THE STRUCTURE OF LIGHT NUCLEI

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STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University. To the best of my knowledge and belief the thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

(T. K. LIM)

ABSTRACT

This thesis presents in detail two methods for the systematic study of the structure of light nuclei. Part I begins with a critical review of other work so that our study may be seen in its correct perspective. Chapter two contains a complete orthogonal classification of the S-state wavefunctions which are present in the ground states of the triton and the alpha particle, and is followed by our treatment of the Schrodinger equations of the three- and four-nucleon systems. Including spin explicitly in the nuclear model of Green, we have solved these equations exactly. The results that accrue from this improvement to Green's model, although interesting in themselves, are only meaningful for shallow and long-ranged Gaussian potentials so these investigations have perforce been limited to the nuclei in the 1s shell.

In part 2, a velocity dependent potential of exponential form, which gives the correct deuteron binding energy and good fits to the two-body scattering data, is determined. By using this interaction, the binding energies of the trinucleon and

the alpha particle are evaluated through a variational-type calculation. The parameters of the trial functions which include short-range two-body correlations, are obtained by fitting the r.m.s. radius and the form factor of each nucleus. The quality of these wavefunctions is also tested by the expectation values of a number of operators in the three- and four-body systems. From our results we are able to conclude that the soft-core nucleon-nucleon interaction is more than adequate as a substitute for the repulsive hard core potentials suggested by high-energy two-body scattering data. It is also obvious that product wavefunctions of analytic form are sufficiently flexible and give good representations of the true eigenfunctions. Other possible conclusions are that the S' state in the triton is unlikely to be present with more than 1·5% probability in the ground state, the Serber and Biel force mixtures are favoured in nuclear photo-disintegration calculations and the sum-rules of Bethe and Levinger are essentially correct.

We conclude with a brief report of the application of our methods to the alpha particle model of ^{12}C and to the trineutron and suggestions for further work.

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CHAPTER 1. INTRODUCTION

The study of systems involving few nucleons has played an important role in the history of nuclear physics. In this field of work the determination of the nucleon-nucleon interaction has held the most interest. As early as 1933 Wigner[1] showed on comparing the binding energies of hydrogen and helium isotopes that nuclear forces should have a short-range and great strength within that range. In 1935 Thomas[2] found that the nuclear interaction cannot have zero range; the binding energy of the triton goes to infinity if the range of the two-body force tends to zero. All these features of nuclear forces had been revealed from the study of the few nucleon systems long before reliable nucleon-nucleon scattering measurements could be performed.

The continued absence of a fundamental theory of nuclear forces has led to attempts to describe the nucleon-nucleon interaction in terms of a potential, a concept borrowed from classical physics and atomic quantum theory. Unfortunately a proliferation of such phenomenological potentials exists and since

the two-nucleon system at present does not provide sufficient criteria to select the unique potential, the known properties of the nuclei next in complexity to the deuteron have once again been suggested as possible tests in the selection of such a true internucleon force. If and when such a satisfactory description of the two-nucleon interaction is attained from the large amount of experimental data still being collected, an important adjunct to investigation of systems containing three, four and more nucleons would be the estimation of the presence and importance of parity-dependent and many-nucleon forces.

In the study of the light nuclei the essential problem is the solution of the Schrodinger equation for the system of interacting particles of approximately equal mass. On the basis of the two-nucleon interaction model, the general Schrodinger equation cannot be solved analytically if a phenomenological potential consistent with most two-body data is used. A direct numerical calculation appears beyond reach of the fastest computers today so one must resort to a number of obvious approximations. One simplification is the reduction of the complete few-nucleon wave-

function to more tractable form. In their centre-of-mass coordinates, the triton wave equation is a partial differential equation in six independent variables whilst the alpha particle wave equation has nine such variables. A systematic classification of the angular wave-functions present in the ground states of the two nuclei based on the absolute and auxiliary quantum numbers of these states was proposed by Derrick and Blatt[3], Cohen and Willis[4-6] and Beam[7]. Such a classification has reduced the triton problem to 16 coupled differential equations in three variables and the alpha particle wave equations to 32 coupled equations in six variables. No further simplification is possible. However there is some evidence to suggest that only a small number of the classified states are important and the others can be summarily dropped from consideration without fear that their absence will affect the conclusions of any calculation. Even so the substantially reduced coupled equations are still difficult to solve explicitly and either of two popular procedures must be made for any further progress. The first involves the assumption of an equivalent interaction operator that is sufficiently

simple so that a straightforward solution follows; the second retains the complicated form of the two-nucleon potential and some variational principle is employed. The advent of faster computers with large storage capacities will help matters but until then these two methods will continue to enjoy a large measure of popularity.

The exact-solution method, which is the less practicable of the two, has a number of distinct advantages: it allows an exact solution which exposes features of the problem not revealed by the variational method. These features lend physical insight into the study of the few-nucleon systems and may conceivably be necessary in the variational calculations. The use of modern phenomenological potentials which have repulsive cores and non-central components require detailed and tedious variational calculations. The most ambitious work in this sphere has been that of Blatt and his co-workers[8-12] but even they have been forced to admit that their complicated wave-functions are still not sufficiently flexible to ensure convergence. It seems that the semi-realistic potentials may not have had their day. Familiarity

in their use may yield valuable information about the structure of the trial functions necessary for the more sophisticated treatments, which as of now occupy an inordinate amount of time. Besides, it appears unlikely that their essential results will have to be thrown overboard when the elaborate variational calculations are performed because these simpler phenomenological potentials may resemble realistic forces in their effects on the main properties of the light nuclei. The flow of papers on the few-nucleon systems that was once a trickle is now a constant stream as the emphasis in research is shifted from the two-body system to the nuclei after the deuteron. In the next section we present a summary of some of the major work done by other authors so as to place our own work in its correct perspective.

1.1 Critical Review of Work on the Few-Nucleon Problem

The first full classification of the angular momentum-isospin functions which can be present in the ground state wavefunctions of the three-nucleon

system was given by Derrick and Blatt[3]. Using Euler angle wavefunctions, they were able to write the total wavefunction as a linear superposition of ten states, each of which had a definite value of total orbital angular momentum, spin angular momentum and a definite permutation symmetry. Cohen and Willis[4-6] and Beam's[7] approach was a systemisation of the operator technique used by Gerjuoy and Schwinger[13] and Sachs[14]. Their method gave eleven and fourteen orthogonal states respectively for the triton and alpha particle ground states. Although there is no completely reliable estimate of the relative importance of the various states, magnetic moment measurements indicate the S state of total spatial symmetry to be predominant. (This is the only state of any consequence present if the nucleon-nucleon potential is central and has Majorana and Wigner exchange character). The next most important states will be the so-called S' state (the S state of mixed symmetry) and the D states.

Early work on the triton and alpha particle binding energies were done by Blatt and Weisskopf[15], Irving[16], and Rarita and Present[17] using static

central internucleon potentials which are considered outmoded nowadays. The binding energies were sensitive to the shape of the two-nucleon potential; the more realistic attractive Yukawa and exponential wells gave rise to overbound nuclei. These variational calculations were carried out with the help of regular trial functions which were totally symmetric and which had a single parameter determined by the Rayleigh-Ritz variational principle or by fitting the Coulomb energy of ^3He. (The latter was evaluated as a first order perturbation term in the Hamiltonian of ^3H). The trial functions were chosen to approximate the true wavefunctions and their parameters had no direct connection with the parameters of the nucleon-nucleon potentials. Subsequently an equivalent two-body method was introduced by Feshbach and Rubinow[18] and Morpurgo[19] in which the essential approximation was to give the wavefunctions some particular forms. Thus for the triton, Feshbach and Rubinow assumed the wavefunction to be dependent on a particular symmetric combination of the three interparticle separations but the functional form of the approximate wavefunction was found by solving a differential equation obtained

from a variational principle formulation of the full problem. The solution gave the best trial function for the variable chosen. An improvement to the method in which the variable could be any linear combination of the three interparticle distances was developed recently by Bhaduri et al.[20]. Although these two-body methods are not restricted to attractive potentials, their inability to include two-body correlations is a serious limitation to accuracy when other interactions are considered.

Recent high energy two-body scattering data have suggested the presence of a strong repulsion in the nuclear force at small interparticle distances. This repulsion has generally been represented as an impenetrable hard core in the nucleon-nucleon potential. Such a hard core presents difficulties when one tries to solve the Schrodinger equation. In the variational calculation, the trial function must vanish within the hard-core radius and in the region just outside of the core where the potential is required to be strongly attractive, the wavefunction has to be chosen with extreme care. Here the absolute values

of the kinetic and potential energies are orders of magnitude larger than the total energy and small defects in the function in this area have a deleterious effect on the local total energy. Thus if the wavefunction fails to be a faithful reproduction of the exact wavefunction, the upper bound on the binding energy will be so far away from the eigenvalue as to be of little practical use. The first successful attempt to include hard core potentials in a variational calculation of the triton binding energy was made by Ohmura et al.[21]. Assuming a symmetric radial function which was a product of functions of each interparticle separation, they were able to evaluate all the expectation values analytically. Their results yielded a core radius of about 0·3 fermis, and a value of the Coulomb energy of ^{3}He that was smaller and nearer to the expected magnitude than were obtained from the calculations with central potentials. The next major series of calculations on the three- and four-nucleon systems using hard-core potentials were carried out by Tang et al.[22-24]. Their analytic functions were of product form; each two-particle function

was found following the proposal of Austern and Iano[25]. In the region just outside of the core the function chosen is the solution of a two-body differential equation where the nucleon-nucleon potential is used and the energy is a variational parameter. This function is then matched onto an asymptotic function of form suggested by Pappademos[26]. The calculations are essentially numerical and all integrals arising in the evaluation of the matrix elements have been determined by Monte Carlo methods. The 'equivalent two-body method' of Feshbach and Rubinow can be successfully extended to handle potentials that contain hard cores. If the trial function is assumed to be of product form the triton equation can be cast into a form where the Euler-Lagrange equations can be solved iteratively. This method was defined by Delves and Derrick[27], Bodmer and Ali[28], and Murphy and Rosati[29]. Regrettably, the iterative procedure requires prior knowledge of the energy in the two-body integro-differential equation; otherwise, the iterated function misbehaves at large interparticle separations. Also a considerable amount of computer time is necessary because of the time-consuming pro-

cedures for numerical differentiation, interpolation and integration of second order differential equations. In fact Kok and Wageningen[30,31], in their exhaustive work on the triton problem, believe that it is perhaps more worthwhile to retain and improve the Austern-Iano method which under some circumstances gives the same functions as the 'equivalent two-body method'.

Qualitatively the effect of the hard-core is to make the nucleon potential weaker at higher energies and it is conceivable that one can replace successfully the hard core now present in most of the better nucleon-nucleon potentials, by a velocity dependent potential[32,33]. For example, in any scattering problem a central static potential with hard core can be transformed by means of a unitary transformation to a physically equivalent problem containing velocity dependent forces[34]. Green[35], Levinger[36], Rojo et al.[37,38] Herndon et. al.[39] and Lim[40] have been able to derive velocity dependent potentials of exponential and Gaussian forms that yield excellent fits to most two-body data. Calculations involving

velocity dependent potentials do not have the unpleasant features present in the hard-core calculations. The velocity dependent potential is easier to handle mathematically; it does not give infinite matrix elements and it permits the use of the perturbation treatment of Euler in solving the nuclear many-body problem. A number of authors[39-43] have investigated the feasibility of introducing the velocity dependent interaction into the few-nucleon problem. Their results have proved most encouraging.

When non-central components are present in the nuclear force, states other than the S state assume some importance and must be included in the calculations. Feshbach and Pease[44,45], Abraham, Cohen and Roberts[46], Irving[47] and Mukherjee[48] amongst others have evaluated the binding energies of the triton and alpha particle when the nuclear interaction was a mixture of central-attractive and tensor forces. The results of these authors are an improvement on those of the simple central potentials but are nevertheless too inconclusive. A great disadvantage is that in these works the central forces are still attractive. Recently modern phenomenological potentials[49-53]

have been constructed which have tensor, spin-orbit, quadratic spin-orbit components and hard cores. Blatt et al.[8-12] have performed elaborate and tedious variational calculations using these potentials but computational errors, the inherent difficulty of the problem and the feeling that the trial functions employed were not sufficiently complete to ensure convergence, have not allowed them to draw any precise conclusions about their results. Their list of beliefs in reference 11, engendered by their intimate knowledge of the three-nucleon system include information that has already been derived from studies with the semi-realistic potentials.

On the premise that the correct form of the interaction between nuclear particles is not known and therefore attempts at investigating interactions of all types are equally justified, Yamaguchi[54], Mitra et al.[55-61] and Kharchenko and Sitenko[62] have proposed a completely different approach to the problem. In their work the local potential is replaced by a non-local separable potential. Such a potential, although its use would be considered

unjustified if only because field theoretical potentials are local, has certain mathematical advantages and besides satisfying the usual requirements of translational and time reversal invariance also produces nuclear saturation. It renders the two-body problem exactly soluble and provides a considerable reduction of the three-nucleon problem to an effective two-body problem which is easily solved numerically. Results from using this form of interaction have had mixed success. Some experimental data are closely reproduced but other results have conflicted violently with variational calculations and with experiment[63-66]. The trineutron is now acknowledged to be unbound but the separable calculations of Mitra and Bhasin[63] indicate that this nucleus can be bound by 1 MeV. It would appear that there is a fundamental difference between the variational calculation and the separable potential approach. The work of Fuda[67] has also revealed that some of the equations obtained by Bhakar[58] when tensor terms are included in the potential are in error. This casts doubts on the numerical results of the Indian group and may be responsible for the discrepancies with experiment. The separable method could still

be rehabilitated.

In similar spirit to the separable method are the exact calculations of Kalos[68], Baker et. al.[69], Banville[70], Simonov and Badalyan[71] and Green[72]. The latter two excite more than passing interest. Simonov's method of six-dimensional angular harmonics has been tested with a square well central potential. The results are good and point out that application of the method to other potentials may be possible. Green's method is a rigorous quantal treatment of the many-body problem in which actual two-body potentials are replaced by oscillator-type potentials, the difference being treated as a perturbation.

Although the ability to yield the experimental binding energy has always been regarded as the most significant indication of the flexibility of any wavefunction, the energy as derived from a variational calculation cannot be taken as the sole criterion for faithfulness of the trial function, since there are uncertainties in the nucleon-nucleon potential. Other properties of the bound state must be called to test the quality of the function. These parameters of the

ground state include the r.m.s. radius, the charge and magnetic form factors, the Coulomb energy, the photodisintegration cross-sections, the muon capture rate and the Panofsky ratio (the ratio of the probabilities of the reaction processes $\pi^- + A_N^Z \rightarrow \pi^o + A_{N+1}^{Z-1}$) and $\pi^- + A_N^Z \rightarrow \gamma + A_{N+1}^{Z-1}$). For the triton other quantities of interest are the ^{16}O $(^3H, p)^{18}O$ reaction cross-section and the percentage of the various angular states. The complexity of some of the wavefunctions and potentials used in the variational calculations have seriously hindered their application to the evaluation of these parameters. Thus the photodisintegration cross-sections (integrated and bremsstrahlung-weighted)[73-80], the muon capture rates[81-83] and the Panofsky ratios[84] have only been determined in a small number of calculations. The photodisintegration cross-section of the trinucleon can be found explicitly or through the use of the sum-rules of Bethe and Levinger[85]. These calculations as shown by Fetisov[86], Gibson[87] and Lim[77] are sensitive to the forms of the trial functions and especially their asymptotic behaviour. The other variables are less sensitive to the trial

functions and have been given within a few percent of the experimental values by all reasonable trial functions.

