \zeta(-, h)) dh \right)(z).$$

Therefore, if $\zeta = u \otimes v$, for $u \in L^2(Z)$ and $v \in L^2(G)$, then (5.15) implies that for all $z \in Z$,

$$\begin{aligned}
 & \int_G v(h) \widetilde{\text{TR}}(\kappa)(h)(h^{-1} \cdot u) dh \Big|_z \\
 &= \left(\int_G v(h) (\eta^* \circ W(\kappa_G)(\psi_1(z, e)^{-1} h \psi_1(z, e)) \circ (\eta^{-1})^*)(h^{-1} \cdot u) dh \right)(z).
 \end{aligned}$$

Hence for all $u \in L^2(Z), h \in G$ and $z \in Z$,

$$\begin{aligned}
 (\widetilde{\text{TR}}(\kappa)(h)u)(z) &= \left(\eta^* \circ W(\kappa_G)(\psi_1(z, e)^{-1} h \psi_1(z, e)) \circ (\eta^{-1})^*(u) \right)(z) \\
 &= (\Psi(W(\kappa_G))(h)u)(z).
 \end{aligned}$$

So $\Psi(W(\kappa_G)) = \widetilde{\text{TR}}(\kappa)$, and commutativity of diagram (5.13) implies the claim. □

5.5 Proof of Proposition 5.1

The isomorphism $C^*(X, Z)^G \cong C_r^*G \otimes \mathcal{K}$ used in [18] to identify localised coarse indices with classes in $K_*(C_r^*G)$ is the map

$$W \circ a^{-1} \circ \varphi^{-1},$$

defined on a dense subalgebra and extended continuously. Hence we explicitly have

$$\text{index}_G(\hat{D}) = (W_* \circ a_*^{-1} \circ \varphi_*^{-1})(\text{index}_G^{L^2(E)}(\hat{D}) \oplus 0) \in K_0(C_r^*G). \tag{5.16}$$

Therefore, Lemma 5.2 and Proposition 5.6 (see Remark 5.7) imply that

$$\tau_g(\text{index}_G(\hat{D})) = \tau_g(\widetilde{\text{TR}} \circ (\varphi_E)_*^{-1}([e] - [p_2])).$$

The trace map on the sub-algebra of trace-class operators in $\mathcal{K}(L^2(E|_Z))$ induces the isomorphism

$$K_*(C_r^*G \otimes \mathcal{K}(L^2(E|_Z))) \cong K_*(C_r^*G).$$

Hence Proposition 5.1 follows by Lemma 5.5.

5.6 Proof of Theorem 2.7

Proposition 5.12 *If the operators $e^{-t\hat{D}^2}$ and $e^{-t\hat{D}}\tilde{D}$ and S_0^2 and S_1^2 are g -trace class, then*

$$\text{Tr}_g(S_0^2) - \text{Tr}_g(S_1^2) = \text{Tr}_g(S_0) - \text{Tr}_g(S_1). \tag{5.17}$$

Proof We have $S_0R = RS_1$, and hence

$$\begin{aligned} S_0 - S_0^2 &= S_0(1 - S_0) = RS_1\hat{D}_+; \\ S_1 - S_1^2 &= S_1(1 - S_1) = S_1\hat{D}_+R. \end{aligned}$$

Because $e^{-t\hat{D}^2}$ and $e^{-t\hat{D}}\tilde{D}$ are g -trace class, Lemma 5.3 and Proposition 5.8 in [23] imply that S_0 and S_1 are g -trace class. So the operators $RS_1\hat{D}_+$ and $S_1\hat{D}_+R$ are g -trace class.