1.2 Outline of the Contents

An exact solution of a problem (even if it has been simplified) has certain advantages. Since the actual problem is usually insoluble or involves difficult calculations, the explicit solution serves as exploratory work to develop a feeling for the intricacies of the problem. Thus the n-body Schrodinger equation has been solved exactly by Green using the assumption of a two-body Gaussian potential to which is fitted a harmonic oscillator interaction. With this model it has been possible to study the structural properties of the light nuclei in detail and various features of the collective states of nuclei, which are completely obscured in the independent particle theories, have been studied without approximation. The pair distribution function can be determined for each pair of nucleons and the momentum distribution can be found for each nucleon. In this thesis we have made an obvious improvement to the

model by taking spin into account in a more exact fashion. The three-nucleon system is again exactly soluble and although attempts to do the same for the alpha particle were unsuccessful, an approximation for the four-nucleon wavefunction did allow for a solution. The exact wavefunctions resemble strongly the Irving functions whilst the approximate solutions were similar to the polarised Gaussians of Aranoff[88]. An unexpected result is that values of the binding energy from the approximate calculation are the same as those obtained from a variational treatment with the approximate wavefunction as trial function. The model, although interesting, is nevertheless of somewhat questionable validity. For meaningful results the interaction potential has to be shallow and long-ranged, features which are now rejected in any reasonable two-nucleon potential. Of course it can be argued that the nucleon-nucleon interaction in the bound state differs from that in the free state but there is overwhelming evidence to the contrary.

The second portion of the thesis is a study of the feasibility of replacing the repulsive core interaction by a velocity dependent potential in

investigations of the few-nucleon systems and the necessary structure for the trial radial functions to accommodate such an exchange. Tang and Herndon[39] and Lovitch and Rosati[41] have shown that both forms of interaction give acceptable values of the binding energies and sizes of the three- and four-nucleon systems whilst Kopaleishvili and Machabeli[89] found in π^- and γ absorption reactions in ^{4}He that a comparison with experimental data indicates that the wavefunction corresponding to the velocity dependent potential yields a better fit to the data as a whole for all the considered processes. In the choice of radial functions the elaborate functions of Blatt et al. are more detailed than is necessary for our purpose so we limit ourselves to more tractable forms. These forms suggest themselves from a consideration of the physical characteristics of the nuclei. Blatt and his co-workers have found that wavefunctions which yield the better values of the triton binding energy also give the better fits to the r.m.s. radius and the form factors. Conversely, it may be expected that a properly chosen trial function with parameters determined by fitting the r.m.s. radius and the form factor

will give the more correct binding energy and will be an adequate representation of the true function. Our radial functions are selected following such a procedure and also have the virtue of the right close-in and asymptotic behaviour.

Our functions and the velocity dependent potential are tested in the evaluation of other nuclear parameters. Our results allow us to come to some conclusions about the role of exchange forces in the nuclear interaction, the importance of two-body short-range and asymptotic correlations in the wavefunctions, the percentage of the S' state in the triton, the nuclear sizes, the possible breakdown of charge symmetry in the nuclear force, and the overall correctness of the Bethe-Levinger sum-rules for nuclear photodisintegration.

In chapter two we present a complete classification of the wavefunctions for the S state of the triton and the alpha particle for easy reference. This work is not original but is a concise and clear presentation of Derrick and Blatt, Cohen and Willis and Beam's papers.

Chapter three contains the exact and approximate solutions of the three-nucleon Schrodinger equation when the two-nucleon interaction is a spin-dependent central Gaussian and the model of Green is used. The problem is reduced to one in relative canonical coordinates which allows for a simple solution of the coupled differential equations. The approximation in the solution of the three-body system is also made in chapter four where we treat the Schrodinger equation of the alpha particle. As in the triton problem we use our wavefunctions to evaluate the r.m.s. radius and the form factor of the nucleus.

In chapters five/six and seven we derive a corrected form of the velocity dependent potential of Srivastava. Three trinucleon trial functions are constructed which are then utilised in the determination of the binding energy and other properties of the three-nucleon system.

The alpha particle is considered again in chapter eight. Trial functions in the 'variational' calculation with the velocity dependent potential, do not have product form but are two-parameter

Irving functions. The properties of the four-body system are studied using these functions.

Chapter nine contains two applications of our methods to the alpha particle model of C^{12} and the trineutron. We discuss possible treatment of the quark model of the baryon by both exact and variational methods.

Our conclusions and discussion of the overall accomplishments of this thesis are in chapter ten. Suggestions for further work are considered also.

CHAPTER 2

ORTHOGONAL CLASSIFICATION OF THE THREE- AND FOUR-NUCLEON WAVEFUNCTIONS

In attempting to solve a problem involving several nucleons, one is confronted by various restrictions on the forms of the wavefunctions representing the system under consideration. These restrictions, imposed by the symmetries of the problem, require that the nuclear wavefunctions have definite total angular momentum and parity and must be antisymmetric under the combined interchange of spatial, spin and iso-spin coordinates of any pair of nucleons. For the light nuclei, since iso-spin mixing due to the Coulomb interaction is relatively small, the eigenfunctions must also have definite iso-spin. It follows then that for an exact treatment of these symmetries, it will be a great simplification if the many-nucleon wavefunctions can be separated into four parts: a spin part, an iso-spin part, an orbital angular momentum part and an internal spatial part. For two nucleons

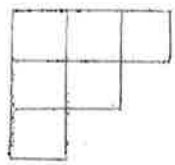
there is no difficulty in carrying out such a separation so that summation over spin and iso-spin leaves relations involving the internal coordinates only. Thus one finds the well-known singlet and triplet states of spin or iso-spin with symmetric and antisymmetric space states, since the dependence on each type of coordinate can only be symmetric or antisymmetric. The construction of a total wavefunction that satisfies the generalised Pauli Principle simply requires multiplication of three antisymmetric functions or the multiplication of one antisymmetric and two symmetric functions.

When the number of nucleons is increased to three or four, the complication in such a procedure climbs considerably. For example in the three-nucleon system, we do not deal with only two types of symmetry but a new symmetry associated with a two-dimensional representation of the permutation group appears. This symmetry introduces more involved ways of forming functions that are totally antisymmetric. As the number of nucleons

rises further, more symmetries are introduced and it becomes apparent that a systematic approach to the problem of constructing eigenfunctions of the few-body systems becomes desirable. For this to be possible, we need to understand the properties of the symmetric group S_N.

2.1 The Symmetric Group S_N.

The group of all permutations of N particles is called the symmetric group S_N and has N! elements. Each irreducible representation of this group may be characterised by a partition $[f_1 f_2 f_3 \cdots\cdots f_k]$ of N into positive integers f_i which satisfy $f_1 \geq f_2 \geq f_3 \geq \cdots \geq f_k$ and $f_1 + f_2 + f_3 + \ldots f_k = N$. Such a partition is conveniently described by a Young Tableau consisting of k contiguous rows of squares, the k-th row containing f_k contiguous squares and such that, looking from top to bottom, no row overjuts the preceding row. Thus the partition [321] is



and the total number of irreducible representations of S_N, corresponds to the number of ways of partitioning N.

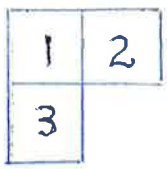
The dimension N_k of the irreducible representation corresponding to the partition $[f_1 f_2 f_3 \ldots . f_k]$ is given by the relation

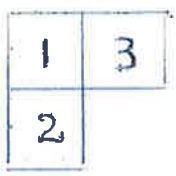
$$N_k = N! \prod_{i<j=1}^{k} (h_i - h_j) \Big/ \prod_{i=1}^{k} h_i \qquad (1)$$

where

$$h_m = f_m + k - m \quad (m=1,2,\ldots .k) \qquad (2)$$

and is in fact equal to the number of ways of numbering in the squares of the partition such that the numerals increase from left to right and from top to bottom. For example [21] has dimension $N_2 = 2$ and the two numbered tableaux are



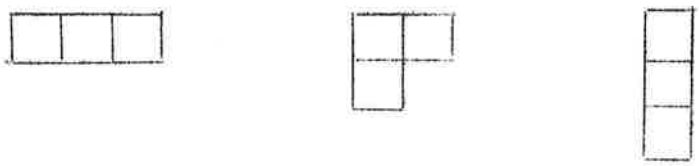


The procedure for determining the representation matrices of the interchange operator P_{ij} and the methods for decomposing the direct product of any number of representations, are beautifully explained

in Hamermesh's book[90]. Matrices and rules for decomposition relevant to the three- and four-nucleon systems are collected together in Tables 1 and 2.

2.2 The Triton Wavefunctions

The group S_3 has three irreducible representations: the one-dimensional symmetric S, the two-dimensional mixed symmetric MS and the one-dimensional antisymmetric A. These are represented respectively by the partitions [3], [21], [111] and the Young Tableaux



The triton ground state has quantum numbers $J^{\pi} = \frac{1}{2}^+$, $T = \frac{1}{2}$. Since the three nucleons can have a maximum spin of $\frac{3}{2}$, the most general wavefunction with these quantum numbers is a linear superposition of $^2S_{\frac{1}{2}}$, $^2P_{\frac{1}{2}}$, $^4P_{\frac{1}{2}}$ and $^4D_{\frac{1}{2}}$ functions. As we shall be dealing exclusively with central forces we shall need the $^2S_{\frac{1}{2}}$ functions only and hence the other states will

TABLE 1.

REPRESENTATION MATRICES AND PRODUCT DECOMPOSITION FOR S_3

INTERCHANGE OPERATOR	REPRESENTATION			
	[3]	[21]		[111]
P_{12}	1	-1	0	-1
		0	+1	
P_{13}	1	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1
		$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	
P_{23}	1	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	-1
		$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	

$$
\begin{aligned}
[3] \otimes [3] &= [3] \\
[3] \otimes [21] &= [21] \\
[3] \otimes [111] &= [111] \\
[21] \otimes [21] &= [3] \oplus [21] \oplus [111] \\
[21] \otimes [111] &= [21] \\
[111] \otimes [111] &= [3]
\end{aligned}
$$

TABLE 2.

REPRESENTATION MATRICES AND PRODUCT DECOMPOSITION FOR S_4

INTERCHANGE OPERATOR	REPRESENTATION					
	[4]	[31]	[22]	[211]	[1111]	
P_{12}	1	$\begin{matrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{matrix}$	$\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{matrix}$	-1	
P_{13}	1	$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{matrix}$	$\begin{matrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{matrix}$	$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{matrix}$	-1	
P_{14}	1	$\begin{matrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{matrix}$	$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{matrix}$	-1	

$$
\begin{aligned}
[4] \otimes [4] &= [4] \\
[4] \otimes [31] &= [31] \\
[4] \otimes [22] &= [22] \\
[4] \otimes [211] &= [211] \\
[4] \otimes [1111] &= [1111] \\
[31] \otimes [31] &= [4] \oplus [31] \oplus [22] \oplus [211] \\
[31] \otimes [22] &= [31] \oplus [211] \\
[31] \otimes [211] &= [31] \oplus [22] \oplus [211] \oplus [1111] \\
[31] \otimes [1111] &= [211] \\
[22] \otimes [22] &= [4] \oplus [22] \oplus [1111] \\
[22] \otimes [211] &= [31] \oplus [211] \\
[22] \otimes [1111] &= [22] \\
[211] \otimes [211] &= [4] \oplus [31] \oplus [22] \oplus [211] \\
[211] \otimes [1111] &= [31] \\
[1111] \otimes [1111] &= [4]
\end{aligned}
$$

be omitted from this classification. With the total orbital angular momentum L equal to zero, the $^2S_{\frac{1}{2}}$ functions will have no dependence on the Euler angles and the orbital angular momentum part of the nuclear wavefunctions will be a simple constant which can be dropped. From the exclusion principle, any system of N spin $\frac{1}{2}$ particles can have spin functions represented by Young Tableaux not having more than two rows. Thus if the spin quantum number is S, the spin eigenfunction has permutation symmetry described by the partition [N/2 + S, N/2 - S]. For the triton $S = \frac{1}{2}$ and hence the spin functions must correspond to the two-dimensional irreducible representation MS. The two spin functions which are the basis functions defining this representation are chosen to be

$$\chi' = \frac{1}{\sqrt{2}} (\alpha_1\beta_2 - \alpha_2\beta_1)\alpha_3 \qquad (3)$$

$$\chi'' = \frac{1}{\sqrt{3}} \sigma_1 \cdot \sigma_3 \chi' \qquad (4)$$

where the components of σ_j are the Pauli spin matrices for nucleon j. The iso-spin functions span a similar [21] representation and are defined as for the spin

functions, with iso-spin vectors replacing the spin vectors. The complete spin-isospin functions therefore span the basis of the direct product of the representation [21] with itself. From Table 1, these are the representations [3], [21], [111]. The basis functions are then

$$A \quad \chi'' \eta' - \chi' \eta'' \tag{5}$$

$$MS \quad \chi' \eta'' + \chi'' \eta' \quad , \chi' \eta' - \chi'' \eta'' \tag{6}$$

$$S \quad \chi' \eta' + \chi'' \eta'' \tag{7}$$

Since the generalised Pauli Principle requires that the internal spatial functions span an irreducible representation that is adjoint to the spin-isospin functions, it follows that the complete S-state of the triton ground state can be written as

$$\Psi(^2S_{\frac{1}{2}}) = \Phi_1^t + \Phi_2^t + \Phi_3^t \tag{8}$$

where

$$\Phi_1^t = \Phi^a \psi^s = (\chi'' \eta' - \chi' \eta'') \psi^s \tag{9}$$

$$\Phi_2^t = \Phi' \psi'' + \Phi'' \psi' = (\chi' \eta'' + \chi'' \eta') \psi'' + (\chi'' \eta'' - \chi' \eta') \psi' \tag{10}$$

$$\Phi_3^t = \Phi^s \psi^a = (\chi' \eta' + \chi'' \eta'') \psi^a \tag{11}$$

The spin-isospin functions are orthogonal in spin-isospin space so Φ_1^t, Φ_2^t and Φ_3^t must be mutually orthogonal. The internal functions ψ^s, ψ', ψ'' and ψ^a can only be functions of the interparticle distances.

2.3 The Alpha Particle Wavefunctions

The alpha particle ground state has quantum numbers $J^\pi = 0$, $T = 0$, and in the absence of non-central forces would be pure $^1S_0$. In order to find a parametrised wavefunction consistent with the symmetries of the four-nucleon system we need the properties of the group S_4. The group S_4 has five irreducible representations: the one-dimensional symmetric S, the three-dimensional mixed-symmetric MS, the two-dimensional mixed M, the three-dimensional mixed-antisymmetric MA and the one-dimensional antisymmetric A. These are represented respectively by the partitions [4], [31], [22], [211], and [1111]. As with the triton wavefunctions the S-state has no dependence on the Euler angles i.e. the wavefunctions are independent of the orientation in space of the

tetrahedron made by the six interparticle separations. The totally antisymmetric wavefunctions will be products of spin, iso-spin and spatial parts only.

Since S = 0, the spin functions correspond to the two-dimensional representation M and are chosen to be

$$\chi' = \tfrac{1}{2}(\alpha_1\beta_2 - \alpha_2\beta_1)(\alpha_3\beta_4 - \alpha_4\beta_3) \qquad (12)$$

$$\chi'' = \frac{1}{\sqrt{3}} \sigma_1 \cdot \sigma_3 \chi' \qquad (13)$$

The iso-spin functions span a similar [22] representation and assume a form identical to that for the spin functions. The complete spin-isospin functions therefore span the basis of the direct product of two [22] representations. From Table 2 these are the representations [4], [22], [1111]. The basis functions are then

$$A \quad \chi''\eta' - \chi'\eta'' \qquad (14)$$

$$M \quad \chi'\eta'' + \chi''\eta' \quad , \quad \chi'\eta' - \chi''\eta'' \qquad (15)$$

$$S \quad \chi'\eta' + \chi''\eta''$$

These are identical to those for the triton spin-isospin functions and suggest that there are only three possible

$^1S_0$ ($S = T = 0$) functions. These are

$$\Psi(^1S_0) = \Phi_1^\alpha + \Phi_2^\alpha + \Phi_3^\alpha \tag{17}$$

where

$$\Phi_1^\alpha = (\chi'' \eta' - \chi' \eta'')\psi^s \tag{18}$$

$$\Phi_2^\alpha = (\chi' \eta'' + \chi'' \eta')\psi'' - (\chi' \eta' - \chi'' \eta'')\psi' \tag{19}$$

$$\Phi_3^\alpha = (\chi' \eta' + \chi'' \eta'')\psi^a \tag{20}$$

CHAPTER THREE

AN EXACTLY SOLUBLE TRITON MODEL

In the nuclear model proposed by Green[72], the nucleus is considered to be made up of n nucleons interacting through central-attractive two-body potentials. These potentials are assumed to have Gaussian spatial dependence. For two particles i and j, an oscillator potential is fitted to the Gaussian around r_{ij} (the distance variable) equal to zero and the difference is taken into account by perturbation theory. This procedure allows one to transform the many-nucleon system into normal coordinates which lead to analytic solutions. The validity of this approach may be questioned in general but for light nuclei, the effect of the infinite wall of the potential would not be decisive since the probability of the distance of two nucleons becoming larger than the range of true nuclear forces is small. When one uses this oscillator description, the extent of the nuclear wavefunction is proportional to $A^{\frac{1}{4}}$ which is not so different from the empirial $A^{\frac{1}{3}}$ dependence for light nuclei. Besides, when put to the test by

Green, the method showed that one could achieve a satisfactory representation of the systematics of light nuclei. One must of course acknowledge the defects associated with Green's choice of two different forms of the nucleon-nucleon potential, his adoption of scalar spin and the truncation of the perturbation series but the approach has, nevertheless, significant advantages, not the least of which is its simple and systematic prescription for calculating the structure and energy levels of all the light nuclei.

An important open question in the study of light nuclei is whether a single form of nucleon-nucleon interaction is sufficient to give good fits to the binding energies of the nuclei in the 1s shell and beyond. This hypothesis, it appears, can be studied and clarified through a realistic generalisation of Green's model in which spin is included explicitly.

3.1 The Triton Equation With Spin

If 1 and 2 denote neutrons $\underline{r}_i$ the radius vector of the ith nucleon, E the triton binding energy and

M the nucleon mass (in the discussion below and all the work that follows we restrict ourselves to the case of equal nuclear masses), the Schrodinger equation for the triton is

$$\left\{ -\frac{\hbar^2}{2M}\sum_{i=1}^{3}\nabla_i^2 + \sum_{i<j=1}^{3} V(ij) \right\}\Psi = E\Psi \qquad (21)$$

where V(ij), the spin-dependent nuclear potential between the ith and jth particles, is assumed to be of the form

$$V(ij) = V_{ij}\exp(-\mu^2 r_{ij}^2) \qquad (22)$$

with

$$V_{ij} = -V_0(w + bP_{ij}^{B}) \qquad (23)$$

and Ψ the total ground state wavefunction. In our extension of Green's model this equation becomes

$$\left\{ -\frac{\hbar^2}{2M}\sum_{i=1}^{n}\nabla_i^2 - \frac{1}{2}\sum_{i=1}^{n}\sum_{i\neq j=1}^{n} V_{ij}\mu^2 r_{ij}^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{i\neq j=1}^{n} < V_{ij}\left[\exp(-\mu^2 r_{ij}^2) + \mu^2 r_{ij}^2\right] > \right\}\Psi = E\Psi \ , \qquad (24)$$

where the brackets < > denote an expectation value computed with the help of Ψ and n = 3. Since we do not include isobaric spin dependence in our potential, the total wavefunction Ψ is not given by equation (8) but must be antisymmetric only in the two identical nucleons and is therefore

$$\Psi^{t} = \psi' \chi'' + \psi'' \chi' \qquad (25)$$

where χ' and χ'' are the two orthogonal spin functions of the trinucleon given in chapter two; the spatial functions ψ' and ψ'' are analogously, antisymmetric and symmetric respectively in the neutron coordinates. Using equation (25) and the transformation matrices of the Bartlett interchange operators P^{B}_{ij} listed on Table 1, then summing over the spin variables, we can reduce our Schrodinger equation to a pair of coupled equations in ψ' and ψ'' and involving the three inter-particle distances only. Introducing the Jacobi vectors

$$\underline{\xi}_1 = \sqrt{\frac{2}{3}} \left(\underline{r}_3 - \tfrac{1}{2}\underline{r}_1 - \tfrac{1}{2}\underline{r}_2 \right)$$

$$\underline{\xi}_2 = \sqrt{\frac{1}{2}} \left(\underline{r}_2 - \underline{r}_1 \right),$$

$$\underline{\xi}_3 = \sqrt{\frac{1}{3}} \left(\underline{r}_1 + \underline{r}_2 + \underline{r}_3 \right) \qquad (26)$$

we obtain

$$r_{12}^2 + r_{13}^2 + r_{23}^2 = 3(\xi_1^2 + \xi_2^2)$$

$$-r_{12}^2 + \tfrac{1}{2}(r_{13}^2 + r_{23}^2) = \frac{3}{2}(\xi_1^2 - \xi_2^2)$$

$$\sqrt{\frac{3}{4}}\,(r_{13}^2 - r_{23}^2) = 3\,\underline{\xi}_1 \cdot \underline{\xi}_2 \qquad (27)$$

In this new system of generalised coordinates, our coupled partial differential equations are

$$-\frac{\hbar^2}{2M}\left(\nabla^2_{\xi_1} + \nabla^2_{\xi_2}\right)\psi'\left(\underline{\xi}_1, \underline{\xi}_2\right) + 3\alpha\left(\xi_1^2 + \xi_2^2\right)\psi'\left(\underline{\xi}_1, \underline{\xi}_2\right)$$

$$+ \frac{3}{2}\beta(\xi_1^2 - \xi_2^2)\psi'(\underline{\xi}_1, \underline{\xi}_2) + 3\beta\,\underline{\xi}_1 \cdot \underline{\xi}_2\,\psi''(\underline{\xi}_1, \underline{\xi}_2)$$

$$= \epsilon\psi'(\underline{\xi}_1, \underline{\xi}_2) \qquad (28)$$

$$-\frac{\hbar^2}{2M}(\nabla^2_{\xi_1} + \nabla^2_{\xi_2})\,\psi''(\underline{\xi}_1, \underline{\xi}_2) + 3\alpha(\xi_1^2 + \xi_2^2)\,\psi''(\underline{\xi}_1, \underline{\xi}_2)$$

$$-\frac{3}{2}\beta(\xi_1^2 - \xi_2^2)\,\psi''(\underline{\xi}_1, \underline{\xi}_2) + 3\beta\,\underline{\xi}_1 \cdot \underline{\xi}_2\,\psi'(\underline{\xi}_1, \underline{\xi}_2)$$

$$= \epsilon\psi''(\underline{\xi}_1, \underline{\xi}_2) \qquad (29)$$

where we have also taken

$$\alpha = wV_0\mu^2, \quad \beta = bV_0\mu^2 \quad \text{and} \quad \epsilon = E - \tfrac{1}{2}\sum_{i=1}^{3}\sum_{i\neq j=1}^{3}\langle\cdots\rangle$$

To solve these equations we use the ansatz

$$\psi_1 = \psi'' + i\psi' = \frac{\phi}{\left[\left(\underline{\xi}_1 + i\underline{\xi}_2\right)^2\right]^{\frac{1}{2}}} \tag{30}$$

$$\text{and } \psi_2 = \psi'' - i\psi' = \frac{\phi}{\left[\left(\underline{\xi}_1 - i\underline{\xi}_2\right)^2\right]^{\frac{1}{2}}} \tag{31}$$

where ϕ is real.

Then equations (28) and (29) reduce to

$$-\frac{\hbar^2}{2M}\left(\nabla_{\xi_1}^2 + \nabla_{\xi_2}^2\right)\psi_1 + 3\alpha\left(\xi_1^2 + \xi_2^2\right)\psi_1 + \frac{3}{2}\beta\left(\underline{\xi}_1 + i\underline{\xi}_2\right)^2\psi_2 = \epsilon\psi_1 \tag{32}$$

$$-\frac{\hbar^2}{2M}\left(\nabla_{\xi_1}^2 + \nabla_{\xi_2}^2\right)\psi_2 + 3\alpha\left(\xi_1^2 + \xi_2^2\right)\psi_2 + \frac{3}{2}\beta\left(\underline{\xi}_1 - i\underline{\xi}_2\right)^2\psi_1 = \epsilon\psi_2 \tag{33}$$

and finally

$$-\frac{\hbar^2}{2M}\left[\left(\underline{\xi}_1 + i\underline{\xi}_2\right)^2\right]^{\frac{1}{2}}\left(\nabla_{\xi_1}^2 + \nabla_{\xi_2}^2\right)\left\{\left[\left(\underline{\xi}_1 + i\underline{\xi}_2\right)^2\right]^{-\frac{1}{2}}\phi\right\}$$

$$+ 3\alpha\left(\xi_1^2 + \xi_2^2\right)\phi + \frac{3}{2}\beta\left[\left(\underline{\xi}_1 - i\underline{\xi}_2\right)^2\right]^{\frac{1}{2}}\left[\left(\underline{\xi}_1 + i\underline{\xi}_2\right)^2\right]^{\frac{1}{2}}\phi = \epsilon\phi \tag{34}$$

$$-\frac{\hbar^2}{2M}\left[\left(\underline{\xi}_1 - i\underline{\xi}_2\right)^2\right]^{\frac{1}{2}}\left(\nabla^2_{\underline{\xi}_1} + \nabla^2_{\underline{\xi}_2}\right)\left\{\left[(\underline{\xi}_1 - i\underline{\xi}_2)^2\right]^{-\frac{1}{2}}\phi\right\}$$

$$+ 3\alpha(\xi_1^2 + \xi_2^2)\phi + \frac{3}{2}\beta\left[(\underline{\xi}_1 - i\underline{\xi}_2)^2\right]^{\frac{1}{2}}\left[(\underline{\xi}_1 + i\underline{\xi}_1)^2\right]^{\frac{1}{2}}$$

$$= \epsilon\phi \qquad (35)$$

In simplifying these expressions we introduce the two three-dimensional complex vectors

$$\underline{z} = \underline{\xi}_1 + i\underline{\xi}_2, \text{ and } \underline{z}^* = \underline{\xi}_1 - i\underline{\xi}_2 \qquad (36)$$

Then, since we are only considering the S-state of the triton, ϕ will depend exclusively on the three scalars z^2, z^{*2} and $\underline{z}\cdot\underline{z}^*$ formed from $\underline{z}$ and $\underline{z}^*$. Introducing further the new variables

$$u = \xi_1^2 + \xi_2^2 = \underline{z}\cdot\underline{z}^*$$

and

$$v = \left[(\xi_1^2 - \xi_2^2)^2 + 4(\underline{\xi}_1\cdot\underline{\xi}_2)^2\right]^{\frac{1}{2}} = \left[z^2 z^{*2}\right]^{\frac{1}{2}} \qquad (37)$$

we find, after partial differentiation,

$$\left[(\underline{\xi}_1 + i\underline{\xi}_2)^2\right]^{\frac{1}{2}}\left(\nabla^2_{\underline{\xi}_1} + \nabla^2_{\underline{\xi}_2}\right)\left\{\left[(\underline{\xi}_1 + i\underline{\xi}_2)^2\right]^{\frac{1}{2}}\phi\right\} \qquad (38)$$

$$= 4z\,\underset{\sim}{\nabla}_z\cdot\underset{\sim}{\nabla}_{z^*}\left\{z^{-1}\phi\right\} = u(\phi_{uu} + \phi_{vv}) + 2v\phi_{uv} + 3\phi_u$$

and the relation

$$z\underset{\sim}{\nabla}_z \cdot \underset{\sim}{\nabla}_{z*} \left\{ z^{-1}\phi(u,v) \right\} = z^*\underset{\sim}{\nabla}_z \cdot \underset{\sim}{\nabla}_{z^*} \left\{ z^{*-1}\phi(u,v) \right\} \qquad (39)$$

where

$$z = (z^2)^{\frac{1}{2}} \text{ and } z^* = (z^{*2})^{\frac{1}{2}} \qquad (40)$$

(The legitimacy of the operator $\underset{\sim}{\nabla}_z \cdot \underset{\sim}{\nabla}_{z*}$ may be questioned but it should be realised that its application is a formal process. At its application on a function, z and z* may be regarded as independent quantities in an algebraic sense).

Using these results, we see that our coupled equations become a single equation in ϕ with independent variables u and v. This equation is written as

$$-\frac{2\hbar^2}{M}\left[u(\phi_{uu} + \phi_{vv}) + 2v\phi_{uv} + 3\phi_u\right] + 3\alpha u\phi + \frac{3}{2}\beta v\phi = \epsilon\phi \qquad (41)$$

which is obviously satisfied by

$$\phi = \exp(-pu - qv) \qquad (42)$$

with

$$\frac{2\hbar^2}{M}(p^2 + q^2) = 3\alpha \quad , \quad \frac{4\hbar^2}{m} pq = \frac{3}{2}\beta$$

$$\text{and } \epsilon = \frac{6\hbar^2}{M}p = 3\left(\frac{\hbar^2}{2M}\right)^{\frac{1}{2}} \left[(3\alpha+\tfrac{3}{2}\beta)^{\frac{1}{2}} + (3\alpha-\tfrac{3}{2}\beta)^{\frac{1}{2}}\right] \quad (43)$$

Therefore, written explicitly, the triton binding energy is

$$E = 3\left(\frac{\hbar^2}{2M}\right)^{\frac{1}{2}} \left[(3\omega V_o\mu^2 + \tfrac{3}{2} bV_o\mu^2)^{\frac{1}{2}} + (3\omega V_o\mu^2 - \tfrac{3}{2} bV_o\mu^2)^{\frac{1}{2}}\right] + \tfrac{1}{2} \sum_{i=1}^{3} \sum_{i\neq j=1}^{3} < .... > \quad (44)$$

whilst the spatial functions are

$$\psi' = e^{-(pu+qv)} \frac{(v+\xi^2-\eta^2)^{\frac{1}{2}}}{2v} \quad (45)$$

$$\psi'' = e^{-(pu+qv)} \frac{(v+\eta^2-\xi^2)^{\frac{1}{2}}}{2v} \quad (46)$$

which behave like $\frac{e^{-k\Sigma r_{ij}^2}}{(\Sigma r_{ij}^2)^{\frac{1}{2}}}$. ψ' and ψ'' resemble the Irving functions (functions which have been found to be flexible for variational calculations involving central potentials of Yukawa and exponential forms)

and go to zero for large interparticle separations. They have the welcome feature too of being directly related to the two-particle interaction.