By Lemma 3.1,

$$\begin{aligned} S_1\hat{D}_+ &= \varphi_1\tilde{S}_1\tilde{D}_+\psi_1 - \varphi_1\tilde{S}_1\sigma\psi'_1 - \varphi'_1\sigma\tilde{Q}\tilde{D}_+\psi_1 + \varphi'_1\sigma\tilde{Q}\sigma\psi'_1 \\ &\quad - \varphi'_2\sigma Q_C D_{C,+}\psi_2 + \varphi'_2 Q_C \sigma\psi'_2. \end{aligned} \tag{5.18}$$

Since \tilde{S}_1 and $\sigma^{-1}\tilde{S}_1\tilde{D}_+$ are g -trace class by assumption, φ'_j has disjoint support from ψ_j , and all operators occurring are pseudo-differential operators, and therefore have smooth kernels off the diagonal, we find that $\sigma^{-1}S_1\hat{D}_+$ is g -trace class. (And the last four terms on the right hand side of (5.18) have g -trace zero.) And $R\sigma$ is has a distributional kernel, so Lemma 3.4 implies that

$$\text{Tr}_g(RS_1\hat{D}_+) = \text{Tr}_g(R\sigma\sigma^{-1}S_1\hat{D}_+) = \text{Tr}_g(\sigma^{-1}S_1\hat{D}_+R\sigma) = \text{Tr}_g(S_1\hat{D}_+R).$$

Hence (5.17) follows. □

Theorem 2.7 follows from Propositions 4.1, 5.1 and 5.12.

6 Non-invertible D_N

We have so far assumed that the Dirac operator D_N on the boundary N is invertible. We now discuss how that assumption can be weakened to the assumption that 0 is isolated in the spectrum of D_N . The arguments are related to those in Section 6 of [23].

6.1 A shifted Dirac operator

Let $\varepsilon > 0$ be such that $([-2\varepsilon, 2\varepsilon] \cap \text{spec}(D_N)) \setminus \{0\} = \emptyset$. Let $\psi \in C^\infty(\hat{M})^G$ be a nonnegative function such that

$$\begin{aligned} \psi(n, u) &= \begin{cases} u & \text{if } n \in N \text{ and } u \in (1/2, \infty); \\ 0 & \text{if } n \in N \text{ and } u \in (0, 1/4); \end{cases} \\ \psi(m) &= 0 \quad \text{if } m \in M \setminus U. \end{aligned}$$

(Recall that $U \cong N \times (0, 1]$ is a neighbourhood of N in M .)

As in Section 6 of [23], we consider the G -equivariant, odd, elliptic operator

$$\hat{D}_\varepsilon := e^{\varepsilon\psi} \hat{D} e^{-\varepsilon\psi}.$$

The operator \hat{D}_ε is G -equivariant, essentially self-adjoint, odd-graded and elliptic. Its restriction to $\hat{M} \setminus M$ equals

$$\sigma\left(-\frac{\partial}{\partial u} + D_N + \varepsilon\right). \tag{6.1}$$

It therefore satisfies the condition (2.2), and has a well-defined index

$$\text{index}_G(\hat{D}_\varepsilon) \in K_0(C_r^*G).$$

Let a_1 be as in (2.14). Theorem 2.7 generalises as follows.

Theorem 6.1 *Suppose that \hat{D}_ε is g -Fredholm, and that the Schwartz kernels of $e^{-t\hat{D}_\varepsilon^2}$ and $\hat{D}_\varepsilon e^{-t\hat{D}_\varepsilon^2}$ have Gaussian off-diagonal decay behaviour as in (3.8). If either*

- (a) G/Z_g is compact; or
- (b) $G = \Gamma$ is discrete and finitely generated, and (2.13) holds for a $k < \frac{2a_1\varepsilon}{3}$,

then

$$\tau_g(\text{index}_G(\hat{D}_\varepsilon)) = \text{index}_g(\hat{D}_\varepsilon).$$

Conditions for \hat{D}_ε to be g -Fredholm were given in Theorem 6.2 and Corollary 6.3 in [23].

Corollary 2.10 also generalises to this setting. This involves Corollary 6.3 in [23].