3.2 First Approximation to the Ground State Wavefunction

If our nuclear forces are spin-independent, the triplet and singlet nucleon-nucleon forces must accordingly be equal. Then in equation (25) $\psi''\chi'$ should be absent and ψ' should not only be symmetric in the like nucleons but also totally symmetric in the three nucleons. Thus a proper first approximation to the ground state wavefunction would be the dropping of the term $\psi''\chi'$ from Ψ. Such an approximation has been tried successfully by Ohmura et. al.[21], Efimov[91] and Rosati and Barbi [92] amongst others. With this reduction in ψ and, after averaging over the spin variables, the single Schrödinger equation from our model becomes

$$-\frac{\hbar^2}{2M}(\nabla_{\xi_1}^2 + \nabla_{\xi_2}^2)\psi'_a + 3\alpha(\xi_1^2 + \xi_2^2)\psi'_a + \frac{3}{2}\beta(\xi_1^2 - \xi_2^2)\psi'_a = \epsilon\psi'_a \qquad (47)$$

(It is interesting to note that this result is exactly what we would get from neglecting the 'off-diagonal' terms - $3i\beta \underline{\xi}_1 \cdot \underline{\xi}_2 \psi_1$ and $3i\beta \underline{\xi}_1 \cdot \underline{\xi}_2 \psi_2$ and then setting $\psi_1 = \psi_2$ in equations (32) and (33)).

Equation (47) is completely separable and on solution gives

$$E = 3\left(\frac{\hbar^2}{2M}\right)^{\frac{1}{2}} \left[(3\omega V_0 \mu^2 + \frac{3}{2} b V_0 \mu^2)^{\frac{1}{2}} + (3\omega V_0 \mu^2 - \frac{3}{2} b V_0 \mu^2)^{\frac{1}{2}} \right] + \frac{1}{2} \sum_{i=1}^{3} \sum_{i \neq j=1}^{3} \langle \ldots . \rangle \quad (48)$$

and

$$\psi'_a = e^{-\frac{1}{2}(a_1 \xi_1^2 + a_2 \xi_2^2)} \quad (49)$$

where

$$a_1^2 = (3\omega V_0 \mu^2 + \frac{3}{2} b V_0 \mu^2) \left(\frac{2M}{\hbar^2}\right) \quad (50)$$

and

$$a_2^2 = (3\omega V_0 \mu^2 - \frac{3}{2} b V_0 \mu^2) \left(\frac{2M}{\hbar^2}\right) \quad (51)$$

Equations (44) and (48) are identical expressions which differ only quantitatively in the expectation

value term $\frac{1}{2}\sum_{i=1}^{3}\sum_{i\neq j=1}^{3}\langle\ldots.\rangle$. If our first approximation is indeed good this difference can be expected to be small. The solution ψ'_a is a polarised Gaussian similar to those of Aranoff and Percus,[88] and, like ψ' and ψ'', is given directly by the nucleon-nucleon interaction. It is intriguing that our approximation, which uses first order perturbation theory, yields the same results as the variational calculation using $\psi'_a\chi''$ as trial function. This can be shown easily. Thus by the variational technique, the triton binding energy is given by

$$E_{var} = \langle -\frac{\hbar^2}{2M}(\nabla^2_{\xi_1} + \nabla^2_{\xi_2})\rangle_a + \langle \sum_{i<j=1}^{3} V_{ij}\exp(-\mu^2 r_{ij}^2)\rangle_a \qquad (52)$$

compared to the expression

$$E = \frac{\hbar^2}{2M}(3a_1 + 3a_2) + \langle \sum_{i<j=1}^{3}(V_{ij}\exp(-\mu^2 r_{ij}^2) + V_{ij}\mu^2 r_{ij}^2]\rangle_a \qquad (53)$$

that follows from equation (48). The kinetic energy expectation value is evaluated straightforwardly and is

$$< -\frac{\hbar^2}{2M}\left(\nabla^2_{\xi_1} + \nabla^2_{\xi_2}\right)>_a = \frac{\hbar^2}{2M}\left(\frac{3}{2}a_1 + \frac{3}{2}a_2\right) \qquad (54)$$

whilst

$$< \sum_{i<j=1}^{3} V_{ij}\mu^2 r_{ij}^2 >_a = -3\alpha<(\xi_1^2 + \xi_2^2)>_a - \frac{3}{2}\beta<(\xi_1^2 - \xi_2^2)>_a$$

$$= -\frac{\hbar^2}{2M}\left(\frac{3}{2}a_1 + \frac{3}{2}a_2\right) \qquad (55)$$

where we have used the formulae

$$\int d\underline{\tau} = \int_0^\infty\int_0^\infty (4\Pi)^2\ \xi_1^2\xi_2^2\, d\xi_1\, d\xi_2 \qquad (56)$$

and $$\int_0^\infty R^{2n-1}\ e^{-kR^2}\ dR = \frac{\Gamma(n)}{2k^n} \qquad (57)$$

Equality with the variational method suggests a possible means of selecting the nucleon-nucleon potential for this model. Fixing the force mixture constants w and b we can determine $V_0\mu^2$ through the requirement of a sensible wavefunction.

3.3 The r.m.s. Radius and Related Quantities

The r.m.s. radius, the body form factor and other

physical properties of the trinucleon are sensitive to the form and parameters chosen for the ground state wavefunction. As observed in the previous section we can determine the depth and range of the nucleon-nucleon force uniquely by requiring that our wavefunctions describe correctly the body form factor of the nuclear system considered and yield agreement with the experimentally-observed values of the r.m.s. radius and the binding energy. The simplicity of ψ'_a makes analytic evaluation of these important quantities easy. If we write

$$\underline{\xi}_\lambda = \sum_{i=1}^{3} \underline{r}_i \, S_{i\lambda} \quad , \quad \lambda = 1, 2 \text{ and } 3 \qquad (58)$$

$$\text{i.e. } (\underline{\xi}_1 \; \underline{\xi}_2 \; \underline{\xi}_3) = (\underline{r}_1 \; \underline{r}_2 \; \underline{r}_3) \begin{pmatrix} -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{6}} & +\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} \end{pmatrix} \qquad (59)$$

the expectation values of the operators appearing in our calculations can be obtained without difficulty by using the results

$$< \exp(-\mu^2 r_{ij}^2) >_a = \left(1 + \frac{\mu^2}{\alpha_{ij}^2}\right)^{-\frac{3}{2}} \tag{60}$$

$$< r_{ij}^2 >_a = \frac{3}{2\alpha_{ij}^2} \tag{61}$$

$$< \frac{1}{r_{ij}} >_a = \frac{2\alpha_{ij}}{\Pi^{\frac{1}{2}}} \tag{62}$$

where $\alpha_{ij}^{-2} = \sum_{\lambda \neq 3} (S_{i\lambda} - S_{j\lambda})^2 / a_\lambda$ (63)

(These relations have been derived from the integrals in the appendix). We see that

$$\alpha_{12}^{-2} = \frac{2}{a_2} \tag{64}$$

$$\alpha_{13}^{-2} = \frac{3}{2a_1} + \frac{1}{2a_2} = \frac{a_1 + 3a_2}{2a_1 a_2} \tag{65}$$

$$\alpha_{23}^{-2} = \frac{3}{2a_1} + \frac{1}{2a_2} = \frac{a_1 + 3a_2}{2a_1 a_2} \tag{66}$$

The r.m.s. radius of matter distribution is defined as

$$R^2_{r.m.s.} = < \frac{1}{3} \sum_{i=1}^{3} (\underline{r}_i - \sqrt{\frac{1}{3}}\, \underline{\xi}_3)^2 >_a = < \frac{1}{9} \sum_{i<j=1}^{3} r_{ij}^2 >_a \tag{67}$$

and is therefore

$$R^2_{r.m.s.} = \frac{1}{6}\left(\frac{2}{a_2} + \frac{a_1+3a_2}{a_1 a_2}\right) = \frac{a_1+a_2}{2a_1 a_2} \qquad (68)$$

If we assume that the ^{3}He ground state has the same wavefunction as the triton (1 and 2 are protons and 3 is the neutron) the Coulomb energy is

$$< \frac{e^2}{r_{12}} > = e^2\left(\frac{2a_2}{\pi}\right)^{\frac{1}{2}} \qquad (69)$$

The body form factor is the Fourier transform of the nucleon density distribution so

$$F_B(q^2) = \int |\psi'_a|^2 \exp\left(-i\underline{q}\cdot\sqrt{\frac{2}{3}}\,\underline{\xi}_1\right)d\tau$$

$$= \exp\left(-\frac{1}{6}\,q^2R^2\right) = \exp\left(-\frac{1}{6}q^2\,\frac{a_1+a_2}{2a_1a_2}\right) \qquad (70)$$

The structure of the nucleus can also be described by the pair distribution function defined by

$$g_{ij}(\underline{r}-\underline{r}') = \int |\psi'_a|^2\ \delta(\underline{r}_i - \underline{r})\delta(\underline{r}_j - \underline{r}')\,d\tau/_N \qquad (71)$$

where N is the normalisation constant

$$N = \int |\psi'_a|^2\ \delta(\underline{r}_i - \underline{r})\,d\tau \qquad (72)$$

With our solution ψ'_a, we have

$$g_{ij}(r_{ij}) = \left(\frac{\alpha_{ij}^2}{\Pi}\right)^{\frac{3}{2}} \exp(-\alpha_{ij}^2 \, r_{ij}^2) \qquad (73)$$

i.e.

$$g_{12}(r_{12}) = \left(\frac{a_2}{2\Pi}\right)^{\frac{3}{2}} \exp\left(-\frac{a_2}{2} r_{12}^2\right) \qquad (74)$$

$$g_{13}(r_{13}) = \left[\left(\frac{2a_1 a_2}{a_1+3a_2}\right)\left(\frac{1}{\Pi}\right)\right]^{\frac{3}{2}} \exp\left(-\frac{2a_1 a_2}{a_1+3a_2} r_{13}^2\right) \qquad (75)$$

$$g_{23}(r_{23}) = \left[\left(\frac{2a_1 a_2}{a_1+3a_2}\right)\left(\frac{1}{\Pi}\right)\right]^{\frac{3}{2}} \exp\left(-\frac{2a_1 a_2}{a_1+3a_2} r_{23}^2\right) \qquad (76)$$

When our exact solutions are employed, the corresponding results can in all probability be expressed in algebraic form but we have preferred to evaluate them numerically. We simplify the calculations by making the transformation first suggested by Irving.

$$\xi_1 = R \cos\theta \quad , \quad \xi_2 = R \sin\theta \qquad (77)$$

The volume element $d\tau$ is

$$d\tau = \xi_1^2 \xi_2^2 \, d\xi_1 d\xi_2 \, d\cos\phi = R^5 \cos^2\theta \sin^2\theta \, dR d\theta d\cos\phi \qquad (78)$$

since the Jacobian is R, and ϕ is the angle contained

by the two vectors. Integrals over R are obtained explicitly so we are left with straightforward two-dimensional integrals over θ and ϕ. These are evaluated using Miller's quadrature formula (6D) in reference (93).

3.4 Results and Discussion

The approximation given by equations (49) - (51) is used to determine the nucleon-nucleon potential for our model calculations. A variety of special values is assumed for the interaction constants V_0 and μ^2 whilst w and b are taken to be 0.8 and 0.2 respectively since scattering data suggest that q the singlet to triplet ratio is equal to 0.6. The best values of our constants are found to be

$$V_0 = 20.0 \text{ MeV and } \mu^2 = 0.06 \text{ fm}^{-2} \qquad (79)$$

These values secure a least squares fit to the experimental three-body form factor (see Table 3) and yield a binding energy of 8.48 MeV on the basis of the foregoing theory. In Table 4 we list the results obtainable from our exact and approximate solutions to the three-body Schrodinger equation together with the

TABLE 3.

BODY FORM FACTOR OF THE THREE-NUCLEON SYSTEM

q^2	$F_B(q^2)$ Experimental values from Collard[95]	$F_B(q^2)$ From Okamoto and Lucas[94]	$F_B(q^2)$ From Using $\psi'_a \chi''$
1.0	0.644	0.645	0.656
2.0	0.418	0.416	0.430
3.0	0.298	0.268	0.282
4.0	0.206	0.173	0.181
5.0	0.141	0.112	0.121
6.0	0.100	0.072	0.080

TABLE 4.

CALCULATED AND OBSERVED B.E., $R_{r.m.s.}$ AND COULOMB ENERGY

Method	Binding Energy (MeV)	r.m.s. Radius (fm)	Coulomb Energy (MeV)
Approximate	8.48	1.64	0.677
Exact	8.75	1.65	0.661
Variational (Okamoto's)	8.16	1.62	0.708
Experimental	8.48	1.70	0.764

experimental values and the results of a variational calculation using a totally symmetric Gaussian. (Our calculations are model calculations and as such should really be compared with other theoretical approaches).

Whilst the r.m.s. radius and the Coulomb energy are in reasonable agreement the binding energy given by the exact and approximate forms of our model is as much as half an MeV better than the result yielded by Okamoto's[94] realistic Gaussian. This confirms the finding of Aranoff[88] that polarised Gaussians can yield better results than the unpolarised functions. This is not unexpected since it can be seen that using a totally symmetric spatial function corresponds to considering only the dominant S state of the triton. On the other hand our exact and approximate wave-functions can be expressed as the sum of such an S state, a state S' of mixed spatial symmetry together with an admixture of the state with L=0 and isospin $T = \frac{3}{2}$. However for calculations which are preliminary in nature, the S state is still a good approximation as the polarisation is not large; in fact the ratio $\frac{a_2}{a_1}$ is nearly 0.9. The parameters α_{ij}^{-2} which

determine the pair distributions have values:

$$\alpha_{12}^{-2} = 5.734 \text{ fm}^2 \tag{80}$$

$$a_{13}^{-2} = a_{23}^{-2} = 5.229 \text{ fm}^2$$

and $$a_1 = 0.3954 \text{ fm}^{-2} \quad , \quad a_2 = 0.3487 \text{ fm}^{-2} \tag{81}$$

It is obvious then that the approximation on which the theory is based i.e. the replacement of the Gaussian by an oscillator well and the treatment of the difference as a perturbation, is well justified, since every pair of nucleons is separated by a distance well within the range in which the approximation is a good one. As might be expected, the particles furthest apart are the two neutrons but even these are not too far up the slopes of their common potential well.

The most undesirable aspect of the model is the value of μ^2 suggested in equation (79). The restriction on $V_0\mu^2$ demanded by our requirement that the wavefunction reproduces the salient physical features of the three-nucleon system leads to a nucleon-nucleon potential well that is broader and shallower than is suggested by two-body scattering data. Our potential has no repulsive core and it may be that a well,

somewhat too broad, is needed to compensate for this defect. It may conceivably indicate also that the free nucleon-nucleon interaction differs from that in the bound state.

CHAPTER FOUR

THE ALPHA PARTICLE PROBLEM

The satisfactory results of our investigation into the trinucleon in the last chapter suggest the possible extension of the method to the other nucleus in the 1s shell. Green's model is a general one for all the light nuclei so our success with the three-nucleon system poses the logical question of whether the incorporation of spin ihto the four-nucleon problem will also lead to exact results, and the subsidiary one of whether the method has the ability to provide a complete set of 1s shell calculations with the single two-body potential. The purpose of this chapter is to study these questions.