6.2 A shifted parametrix

Let $\tilde{\psi}$ be any smooth, G -invariant extension of $\psi|_M$ to the double \tilde{M} of M . As in Sect. 6.2 of [23], we use the operators

$$\begin{aligned} \tilde{D}_\varepsilon &= e^{\varepsilon\tilde{\psi}} \tilde{D} e^{-\varepsilon\tilde{\psi}}; \\ \tilde{Q}_\varepsilon &:= \frac{1 - e^{-t\tilde{D}_{\varepsilon,-}} \tilde{D}_{\varepsilon,+}}{\tilde{D}_{\varepsilon,-} - \tilde{D}_{\varepsilon,+}} \tilde{D}_{\varepsilon,-}; \\ \tilde{S}_{\varepsilon,0} &:= 1 - \tilde{Q}_\varepsilon \tilde{D}_{\varepsilon,+} = e^{-t\tilde{D}_{\varepsilon,-}} \tilde{D}_{\varepsilon,+}; \\ \tilde{S}_{\varepsilon,1} &:= 1 - \tilde{D}_{\varepsilon,+} \tilde{Q}_\varepsilon = e^{-t\tilde{D}_{\varepsilon,+}} \tilde{D}_{\varepsilon,-}. \end{aligned}$$

Let $D_{C,\varepsilon}$ be the restriction of \hat{D}_ε to $N \times (1/2, \infty)$, and let $Q_{C,\varepsilon}$ be the inverse of its self-adjoint closure, restricted to sections of E_- .

Let the functions φ_j and ψ_j be as in Sect. 3.1, with the difference that they change values between 0 and 1 on the interval $(1/2, 1)$ rather than on $(0, 1)$. Set

$$\begin{aligned} R_\varepsilon &:= \varphi_1 \tilde{Q}_\varepsilon \psi_1 + \varphi_2 Q_{C,\varepsilon} \psi_2; \\ S_{\varepsilon,0} &:= 1 - R_\varepsilon \hat{D}_{\varepsilon,+}; \\ S_{\varepsilon,1} &:= 1 - \hat{D}_{\varepsilon,+} R_\varepsilon. \end{aligned}$$

From this point on, the proof of Theorem 6.1 is analogous to the proof of Theorem 2.7. The starting point is that, as in Lemma 3.1,

$$\begin{aligned} S_{\varepsilon,0} &= \varphi_1 \tilde{S}_{\varepsilon,0} \psi_1 + \varphi_1 \tilde{Q}_\varepsilon \sigma \psi'_1 + \varphi_2 Q_{C,\varepsilon} \sigma \psi'_2; \\ S_{\varepsilon,1} &= \varphi_1 \tilde{S}_{\varepsilon,1} \psi_1 - \varphi'_1 \sigma \tilde{Q}_\varepsilon \psi_1 - \varphi'_2 \sigma Q_{C,\varepsilon} \psi_2. \end{aligned}$$

As noted in Sect. 6.2 of [23], the arguments showing that S_0 and S_1 are g -trace class immediately generalise to show that $S_{\varepsilon,0}$ and $S_{\varepsilon,1}$ are g -trace class. Similarly, Propositions 4.1, 5.1 and 5.12 generalise to the current situation, and imply Theorem 6.1. We now use the assumption that the Schwartz kernels of $e^{-t\hat{D}_\varepsilon^2}$ and $\hat{D}_\varepsilon e^{-t\hat{D}_\varepsilon^2}$ have Gaussian off-diagonal decay behaviour, for example to apply a version of Lemma 3.12. This decay behaviour does not follow from bounded geometry of \tilde{M} , because \hat{D}_ε is not a Dirac-type operator as in Sect. 2.4.

Acknowledgements HW was supported by the Australian Research Council, through Discovery Early Career Researcher Award DE160100525, by the Shanghai Rising-Star Program Grant 19QA1403200, and by NSFC Grant 11801178. PH thanks East China Normal University for funding a visit there in 2018.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Camb. Philos. Soc.* **77**, 43–69 (1975)
2. Atiyah, M.F., Segal, G.B.: The index of elliptic operators. II. *Ann. Math.* **2**(87), 531–545 (1968)
3. Atiyah, M.F., Singer, I.M.: The index of elliptic operators. III. *Ann. Math.* **2**(87), 546–604 (1968)
4. Baum, P., Connes, A., Higson, N.: Classifying space for proper actions and K -theory of group C^* -algebras. In C^* -algebras: 1943–1993 (San Antonio, TX, 1993), volume 167 of *Contemp. Math.*, pages 240–291. American Mathematical Society, Providence, RI (1994)
5. Berline, N., Getzler, E., Vergne, M.: Heat kernels and Dirac operators. Grundlehren Text Editions. Springer, Berlin (2004). **(Corrected reprint of the 1992 original)**
6. Bismut, J.-M.: Hypoelliptic Laplacian and orbital integrals. *Annals of Mathematics Studies*, vol. 177. Princeton University Press, Princeton (2011)
7. Carey, A.L., Gayral, V., Rennie, A., Sukochev, F.A.: Index theory for locally compact noncommutative geometries. *Mem. Am. Math. Soc.* **231**(1085), iv+130 (2014)
8. Chandra, H.: Discrete series for semisimple Lie groups. II: explicit determination of the characters. *Acta Math.* **116**, 1–111 (1966)
9. Cheeger, J., Gromov, M., Taylor, M.: Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differ. Geom.* **17**(1), 15–53 (1982)
10. Chen, X., Wang, J., Xie, Z., Yu, G.: Delocalized eta-invariants, cyclic cohomology and higher rho-invariants (2019). [arXiv:1901.02378](https://arxiv.org/abs/1901.02378)
11. Cheng, S.-Y., Li, P., Yau, S.-T.: On the upper estimate of the heat kernel of a complete Riemannian manifold. *Am. J. Math.* **103**(5), 1021–1063 (1981)
12. Connes, A., Moscovici, H.: Cyclic cohomology, the Novikov conjecture and hyperbolic groups. *Topology* **29**(3), 345–388 (1990)
13. Donnelly, H.: Eta invariants for G -spaces. *Indiana Univ. Math. J.* **27**(6), 889–918 (1978)
14. Donnelly, H.: Asymptotic expansions for the compact quotients of properly discontinuous group actions. *Ill. J. Math.* **23**, 485–496 (1979)
15. Farsi, C.: Orbifold η -invariants. *Indiana Univ. Math. J.* **56**(2), 501–521 (2007)
16. Gromov, M., Blaine, L.H., Jr.: Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.* **58**, 83–196 (1984)
17. Guo, H.: Index of equivariant Callias-type operators and invariant metrics of positive scalar curvature. *J. Geom. Anal.* **31**(1), 1–34 (2021)
18. Guo, H., Hochs, P., Mathai, V.: Equivariant Callias index theory via coarse geometry. *Ann. Inst. Fourier* (2021). <https://doi.org/10.5802/aif.3445>; [arXiv:1902.07391](https://arxiv.org/abs/1902.07391)
19. Guo, H., Hochs, P., Mathai, V.: Positive scalar curvature and an equivariant Callias-type index theorem for proper actions. *Ann. K-theory* **6**(2), 319–356 (2021)
20. Guo, H., Hochs, P., Mathai, V.: Coarse geometry and Callias quantisation. *Trans. Am. Math. Soc.* **374**(4), 2479–2520 (2021)
21. Higson, N., Roe, J.: *Analytic K -Homology*. Oxford Mathematical Monographs, Oxford University Press, Oxford (2000)
22. Hochs, P., Song, Y., Tang, X.: An index theorem for higher orbital integrals. *Math. Annalen*, **to appear**. [arXiv:2005.06119](https://arxiv.org/abs/2005.06119)
23. Hochs, P., Wang, B.-L., Wang, H.: An equivariant Atiyah–Patodi–Singer index theorem for proper actions I: the index formula. *Int. Math. Res. Not.* (2021). <https://doi.org/10.1093/imrn/rnab324>. [arXiv:1904.11146](https://arxiv.org/abs/1904.11146)
24. Hochs, P., Wang, H.: A fixed point formula and Harish–Chandra’s character formula. *Proc. Lond. Math. Soc.* **00**(3), 1–32 (2017)
25. Hochs, P., Wang, H.: Orbital integrals and K -theory classes. *Ann. K-Theory* **4**(2), 185–209 (2019)
26. Kağan Samurkaş, S.: Bounds for the rank of the finite part of operator K -theory. *J. Noncommut. Geom.* **14**(2), 413–439 (2020)
27. Leichtnam, E., Piazza, P.: The b -pseudodifferential calculus on Galois coverings and a higher Atiyah–Patodi–Singer index theorem. *Mém. Soc. Math. Fr., Nouv. Sér.* **68**, 123 (1997)
28. Leichtnam, E., Piazza, P.: Spectral sections and higher Atiyah–Patodi–Singer index theory on Galois coverings. *Geom. Funct. Anal.* **8**(1), 17–58 (1998)
29. Leichtnam, E., Piazza, P.: A higher Atiyah–Patodi–Singer index theorem for the signature operator on Galois coverings. *Ann. Global Anal. Geom.* **18**(2), 171–189 (2000)
30. Lott, J.: Higher eta-invariants. *K-Theory* **6**(3), 191–233 (1992)
31. Lott, J.: Delocalized L^2 -invariants. *J. Funct. Anal.* **169**(1), 1–31 (1999)