4.1 The Binding Energy of ^{4}He

The Schrodinger equation of the alpha particle, when we neglect the Coulomb interaction, is equation (24) with n equal to four. The ground state wave-function Ψ^{α} is required to be antisymmetric under the combined interchange of spatial and spin coordinates

of the like nucleons. Since the total spin of the alpha particle is zero the two spin functions belonging to the system are χ' and χ'' given explicitly by equations (12) and (13), where we have assumed 1 and 2 to be neutrons and 3 and 4 to be protons. Thus Ψ^{α} is

$$\Psi^{\alpha} = \psi' \chi'' + \psi'' \chi' \qquad (82)$$

where ψ' and ψ'', the spatial functions, have a similar symmetry property to χ' and χ''. The transformation properties of the spin interchange operators P_{ij}^{σ} are given by the representation matrices of the two-dimensional irreducible representation listed on Table 2. Using these and summing over spin, we reduce our four-body Schrodinger equation to a pair of coupled equations. If we use the convenient transformation of our coordinates

$$\underline{\eta}_1 = \frac{\underline{r}_1 + \underline{r}_2 - \underline{r}_3 - \underline{r}_4}{2}$$

$$\underline{\eta}_2 = \frac{\underline{r}_1 - \underline{r}_2}{\sqrt{2}}$$

$$\underline{\eta}_3 = \frac{\underline{r}_3 - \underline{r}_4}{\sqrt{2}}$$

$$\underline{\eta}_4 = \tfrac{1}{2}(\underline{r}_1 + \underline{r}_2 + \underline{r}_3 + \underline{r}_4) \qquad (83)$$

our coupled equations are, in the new set of coordinates,

$$-\frac{\hbar^2}{2M}(\nabla_{\eta_1}^2+\nabla_{\eta_2}^2+\nabla_{\eta_3}^2)\psi' + 4\alpha(\eta_1^2+\eta_2^2+\eta_3^2)\psi' + \beta(\eta_2^2+\eta_3^2-2\eta_1^2)\psi' + 2\sqrt{3}\beta\,\underline{\eta}_2\cdot\underline{\eta}_3\psi'' = \epsilon\psi' \qquad (84)$$

$$-\frac{\hbar^2}{2M}(\nabla_{\eta_1}^2+\nabla_{\eta_2}^2+\nabla_{\eta_3}^2)\psi'' + 4\alpha(\eta_1^2+\eta_2^2+\eta_3^2)\psi'' - \beta(\eta_2^2+\eta_3^2-2\eta_1^2)\psi'' + 2\sqrt{3}\underline{\eta}_2\cdot\underline{\eta}_3\psi' = \epsilon\psi'' \qquad (85)$$

We have been unable to solve these expressions exactly but the approximation of chapter three can be equally well applied here. Thus assuming

$$\Psi^{\alpha} = \psi''\chi' \qquad (86)$$

we derive the single partial differential equation

$$-\frac{\hbar^2}{2M}(\nabla_{\eta_1}^2+\nabla_{\eta_2}^2+\nabla_{\eta_3}^2)\psi'' + 4\alpha(\eta_1^2+\eta_2^2+\eta_3^2)\psi'' - \beta(\eta_2^2+\eta_3^2-2\eta_1^2)\psi'' = \epsilon\psi'' \qquad (87)$$

which is solved by separation of variables.

The solution is easy to write down

$$\psi'' = e^{-\frac{1}{2}(b_1\eta_1^2 + b_2\eta_2^2 + b_3\eta_3^2)} \tag{88}$$

$$\epsilon = \left(\frac{\hbar^2}{2M}\right)3(b_1 + b_2 + b_3) \tag{89}$$

where

$$b_1^2 = (4\omega V_0\mu^2 + 2bV_0\mu^2)\left(\frac{2M}{\hbar^2}\right) \tag{90}$$

$$b_2^2 = (4\omega V_0\mu^2 - bV_0\mu^2)\left(\frac{2M}{\hbar^2}\right) = b_3^2 \tag{91}$$

4.2 Properties of ^{4}He

As with our treatment of the triton, we let

$$\underline{\eta}_\lambda = \sum_{i=1}^{4} \underline{r}_i\, T_{i\lambda}\ , \quad \lambda = 1,\ 2,\ 3 \text{ and } 4 \tag{92}$$

$$(\underline{\eta}_1, \underline{\eta}_2, \underline{\eta}_3, \underline{\eta}_4) = (\underline{r}_1 + \underline{r}_2 + \underline{r}_3 + \underline{r}_4) \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \tag{93}$$

We then use the results

$$\langle \exp(-\mu^2 r_{ij}^2) \rangle = \left(1 + \frac{\mu^2}{\alpha_{ij}^2}\right)^{-\frac{3}{2}} \tag{94}$$

$$\langle r_{ij}^2 \rangle = \frac{3}{2\alpha_{ij}^2} \tag{95}$$

$$\left\langle \frac{1}{r_{ij}} \right\rangle = \frac{2\alpha_{ij}}{\pi^{\frac{1}{2}}} \tag{96}$$

where

$$\alpha_{ij}^{-2} = \sum_{\lambda \neq 4} (T_{i\lambda} - T_{j\lambda})^2 / b_\lambda \tag{97}$$

to determine explicitly the expectation values of the four-nucleon operators. We have

$$\alpha_{12}^{-2} = \frac{2}{b_2} \tag{98}$$

$$\alpha_{13}^{-2} = \frac{1}{b_1} + \frac{1}{b_2} = \frac{b_1 + b_2}{b_1 b_2} \tag{99}$$

$$\alpha_{14}^{-2} = \frac{1}{b_1} + \frac{1}{b_2} \tag{100}$$

$$\alpha_{23}^{-2} = \frac{1}{b_1} + \frac{1}{b_2} \tag{101}$$

$$\alpha_{24}^{-2} = \frac{1}{b_1} + \frac{1}{b_2} \tag{102}$$

$$\alpha_{34}^{-2} = \frac{2}{b_2} \tag{103}$$

The physical properties of interest are those already considered in the last chapter. Thus the r.m.s. radius for this system is

$$R^2_{r.m.s.} = \left\langle \frac{1}{4} \sum_{i=1}^{4} (\underline{r}_i - \tfrac{1}{2}\,\underline{\eta}_4)^2 \right\rangle = \left\langle \frac{1}{16} \sum_{i<j=1}^{4} r_{ij}^2 \right\rangle$$

$$= \frac{3}{8}\left(\frac{1}{b_1} + \frac{2}{b_2}\right) \qquad (104)$$

The expressions for the other quantities are as follows:

$$\left\langle \frac{e^2}{r_{34}} \right\rangle = e^2\left(\frac{2b_2}{\pi}\right)^{\frac{1}{2}} \qquad (105)$$

and the body form factor

$$F_B(q^2) = \tfrac{1}{2}\int |\psi|^2 \left\{ \exp\left[- i\underline{q}\cdot\left(\underline{r}_3 - \frac{\underline{\eta}_4}{2}\right)\right] + \exp\left[- i\underline{q}\cdot\left(\underline{r}_4 - \frac{\underline{\eta}_4}{2}\right)\right] \right\} d\tau$$

$$= \exp\left[- \frac{1}{6}\, q^2 \left(\frac{3}{8}\right)\left(\frac{1}{b_1} + \frac{2}{b_2}\right) \right] \qquad (106)$$

The alpha particle pair distribution functions are

$$g_{12}(r_{12}) = \left(\frac{b_2}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{b_2}{2} r_{12}^2\right) \tag{107}$$

$$g_{13}(r_{13}) = \left(\frac{b_1 b_2}{\pi(b_1+b_2)}\right)^{\frac{3}{2}} \exp\left(-\frac{b_1 b_2}{(b_1+b_2)} r_{13}^2\right) \tag{108}$$

$$g_{14}(r_{14}) = \left(\frac{b_1 b_2}{\pi(b_1+b_2)}\right)^{\frac{3}{2}} \exp\left(-\frac{b_1 b_2}{(b_1+b_2)} r_{14}^2\right) \tag{109}$$

$$g_{23}(r_{23}) = \left(\frac{b_1 b_2}{\pi(b_1+b_2)}\right)^{\frac{3}{2}} \exp\left(-\frac{b_1 b_2}{(b_1+b_2)} r_{23}^2\right) \tag{110}$$

$$g_{24}(r_{24}) = \left(\frac{b_1 b_2}{\pi(b_1+b_2)}\right)^{\frac{3}{2}} \exp\left(-\frac{b_1 b_2}{(b_1+b_2)} r_{24}^2\right) \tag{111}$$

$$g_{34}(r_{34}) = \left(\frac{b_2}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{b_2}{2} r_{12}^2\right) \tag{112}$$

4.3 Results and Discussion

The internucleon potential obtained in the three-body calculations is applied to the evaluation of the physical properties of the alpha particle. In tables 5 and 6 we record the body form factors, the binding energies, the r.m.s. radius and the Coulomb energies derived from our calculations,

those of a variational calculation with a simple Gaussian and the experimental data. From these tables it can be seen that the fit to the experimental form factor is poor; the r.m.s. radius and the Coulomb energy values are just reasonable. The values of the parameters α_{ij}^{-2} are

$$\alpha_{12}^{-2} = \alpha_{34}^{-2} = \frac{2}{0\cdot4168}\ \text{fm}^2 = 4\cdot798\ \text{fm}^2 \qquad (113)$$

$$\alpha_{13}^{-2} = \alpha_{14}^{-2} = \alpha_{23}^{-2} = \alpha_{24}^{-2} = \left(\frac{1}{0\cdot4168} + \frac{1}{0\cdot4565}\right)\text{fm}^2$$

$$= 4\cdot590\ \text{fm}^2 \qquad (114)$$

They show that the like particles are again separated by larger distances than are unlike particles. These distances are all smaller than those in the triton which is in agreement with the more compact structure of the alpha particle. The improvement in the binding energy from our calculations parallels that in the triton, but the four-nucleon system is still underbound. The latest electron scattering experiments [98] indicate an r.m.s. radius of 1.50 fm. for the alpha particle and the simple Gaussian which

TABLE 5

BODY FORM FACTOR OF THE FOUR-NUCLEON SYSTEM

q^2	$F_B(q^2)$ (From Hofstadter and Repellin[97,98])	$F_B(q^2)$ (Variational)	$F_B(q^2)$ (From using $\psi''\chi'$)
1.0	0.703	0.714	0.646
2.0	0.490	0.512	0.417
3.0	0.340	0.366	0.270
4.0	0.234	0.262	0.174
5.0	0.160	0.187	0.113
6.0	0.110	0.134	0.073

TABLE 6

CALCULATED AND OBSERVED B.E., $R_{r.m.s.}$ AND COULOMB ENERGY

Method	Binding energy (MeV)	r.m.s. Radius (fm)	Coulomb energy (MeV)
Approximate	23.10	1.62	0.740
Variational	19.51	1.42	0.858
Experimental	28.3	1.5	

gives this also yields a binding energy identical to ours viz. 23.10 MeV. These results differ from those usually obtained by other authors who have worked with central potentials. They have found in their calculations that any potential which gives the experimental binding energy of the triton invariably yields an overbound alpha particle. This fact has been ascribed to the non-saturating character of the attractive potentials and the omission of non-central forces. It appears then that our shallow and longer-ranged potential can be attributed to the necessity to compensate for non-saturation. However, although these arguments are interesting, they do not remove the deficiencies in the potential and although our results for the triton and the alpha particle are satisfactory it does not seem worthwhile extending these calculations to nuclei beyond the 1s shell.

PART 2

CHAPTER FIVE

SHORT-RANGE CORRELATIONS AND THE TRINUCLEON

Although the exact method discussed in the first part of the thesis is attractive, it is still undeniable that the most practicable method for treating the N-body Schrodinger equation is the variational method. Thus, if we assume a phenomenological point of view and choose a nuclear potential composed only of two-body forces that are approximately consistent with two-nucleon scattering data up to a few hundred MeV, give the correct deuteron binding energy and have the proper saturation character, the variational calculation is probably the only useful and general method for solving the N-nucleon problem. In this chapter and the following three, we develop a flexible but consistent method for the approximate solution of the three and four-nucleon systems. The method is of some generality and is one in which we systematically use our physical intuition to include every effect that might help determine the character of the nuclear wavefunction and then quantitatively test and improve the approximate wavefunction that our intuition has led us to.

5.1 The Nucleon-Nucleon Interaction

In recent years a number of realistic interaction potentials have been obtained which fit all the two-nucleon data at low and intermediate energies. Whilst these forces have been employed for the treatment of some nuclear systems, it remains true that calculations with them are difficult and lengthy, and it appears that it will be a long time before any can be carried out for the more complex spectra without greatly over-simplifying the dynamical situation. In the meantime, and before the advent of techniques for handling the more sophisticated forces, a great deal of work must still be done using simplified forms of two-body potentials. These potentials are chosen to contain a number of parameters which are fitted to nuclear data and to have no singular features so as to make it possible to apply perturbation theory. The approach using these semi-realistic potentials is capable of providing insight into the dynamics of nuclei and the structure of nuclear states. To justify such an approach, it is of course not sufficient to know that a more realistic treatment would be prohibitively complicated; there must be reasons to believe that the

simplified model reproduces the main qualitative features of the problem. These reasons can be found in the two related facts that

a) the dynamical features are insensitive to the detailed form of the two-nucleon interaction, and

b) the semi-realistic forces, in spite of their different appearance, may closely resemble realistic forces in their effect on the main properties of low-lying states.

These facts are borne out for example by the calculations of Blatt and Delves[11], Barrett[99], Gupta, Bhakar and Mitra[60] and Okamoto and Lucas[94]. There are some properties of the nuclear forces which must be common to all the phenomenological potentials. The interaction must be short-ranged, strongly attractive over most of this range and extremely repulsive at very short interparticle separations. The necessity to describe this repulsion has led to the extensive use of the hard core at small distances. Unfortunately these hard cores do not lend themselves readily to perturbation calculations in nuclear matter and non-singular velocity-dependent

potentials have been suggested as possible substitutes. Recently it has been found that one can reproduce the experimentally deduced nucleon-nucleon phase shifts with a number of velocity-dependent potentials and that one can simulate the effect of the hard core with such forces. We have therefore selected a velocity-dependent potential to describe the two-particle interaction. The two-body potential is taken to be of the form

$$V(r_{ij}) = (\omega P^{W}_{ij} + bP^{B}_{ij} + hP^{H}_{ij} + mP^{M}_{ij})\ (V)_{static}(r_{ij}) + V_{vel.dep.}(r_{ij}) \quad (115)$$

where w,b,h,m are the exchange force constants and P^{W}_{ij}, P^{B}_{ij}, P^{H}_{ij} and P^{M}_{ij} the usual interchange operators. Here $(V)_{static}(r_{ij})$ is given by

$$(V)_{static}(r_{ij}) = X_{static}(V_o)_{static}\ \exp(-2r_{ij}/\beta s) \quad (116)$$

where the ratio of the triplet static potential to the singlet static potential

$$X_{static} = \frac{w + b + h + m}{w - b - h + m} = \frac{1}{q} \quad (117)$$

$$(V)_{\text{vel.dep.}}(r_{ij}) = (V_s)_{\text{vel.dep.}}(r_{ij}) + (V_t)_{\text{vel.dep.}}(r_{ij}) \tag{118}$$

$$(V_s)_{\text{vel.dep.}}(r_{ij}) = [(V_o)_{\text{vel.dep.}}/2]\,[\not{p}^2_{ij}\,\omega_s(r_{ij}) + \omega_s(r_{ij})\not{p}^2_{ij}] \tag{119}$$

$$(V_t)_{\text{vel.dep.}}(r_{ij}) = [X_{\text{vel.}}(V_o)_{\text{vel.dep.}}/2]\,[\not{p}^2_{ij}\,\omega_t(r_{ij}) + \omega_t(r_{ij})\not{p}^2_{ij}] \tag{120}$$

with

$$\omega(r_{ij}) = \exp\,(-2^{r_{ij}/\beta'}) \tag{121}$$

(The subscripts s and t refer to the singlet and triplet states, respectively). The effective nuclear potential is selected to be of exponential form in both the ^1S and ^3S states because the ground states of the few-nucleon systems are better described by exponential wavefunctions. The singlet-even interaction is taken to be v_4 in Rojo's thesis[38] and has the values

$$\begin{aligned} (V_o)_{\text{static}} &= -100 \text{ MeV}, \quad {}^1/_{\beta_s} = 0.625 \text{ fm}^{-1} \\ (V_o)_{\text{vel.dep.}} &= 82.94 \text{ MeV}, \quad {}^1/_{\beta'_s} = 1.40 \text{ fm}^{-1} \end{aligned} \tag{122}$$

This gives a reasonably good fit to the singlet scattering length and effective range (a_s = 232 and r_s = 3.06 fms. as against the experimental -23.69 and 2.82 fms. respectively), and to the ^1S and ^1D phase shifts up to 340 MeV laboratory energy of the incident proton in p-p scattering. The parameters of the triplet potential are determined by fitting the binding energy of the deuteron and Breit's ^3S phase shifts at E_{lab} = 270 MeV. Our procedure for evaluating β'_t X_{static} and $X_{vel.}$ follows that of Srivastava[40a]. We assume $\frac{1}{\beta'_t}$ to be 1.0 fm^{-1} and taking a three-parameter trial function of form $\exp(-xr) - z\exp(-yr)$ for the deuteron wavefunction, we calculate the values of x,y and z which give a binding energy of 3.49 MeV with Srivastava's triplet even potential (this result was obtained by Lovitch and Rosati[41] through a direct humerical evaluation of the Schrodinger equation). With our trial function the expression for the binding energy of the deuteron is given by

$$B.E.(D) = N^{-2}\left\{ BE(x,x) + zBE(x,y) + zBE(y,x) + z^2 BE(y,y)\right\} \qquad (123)$$

where

$$BE(\alpha,\beta) = KE(\alpha,\beta) + VSTAT(\alpha,\beta) + VVEL(\alpha,\beta) \tag{124}$$

$$KE(\alpha,\beta) = -\frac{\hbar^2}{2m}\int_0^\infty \left[e^{-\alpha r}\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right\}e^{-\beta r}\right] 4\Pi r^2 \, dr$$

$$= \left(-\frac{\hbar^2}{2m}\right) 8\Pi \left\{\frac{\beta^2}{(\alpha+\beta)^3} - \frac{\beta}{(\alpha+\beta)^2}\right\} \tag{125}$$

$$VSTAT(\alpha,\beta) = \int_0^\infty \left[e^{-\alpha r}\left\{V_{static}(r)\right\}e^{-\beta r}\right] 4\Pi r^2 \, dr$$

$$= X_{static}\,(V_0)_{static}\,\Pi\left(\frac{2}{\alpha+\beta+{}^2/_{\beta_s}}\right)^3 \tag{126}$$

$$VVEL(\alpha,\beta) = \left(-\frac{2m}{\hbar^2}\right) X_{vel.}\,(V_0)_{vel.}\left[KE\left(\alpha,\beta+\frac{2}{\beta'_s}\right) + KE\left(\alpha+{}^2/_{\beta'_s},\,\beta\right)\right] \tag{127}$$

$$\int x^n e^{-tx}\,dx = \frac{-e^{-tx}}{t^{n+1}}\, n!\left(1+tx+\frac{t^2x^2}{2!} + \cdots + \frac{t^n x^n}{n!}\right) \tag{128}$$

and the normalisation constant N^2 is

$$N^2 = \Pi\left[\frac{1}{(2x)^3} + \frac{2z}{(x+y)^3} + \frac{z^2}{(2y)^3}\right] \tag{129}$$

With our values of x,y and z (x = 0.16, y = 1.00 and z = 1.0), we plot the sets of X_{static} and $X_{vel.}$ which give the true deuteron binding energy against those of Srivastava which fit the ^3S phase shifts at E_{lab} = 270 MeV. These phase shifts are derived using the Born Approximation and are given explicitly by

$$\tan \delta_\ell = -k \int_o^\infty [j_\ell(kr)]^2 U_{eff}\, r^2\, dr \qquad (130)$$

$$U_{eff} = \frac{1}{1 + 2w(r)} \left\{ 2k^2 w(r) + \frac{V_{static}(r)}{41.5} - \frac{[w'(r)]^2}{1+2w(r)} \right\} \qquad (131)$$

With our expressions for $V_{static}(r)$ and $w(r)$ and substituting for the Bessel function $j_o(kr)$, we get

$$\tan \delta_o = -4kX_{vel.} \int_o^\infty \frac{[\sin^2(kr)]e^{-2kr}}{1 + 4X_{vel.}e^{-2r}}\, dr + \frac{100\, X_{static}}{41.5k} \int_o^\infty \frac{[\sin^2(kr)]e^{-1.25r}}{1+4\, X_{vel.}e^{-2r}}\, dr + \frac{16\, X_{vel.}^2}{k} \int_o^\infty \frac{[\sin^2(kr)]e^{-4r}}{(1-4\, X_{vel.}e^{-2r})^2}\, dr \qquad (132)$$

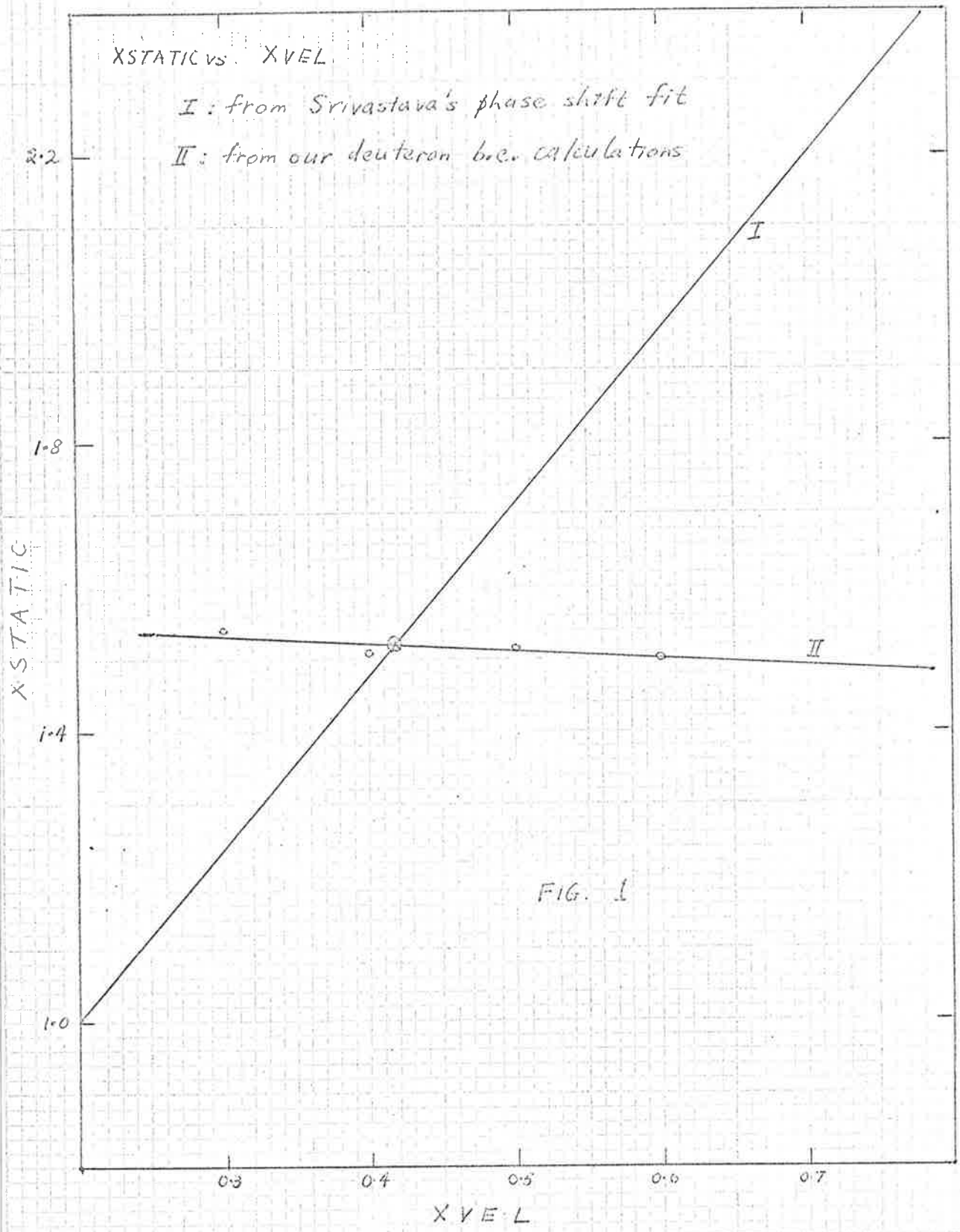

XSTATIC vs XVEL
I : from Srivastava's phase shift fit
II : from our deuteron b.e. calculations
2·2
1·8
1·4
1·0
XSTATIC
I
II
0·3
0·4
0·5
0·6
0·7
XVEL

FIG. 1

The point of intersection of the two curves (see Fig. 1) yields the correct set of force constants X_{static} and $X_{vel.}$. These are

$$X_{static} = 1.52 \quad \text{and} \quad X_{vel.} = 0.415$$

A second quadratically velocity dependent potential that will be used in our investigations is the Herndon-Tang[39] Gaussian potential, which has the form

$$V(\underline{r},\underline{p}) = V_o \exp(-\alpha r^2) + p^2 w_o \exp(-\beta r^2) + w_o \exp(-\beta r^2)\underline{p}^2 \quad (133)$$

The triplet parameters are

$$V_o = -111.5 \text{ MeV}, \ \alpha = 0.65 \text{ fm}^{-2}, \ w_o = 37.323 \text{ MeV}$$

$$\text{and } \beta = 2.2 \text{ fm}^{-2} \quad (134)$$

whilst the singlet parameters are

$$V_o = -34.8 \text{ MeV}, \ \alpha = 0.35 \text{ fm}^{-2}, \ w_o = 37.323 \text{ MeV}$$

$$\text{and } \beta = 2.2 \text{ fm}^{-2} \quad (135)$$

It is pertinent to point out that there are two interpretations of $\underline{p}$ in the velocity dependent potentials. Srivastava's calculations[40a, 79] are based on the assumption that for the three body system $\underline{p}$ is in

each instance the momentum conjugate to the inter-particle separations, r_1, r_2 and r_3 (to be referred to as case 1). Lovitch and Rosati, however, interpret $\underline{p}$ to be the momentum canonically conjugate to the two-particle relative coordinates in the center-of-mass frames of the particles taken two at a time as if the third one did not exist (case 2). Although we believe Lovitch and Rosati's interpretation to be the correct one, we nevertheless, use both in our three-body calculations so as to derive a quantitative comparison of the two.

The third potential in this thesis is the Rarita-Present[17] potential which is

$$V(r_{ij}) = V_t(r_{ij}) = -123.3 \exp(-1.156\ r_{ij}) \left\{ wP_{ij}^{\ W} + bP_{ij}^{\ B} + hP_{ij}^{\ H} + mP_{ij}^{\ M} \right\} \qquad (136)$$

$$V_s(r_{ij}) = qV_t(r_{ij}) \qquad (137)$$

(where q, the ratio of the singlet even to triplet even potentials is 0.6, and the subscripts s and t refer to singlet and triplet respectively).

5.2 The Trinucleon Wavefunction

To obtain the trinucleon binding energy, we solve the Schrodinger equation of the system by a form of variational calculation. The selection of the trial functions is, in the final analysis, a trial and error process. In order to contain the errors, we formulate our trial functions through a flexible but consistent method, in which we allow our physical intuition to help us determine the character of the nuclear wavefunctions. Two important physical features must be borne in mind in the determination of the form of the wavefunctions: the effects of the Pauli Exclusion Principle (P.E.P.) in systems of nuclear dimensions and the nature of the nuclear force.

The P.E.P. and the symmetries of the three-nucleon system predict a total of ten states present in the ground state of the triton. However it is expected that only a small number contribute to any appreciable extent. Thus from measurements of the magnetic moments, the capture of neutrons by deuterium and various other evidence, including variational calculations of the binding energy, we believe these states to be the predominant S-state Φ_1^t, which is completely symmetric

under spatial exchange of the nucleons, together with small admixtures of the S-state of mixed symmetry (the so-called S′ state) Φ_2^t and the D states (which we label collectively by D).

For central forces, the only state of any consequence in binding energy calculations is the S-state and in our work, the ground state of the three-nucleon system with $J=T=\frac{1}{2}$ is approximated by the spatially symmetric S-state, Φ_1^t (see equations 3 , 4 and 9).

The 1s shell nuclei have fairly large binding energies and are relatively compact structures. These physical properties suggest the component nucleons of these nuclei spend a considerable amount of time well within the range of their mutual nuclear forces. Such a physical situation compounded with our use of the velocity dependent potential which is repulsive at small interparticle separations, is indicative of the importance of short-range two-body correlations between the nucleons for these light nuclei. Our spatial wavefunction ψ^S is therefore taken to be a product of two-particle exponentials, each suitably modified by a two-body correlation function.

$$\psi^S = \prod_{i<j}^{3} g(r_{ij})\, f(r_{ij}) \qquad (138)$$

where $g(r_{ij})$ and $f(r_{ij})$ are chosen to have one of the forms

$$\text{(i)}\quad g(r_{ij}) = \exp(-a r_{ij}^2), \qquad f(r_{ij}) = 1 - c\exp(-b r_{ij}^2) \qquad (139)$$

$$\text{(ii)}\quad g(r_{ij}) = \exp(-a'|r_{ij}|), \qquad f(r_{ij}) = 1 - c'\exp(-b'|r_{ij}|)$$

$$\text{(iii)}\quad g(r_{ij}) = |r_{ij}|^{-\frac{1}{2}} \exp(-a''|r_{ij}|), \qquad (140)$$

$$f(r_{ij}) = 1 - c''\exp(-b''|r_{ij}|) \qquad (141)$$

These functions have considerable flexibility and are capable of giving a good representation of the principal features of the bound system with short-range forces. They manifest the correct close-in and asymptotic behaviour, especially functions (ii) and (iii), and are still tractable enough to allow analytic evaluation of all the quantities of interest in the trinucleon. The parameters $a, b, c, a', b', c', a'', b''$ and c'' are obtained by fitting the body form factor $F_1(q^2)$ given by

$$F_1(q^2) = \int |\psi^S|^2 \exp(i\underline{q}.\tfrac{2}{3}\underline{R})\, d\tau \qquad (142)$$

and the r.m.s. radius $R_{r.m.s.}$ given by

$$R^2_{r.m.s.} = \int |\psi^S|^2 \left\{ \frac{1}{9} \left[2(r_1^2 + r_2^2) - r_3^2 \right] \right\} d\tau = \frac{1}{3} \int |\psi^S|^2 r_1^2 \, d\tau \quad (143)$$

where $\underline{R} = \underline{x}_1 - \frac{1}{2}(\underline{x}_2 + x_3)$, r_1, r_2, r_3 are the inter-particle separations and $d\tau = 8\pi^2 \, r_1 r_2 r_3 dr_1 dr_2 dr_3$.