32. Miščenko, A.S., Fomenko, A.T.: The index of elliptic operators over C^* -algebras. *Izv. Akad. Nauk SSSR Ser. Mat.* **43**(4), 831–859 (1979)
33. Palais, R.S.: On the existence of slices for actions of non-compact Lie groups. *Ann. Math.* **2**(73), 295–323 (1961)
34. Piazza, P., Schick, T.: Rho-classes, index theory and Stolz' positive scalar curvature sequence. *J. Topol.* **7**(4), 965–1004 (2014)
35. Piazza, P., Posthuma, H., Song, Y., Tang, X.: Higher orbital integrals, rho numbers and index theory (2021). [arXiv:2108.00982](https://arxiv.org/abs/2108.00982)
36. Puschnigg, M.: New holomorphically closed subalgebras of C^* -algebras of hyperbolic groups. *Geom. Funct. Anal.* **20**(1), 243–259 (2010)
37. Ramachandran, M.: Von Neumann index theorems for manifolds with boundary. *J. Differ. Geom.* **38**(2), 315–349 (1993)
38. Roe, J.: Positive curvature, partial vanishing theorems and coarse indices. *Proc. Edinb. Math. Soc.* (2) **59**(1), 223–233 (2016)
39. Song, Y., Tang, X.: Higher orbit integrals, cyclic cocycles, and K -theory of reduced group C^* -algebra. [arXiv:1910.00175](https://arxiv.org/abs/1910.00175)
40. Wang, B.-L., Wang, H.: Localized index and L^2 -Lefschetz fixed-point formula for orbifolds. *J. Differ. Geom.* **102**(2), 285–349 (2016)
41. Weinberger, S., Guoliang, Y.: Finite part of operator K -theory for groups finitely embeddable into Hilbert space and the degree of nonrigidity of manifolds. *Geom. Topol.* **19**(5), 2767–2799 (2015)
42. Xie, Z., Guoliang, Y.: Positive scalar curvature, higher rho invariants and localization algebras. *Adv. Math.* **262**, 823–866 (2014)
43. Xie, Z., Yu, G.: Delocalized eta invariants, algebraicity, and K -theory of group C^* -algebras. *Int. Math. Res. Not.* **2021**(15), 11731–11766 (2021)
44. Zeidler, R.: Positive scalar curvature and product formulas for secondary index invariants. *J. Topol.* **9**(3), 687–724 (2016)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.