These integrals and the others which arise in our three-body calculations are taken over the domain consistent with the triangular inequalities $r_1 + r_2 \geq r_3$, $r_1 + r_3 \geq r_2$ and $r_2 + r_3 \geq r_1$ and are given in the appendix. Our fitting procedures are expected to produce good wavefunctions. Blatt and his co-workers have found that potentials that yield the better binding energy in a variational calculation also give wavefunctions which yield the better fits to the form factor and the r.m.s. radius. Conversely, we can confidently expect that satisfactory binding energy values and wavefunctions can be obtained by fitting our waveforms to these quantities. The experimental values of $F_1(q^2)$ selected are those of Levinger and Srivastava[100] and Okamoto and Lucas[94], whilst the r.m.s. radius is taken to be 1.66 fm. for (i) to facilitate comparison with the results of reference 94, and 1.70 fm. for (ii) and (iii). For the Gaussian, the parameters b and c

are varied from 0.2 to 6.0 fm^{-2} and 0 to 1.0 respectively, with a = 0.058, 0.062, 0.066 and 0.070 fm^{-2}. The best fit to $R_{r.m.s.}$ and $F_1(q^2)$ (see Fig. 2 and Table 7) is given by a = 0.062 fm^{-2}, b = 1.938 fm^{-2} and c = 0.4. For (ii) the parameters b′ and c′ are varied from 0.5 to 6.0 fm^{-1} and 0. to 1.0 respectively, with a′ = 0.36, 0.38, 0.40 and 0.42 fm^{-1}. The best fit to the selected $R_{r.m.s.}$ and $F_1(q^2)$ is given by a′ = 0.40 fm^{-1}, b′ = 1.60 fm^{-1} and c′ = 0.4. For (iii), b′′ and a′′ are varied from 0.5 to 6.0 fm^{-1} and 0.20 to 0.35 fm^{-1} respectively with c′′ fixed at 1.0. Good fits to $R_{r.m.s.}$ and $F_1(q^2)$ are obtained with a′′ = 0.275 fm^{-1}, b′′ = 1.525 fm^{-1} and c′′ = 1.0.

Table 7 Overleaf...

TABLE 7

THE FORM FACTOR OF THE TRINUCLEON ASSUMING A PURE S-STATE ($F_1(q^2)$)

	OBSERVED		CALCULATED					
					3-parameter			
q^2	OL [a]	LS [b]	Hard Core [c]		exponential [d]	(i)	(ii)	(iii)
(fm^{-2})			0.45 fm	0.50 fm				
1.0	0.645	0.639	0.645	0.633	0.563	0.630	0.636	0.644
2.0	0.420	0.412	0.438	0.423	0.337	0.395	0.424	0.439
3.0	0.300	0.282	0.307	0.293	0.211	0.246	0.293	0.312
4.0	0.210	0.193	0.220	0.207	0.138	0.151	0.208	0.228
5.0	0.144	0.125	0.160	0.149	0.091	0.092	0.152	0.170
6.0	0.103	0.091	0.118	0.107	0.063	0.055	0.113	0.129
7.0			0.087	0.078		0.031	0.085	0.099
8.0	0.041	0.029			0.029	0.018	0.065	0.078

[a]) From the analysis of Okamoto and Lucas [94])

[b]) Calculated from the two sets of experimental form factors obtained by Levinger and Srivastava [100])

[c]) From the work of Tang and Herndon [24]) with repulsive-core potentials. The figures refer to the core radii.

[d]) From Levinger and Srivastava's calculations with a three-parameter exponential wavefunction.

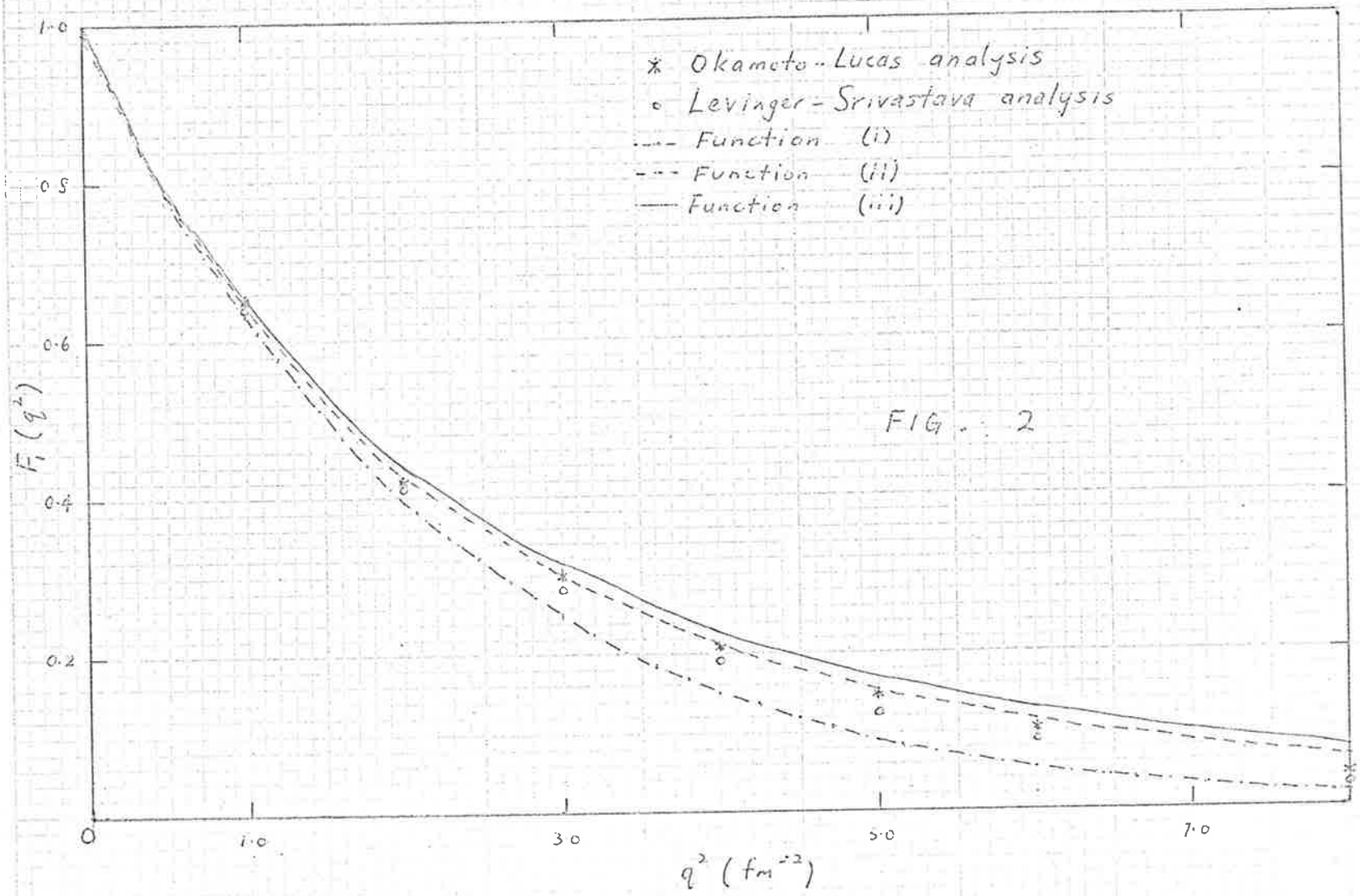

* Okamoto - Lucas analysis
° Levinger - Srivastava analysis
Function (i)
Function (ii)
Function (iii)
1·0
0·5
0·6
0·4
0·2
0
1·0
3·0
5·0
7·0
$F_1(q^2)$
q^2 (fm^{-2})

FIG. 2

5.3 Triton Binding Energy and Coulomb Energy of ^{3}He

With our ground state wavefunction, we can write the binding energy of the triton as

$$E(^3H) = \frac{T(\psi^S,\psi^S) + V(\psi^S,\psi^S)}{N(\psi^S,\psi^S)} \tag{144}$$

where the expectation values

$$T(\phi,\xi) = -\int\left\{\frac{\hbar^2}{M}\left(\frac{\partial\phi^*}{\partial r_1}\frac{\partial\xi}{\partial r_1} + \frac{\partial\phi^*}{\partial r_2}\frac{\partial\xi}{\partial r_2} + \frac{\partial\phi^*}{\partial r_3}\frac{\partial\xi}{\partial r_3}\right)\right.$$

$$\left. + \frac{\hbar^2}{M}\left[t(123) + t(231) + t(312)\right]\right\}d\tau \tag{145}$$

$$V(\phi,\xi) = \int\phi^*\left[3\,V_{eff}(r_{ij})\right]\xi\;d\tau \tag{146}$$

$$N(\phi,\xi) = \int\phi^*\xi\;d\tau \tag{147}$$

$$\text{with } t(ijk) = \frac{r_i^2+r_j^2-r_k^2}{4r_ir_j}\left(\frac{\partial\phi^*}{\partial r_i}\frac{\partial\xi}{\partial r_j} + \frac{\partial\phi^*}{\partial r_j}\frac{\partial\xi}{\partial r_i}\right) \tag{148}$$

and

$$V_{eff}(r_{ij}) = \tfrac{1}{2}\left[V_t(r_{ij}) + V_s(r_{ij})\right] \tag{149}$$

For functions of the product type

$$\psi^S = f(r_{12})\, f(r_{13})\, f(r_{23}) \qquad (150)$$

the expectation value of the kinetic energy operator may be written in the simple form

$$T(\psi^S,\psi^S) = \frac{\hbar^2}{M} \sum_{\text{et cycl}} \int \psi^S \left[- \frac{\partial^2 \psi^S}{\partial r_3^2} - \frac{2}{r_3} \frac{\partial \psi^S}{\partial r_3} - \frac{r_1^2 + r_2^2 - r_3^2}{2 r_1 r_2} \frac{\partial^2 \psi^S}{\partial r_1 \partial r_2} \right] d\tau$$

$$= \frac{3\hbar^2}{2M} \int \left[f'(r_3))^2 - f(r_3) f''(r_3) - \frac{2}{r_3} f(r_3) f'(r_3) \right] f^2(r_1) f^2(r_2)\, d\tau \qquad (151)$$

The two interpretations of $\underline{p}$ alluded to earlier in the chapter result only in a difference in the expectation value of the velocity-dependent part of the potential energy. The expression obtained by Srivastava becomes

$$V_{\text{vel.dep.}}(\psi^S,\psi^S) = \frac{3}{2} \int \psi^{S*} \left\{ (V_0)_{\text{vel.dep.}} \right\} \left\{ [p_{12}^2\ \omega_s(r_3) + \omega_s(r_3) p_{12}^2] + X_{\text{vel.}} [p_{12}^2 \omega_t(r_3) + \omega_t(r_3) p_{12}^2] \right\} \psi^S\, d\tau \qquad (152)$$

where

$$p_{12}^2 = -\left(\frac{\partial^2}{\partial r_3^2} + \frac{2}{r_3}\frac{\partial}{\partial r_3}\right) \qquad (153)$$

while that of Lovitch and Rosati is

$$V_{vel.dep.}(\psi^s\psi^s) = \frac{3}{2}\int\left\{\left[\omega_{eff}(r_3) + \frac{1}{4}\omega_{eff}(r_1) + \frac{1}{4}\omega_{eff}(r_2)\right]\right.$$

$$\left. t_3(g_3f_3)g_1^2f_1^2g_2^2f_2^2\right\}d\tau \qquad (154)$$

where

$$\omega_{eff}(r) = \omega_s(r) + \omega_t(r)\ ,\ t_3(u) = \left(\frac{\partial u}{\partial r_3}\right)^2 - \frac{u}{r_3}\frac{\partial^2(r_3u)}{\partial r_3^2} \qquad (155)$$

To evaluate the Coulomb energy of ^{3}He, we assume for this nucleus the same ground state wavefunction as in ^{3}H and use the potential $V_{pp}^{el}(r)$ of Schneider and Thaler[101], which takes account of the finite nucleon size. Thus

$$C.E. = C(\psi^s,\psi^s) \qquad (156)$$

with

$$C(\phi,\xi) = \int\phi^*\left\{\frac{e^2}{r_3}\left[1-e^{-3.36r_3}(0.582r_3-2.776) - e^{-2.97r_3}\right.\right.$$

$$\left.\left.(0.644r_3+3.639)\right]\right\}\xi\ d\tau \qquad (157)$$

A direct comparison between C.E. and the binding energy difference of the mirror nuclei ^3H and ^3He may provide important information about the existence of charge asymmetry of nuclear forces.

5.4 The Photodisintegration Cross-Sections of ^3H and ^3He

The investigation of the static characteristics of ^3H and ^3He is by no means sufficient for establishing the complicated nature of the wavefunction and our choice of interaction. Nuclear reactions involving the three-nucleon system must be studied to furnish complementary information. An important source of such information about the structure and interactions of the nucleons is the nuclear photoeffect. Unfortunately the calculation of the differential and total cross-sections of different photonuclear processes requires an intimate knowledge of the wavefunctions of the initial and final states of the system, which are known poorly. The situation is partially obviated if the cross-sections are summed over all possible excited states, using closure for the matrix elements, so that our results will depend only on the ground state. Two quantities are of

interest here, the bremsstrahlung-weighted cross-section and the integrated cross-section. General expressions for these cross-sections have been derived by Levinger and Bethe[85] and Rustgi and Levinger[102] on the basis of the sum-rules of Thomas, Reiche and Kuhn. The bremsstrahlung-weighted cross-section σ_b is defined as

$$\sigma_b = \int_0^\infty \frac{\sigma(W)}{W} \, dW \qquad (158)$$

where $\sigma(W)$ is the electric dipole absorption cross-section, when we neglect retardation of the nucleus and average over all orientations of the nucleus and W is the photon energy. If the spatial part of the ground state wavefunction is totally symmetric, σ_b is simply related to the r.m.s. radius through the expression

$$\sigma_b = \frac{4}{3} \pi^2 \frac{e^2}{\hbar c} \frac{NZ}{A-1} R^2_{r.m.s.} \qquad (159)$$

Thus to determine $\sigma_b(^3He)$, we need only multiply our mean square radius by the constant in equation (159). As we have used a spatially symmetric wavefunction, $R_{r.m.s.}$ is also the charge radius for point nucleons,

a fact not realised by Srivastava[75] and Rustgi[74], who have equated it to the true charge radius. It is obvious that σ_b does not depend on the character of the nucleon-nucleon forces but is determined solely by the properties of the wavefunction of the ground state of the system.

The integrated photodisintegration cross-section

$$\sigma_{int} = \int_0^\infty \sigma(W)\,dW \tag{160}$$

can be written as

$$\sigma_{int} = \left(\frac{2\Pi^2 e^2 \hbar}{Mc}\right)\left[\left(\Sigma_n f_{on}\right)_T + \left(\Sigma_n f_{on}\right)_{static} + \left(\Sigma_n f_{on}\right)_{vel.dep.}\right] \tag{161}$$

where $\Sigma_n f_{on}$ is the summed oscillator strength, and T, static and vel. dep. refer to the contributions of the kinetic energy, the static potential and the velocity dependent part of the potential. If we assume a spatially constant electric field along the z-axis

$$\Sigma_n f_{on} = -\frac{M}{\hbar^2}\int \psi^s{}^* [[0, D_z], D_z]\psi^s d\tau \tag{162}$$

where the operator O is

$$-\frac{\hbar^2}{2M}\sum_{i}^{3}\nabla_i^2,\ \sum_{i<j}^{3}(\omega P_{ij}^{W} + bP_{ij}^{B} + hP_{ij}^{H} + mP_{ij}^{M})V_{static}(r_{ij})$$

$$\text{or}\ \sum_{i<j}^{3} V_{vel.dep.}(r_{ij})$$

and

$$D_Z = \frac{N}{A}\Sigma_p\, z_p - \frac{Z}{A}\Sigma_n\, z_n \qquad (163)$$

On explicit evaluation,

$$(\Sigma_n f_{on})_T = \frac{N^2}{A^2} Z + \frac{Z^2}{A^2} N = \frac{NZ}{A} \qquad (164)$$

$$(\Sigma_n f_{on})_{static} = -\left(\frac{M}{3\hbar^2}\right)\left\{\int \psi^{s*}\left[\Sigma_i\Sigma_j r_{ij}^2 (V)_{static}(r_{ij})\right.\right.$$

$$\left.\left.(mP_{ij}^{M} + hP_{ij}^{H})\right]\psi^{s} d\tau\right\} \qquad (165)$$

$$(\Sigma_n f_{on})_{vel.dep.} = \frac{4}{9}\frac{M}{\hbar^2}(V_o)_{vel.dep.}\left\{\int \psi^{s*}\left[\omega_s(r_{ij}) + X_{vel.}\right.\right.$$

$$\left.\left.\omega_t(r_{ij})\right]\psi^{s}\, d\tau\right\} \qquad (166)$$

In equation (165), i and j denote protons and neutrons, respectively.

In the derivation of these relations we have used the property that

$$\left[\left[\Sigma_i\Sigma_j P^M_{ij},\ \Sigma_i z_i\right],\ \Sigma_i z_i\right] = \left[\Sigma_i\Sigma_j (z_j - z_i) P^M_{ij},\ \Sigma_i z_i\right]$$

$$= \Sigma_i\Sigma_j (z_j - z_i)^2\ P^M_{ij} \qquad (167)$$

$$(z_j - z_i)^2 = \frac{1}{3}\ r^2_{ij} \qquad (168)$$

$$D_z = \frac{1}{3}\ z_{13} + \frac{1}{3}\ z_{23} \qquad (169)$$

and

$$\left[\left[P^2_{23}\ \omega(r_1) + \omega(r_1) P^2_{23},\ z_{23}\right],\ z_{23}\right] = -\ 4\hbar^2\,\omega(r_1) \qquad (170)$$

Application of the Thomas-Reiche-Kuhn sum-rules has eliminated the terms in w and b from $(\Sigma_n f_{on})_{static}$ since the corresponding Wigner and Bartlett operators commute with the space coordinates. The expectation value of the Heisenberg operator with our S-state wave-function is

$$\langle P^H_{ij} \rangle = \tfrac{1}{2} \langle P^M_{ij} \rangle . \qquad (171)$$

Since

$$(V)_{static}(r_{ij}) = \frac{1}{(\omega+m)}\ [V_{eff}(r_{ij})]_{static} \qquad (172)$$

our expression for σ_{int} finally reduces to

$$\sigma_{int} = \frac{4\Pi^2 e^2 \hbar}{3Mc}\left[1 - \frac{M(m+\frac{1}{2}h)}{(\omega+m)\hbar^2}\int \psi^{s*}\left\{[V_{eff}(r_{ij})]_{static}\right.\right.$$

$$\left.r_{ij}^2\ P_{ij}^M\right\}\psi^s\ d\tau + \frac{2}{3}\frac{M}{\hbar^2}\ (V_o)_{vel.dep.}$$

$$\left.\int \psi^{s*}\ [\omega_s(r_{ij}) + X_{vel.}\ \omega_t(r_{ij})]\ \psi^s d\tau\right] \qquad (173)$$

5.5 Results and Discussion

Our results are shown in Figure 1 and Tables 7 to 9. It is obvious that the best overall fit to $F_1(q^2)$ is given by (ii). The fitting procedure has been slightly biased in favour of a good fit at small q^2 and although a rigid adherence to a least squares fit may improve the results for (i) and (ii), the experimental results at high momentum transfer are not known accurately enough to warrant it. Our correlated Gaussian gives almost the same values of $F_1(q^2)$, $R_{r.m.s.}$ and the Coulomb energy as the soft core wavefunction of Okamoto and Lucas[94]. This suggests little difference between their trial function and ours.

TABLE 8

THE BINDING ENERGY, COULOMB ENERGY

AND THE R.M.S. RADIUS OF THE THREE-NUCLEON SYSTEM

Wave-function	Potential	B.E. (^{3}H) (MeV)	C.E. (point) (MeV)	C.E. (finite) (MeV)	r.m.s. radius (fm)
Uncorrelated Gaussian [a]	Herndon-Tang Case 2	2.50	0.708	0.66	1.62
Best Variational Case A [b]	Herndon-Tang Case 2	7.623	0.771		1.63
Case B	Herndon-Tang Case 2	8.240	0.717		1.67
(i)	Herndon-Tang Case 2	3.30	0.67	0.62	1.66
Uncorrelated Exponential [c]	Srivastava Case 1	6.08	0.597		
Three-parameter Exponential [c]	Srivastava Case 1	7.17	0.663		1.92
Best Variational Case A [b]	Srivastava Case 2	10.903	0.748		1.65
Case B	Srivastava Case 2	11.201	0.706		1.69
(ii)	Srivastava Case 1	9.30	0.74	0.69	1.70

Table cont'd .....,

TABLE 8 Cont'd.

	Srivastava				
(ii)	Case 2	10.20	0.74	0.69	1.70
	Our potential				
(ii)	Case 1	5.92	0.74	0.69	1.70
	Our potential				
(ii)	Case 2	6.76	0.74	0.69	1.70
(ii)	Rarita-Present	7.20	0.74	0.69	1.70
Best Variational	Rarita-Present				
Case A[d])		7.69	0.81		1.68
Case B	Rarita-Present	7.95	0.75		1.70
	Our Potential				
(iii)	Case 2	3.53	0.76	0.72	1.70
(iii)	Rarita-Present	6.20	0.76	0.72	1.70
Hard-core					
0.45fm [e])	Tang-Herndon	7.42		0.69	1.68
0.50fm [e])	Tang-Herndon	7.05		0.67	1.72
Experimental		8.49	0.764		1.66 ($\langle r^2 \rangle^{\frac{1}{2}}$)
					1.70 ($R_c(^3H)$)
					1.87 ($R_c(^3He)$)

[a]) This is the simple Gaussian of reference 94.

[b]) Calculations from reference 41.

[c]) Evaluated by Srivastava[40a]) using an identical form to our velocity dependent potential but with $X_{static} = 1.84$ and $X_{vel.} = 0.55$.

[d]) Calculations from reference 92.

[e]) From the paper of Tang and Herndon[24]). The figures refer to the core radii of the repulsive-core potentials used by them.

TABLE 9

PHOTODISINTEGRATION CROSS-SECTIONS OF ^3He

Wave function	Potential	σ_b (mb)	$\sigma_{int}(S)^d$) (Mev.mb)	$\sigma_{int}(B)^d$) (Mev.mb)	$\sigma_{int}(RI)^d$) (Mev.mb)
Hard-Core 0.40fm[a])	Ohmura, et al.	2.29	56.9	60.3	67.0
Three-parameter exponential[b])	Our Potential	3.5	56.0	59.6	66.9
(i)	Our Potential	2.62	52.0	54.8	60.6
(ii)	Our Potential	2.74	57.8	62.0	70.5
(iii)	Our Potential	2.74	62.6	70.5	86.9
Experimental[c])		2.53±0.19		62±6	

a) Extracted from reference 111.

b) From the calculations of Srivastava[75]) but with our potential parameters.

c) The experimental results of Gorbunov and Spiridonov[115]).

d) The letters S, B and RI refer to Serber, Biel, Rosenfeld and Inglis force mixtures.

However, an advantage with our modified Gaussian is that it is perhaps more tractable for binding energy calculations. The triton binding energy obtained using Herndon and Tang's velocity dependent potential is much smaller than that of case A and case B of Lovitch and Rosati's 'best' variational calculation. This may be attributed to the poor fit to $F_1(q^2)$ for large momentum transfers with (i). Besides, the Gaussian form is known to be unsuitable for asymptotic regions, and this is partially confirmed by the smallness of $\sigma_{int}(^3He)$. From these considerations, it appears the correlated Gaussian function falls short as a realistic trinucleon wavefunction. It would seem therefore that Okamoto and Lucas's 'accurate' estimate of C.E. is subject to doubt.

For the true exponential wavefunction (ii), we can expect the results to be more trustworthy. The experimental form factor is fitted well at all values of q^2 and it exhibits the correct asymptotic behaviour (the asymptotic form of (iii), $\exp(-\mu r_{ij})/r_{ij}^{\frac{1}{2}}$ with $\mu = \left[\frac{M}{3h^2} (B.E.(^3H) - B.E.(D)) \right]^{\frac{1}{2}}$, is suggested by the work of Pappademos and it is encouraging to note

that (ii) is almost identical to (iii) for r_{ij} greater than 5 fm.). Also our results for the binding energy are sufficiently close to Lovitch and Rosati's to encourage us to believe that our trial function (ii) is near to the best obtained variationally. Using our velocity dependent potential, we find B.E.(^{3}H) to be very reasonable (especially if a direct comparison is made with the hard-core calculations of Tang and Herndon). Together with the excellent agreement between our σ_{int} and experiment the suggestion is that our potential is quite accurate. The two interpretations of $\underline{p}$ lead to a small quantitative difference (less than 1 Mev) in the calculated values of the binding energy, so it can be concluded that the discrepancy between Srivastava and Lovitch and Rosati's binding energy results is almost wholly attributable to the poor trial function used by Srivastava. σ_{int} is little affected by our use of Srivastava's interpretation of $\underline{p}$ since the velocity dependent part of our potential contributes almost negligibly.

Although function (iii) has the right asymptotic properties, it yields a low B.E. value and must be inadequate for small interparticle separations. The velocity dependent potential has only a soft repulsive core and it may be that our wavefunction should not vanish for zero interparticle distances; Lovitch and Rosati's variational functions and (ii) remain finite for r_{ij} equal to zero.

Summarising, we record that:

1) Our exponential velocity dependent potential yields results equivalent to a hard-core potential.

2) When a soft repulsive internucleon force is assumed, short-range two-body correlations are essential in binding energy calculations.

3) Our fitting procedures for obtaining trial functions for the trinucleon provide realistic wavefunctions if a suitable spatial form is selected at the outset.

4) The Lovitch-Rosati and Srivastava interpretations of $\underline{p}$, the momentum vector in the velocity dependent interaction, result in only a

small quantitative difference in the calculated binding energies and little or no difference in the integrated photodisintegration cross-sections.

5) The photodisintegration cross-section is sensitive to the tail of the ground state wave-function, which explains the almost negligible contribution of the velocity dependent part of the nuclear force, and the small value of σ_{int} for the Gaussian.

6) The Serber and Biel force mixtures are favoured in σ_{int} calculations.

7) Since our correlated exponential function provides good fits to $F_1(q^2)$ and σ_{int} simultaneously, the charge distribution in the trinucleon as measured by high energy electron scattering is the same as that which gives rise to electric dipole absorption.

8) Our evaluation of the Coulomb energy of ^3He, taken with those of other authors, indicates that C.E. is less than the binding energy difference in the two three-nucleon systems and that charge asymmetry may be present to the order of 0.1 MeV (Okamoto and Lucas's estimate of 0.13 MeV may be slightly too large).

9) Our calculations can be profitably extended to other light nuclei.

CHAPTER SIX

THE S′ STATE IN THE TRINUCLEON

Two important tools for extracting information about the structure of the three-nucleon system are the high-energy scattering of electrons by ^3H and ^3He[95] and the muon capture rate of ^3He[81,82]. Considerable data exists for these experiments and a satisfactory theoretical analysis of these reaction processes should yield a clearer picture of the detailed structure of the ground state of the trinucleon. Many authors[60,82,87,100,103-105] have made estimates of the relative probabilities of the various states expected to be present in the ground state by fitting such experimental data. Unfortunately these estimates span a large range; for example the probability of the S′ state $P_{S'}$, assumes values from 0.5 to 4.0%. Lately two groups have come to agree among themselves on the magnitude of $P_{S'}$. Thus Schiff and his co-workers take $P_{S'}$ to be 2% while Mitra et al.[60], Levinger and Srivastava[100] and Davies[112] have obtained a value of about 1%. Both sets of

investigations possess shortcomings however; the variational calculations of Schiff's group are obscured by their use of simple wavefunctions that do not exhibit the correct asymptotic as well as close-in behaviour for the three-particle system while the separable non-local method of Mitra gives rise to values of the trineutron binding energy and the Coulomb energy of ^3He [63-66,94] which are inconsistent with the general variational method and with experiment. As some of the expectation values of operators in the trinucleon are extremely sensitive to the form of the wavefunction assumed, these doubts on the accuracy of the results of both groups do not allow us to make any firm conclusions about the actual probabilities of the states. In this chapter we report a careful estimate of $P_{S'}$ [83], assuming for simplicity that only the S and S' states are present in the ground state. Insofar as it can affect our conclusions the D state is also considered qualitatively.

6.1 The Trinucleon Wavefunction

From chapter two, our ground state wavefunction is approximated by

$$\Psi(^2S_{\frac{1}{2}}) = \Phi_1^t + \Phi_2^t \tag{174}$$

where ψ^s is function (ii) in chapter five and ψ' and ψ'' take the forms

$$\psi' = \frac{1}{\sqrt{6}}\left[g(12,3) + g(13,2) - 2g(23,1)\right] \tag{175}$$

$$\psi'' = \frac{1}{\sqrt{2}}\left[g(12,3) - g(13,2)\right] \tag{176}$$

with

$$g(ij,k) = Bg(ik)g(jk)h(ij) \tag{177}$$

and

$$\begin{aligned} g(ij) &= \exp(-\alpha r_{ij}), \quad \alpha = 0.37\ \text{fm}^{-1} \\ h(ij) &= \exp(-\beta r_{ij}) \end{aligned} \tag{178}$$

The two mixed symmetry functions of the S′ state are assumed to have simple exponential form as it is unlikely that experimental data are sensitive to the interior forms of the S′ state functions. In fact correlations are indirectly introduced into these functions since

the most significant contributions of ψ' and ψ'' result from the overlap of the S′ state components with the correlated S state. The results of Gibson[87] and Okamoto and Lucas[126] also support the neglect of correlations in ψ' and ψ''. The value assumed for α stems from form factor and binding energy fits with unmodified exponentials for ψ^S. In terms of the functions g(ij,k), $P_{S'}$ is

$$P_{S'} = 2\int[g^2(12,3) - g(12,3)g(13,2)]\,d\tau \qquad (179)$$

which becomes

$$P_{S'} = 2\left[1 - \frac{(2\alpha)^3(\alpha+\beta)^2\left(\frac{11}{2}\alpha^2+2\beta^2+\frac{13}{2}\alpha\beta\right)}{(8\alpha^2+5\alpha\beta+\beta^2)\left(\frac{3}{2}\alpha+\frac{1}{2}\beta\right)^5}\right] \qquad (180)$$

if we normalise g(ij,k) to unity. Thus having fixed the value of α, we can determine β from the magnitude of $P_{S'}$; B is the product of $(P_{S'})^{\frac{1}{2}}$ and the normalisation constant of the g(ij,k). Our wavefunction ψ^S is normalised to unity and is then multiplied by $(P_S)^{\frac{1}{2}}$.

6.1 Charge Form Factors

The experimental data on the elastic scattering

of electrons from ^3H and ^3He can be analysed in terms of the electric charge form factors by means of the Rosenbluth equation for spin $\frac{1}{2}$ systems. The basic formulae, which take into account the charge and the spin of the nuclei, have been used to express $F_{ch}(^3H)$ and $F_{ch}(^3He)$ as functions of the four-momentum transfer q. Using non-relativistic kinematics and the impulse approximation, we obtain the relevant charge form factors as the three-dimensional Fourier transforms of the expectation values of the nuclear charge density. If we assume that the three nucleons contribute without mutual interference or distortion, the charge density operator is

$$\rho_c(\underline{r},\underline{r}_i) = \sum_{i=1}^{3}\left[\tfrac{1}{2}(1+\tau_{iz})f^p_{ch}(\underline{r}-\underline{r}_i)+\tfrac{1}{2}(1-\tau_{iz})f^n_{ch}(\underline{r}-\underline{r}_i)\right] \quad (181)$$

The τ's are isospin matrices and the quantities $f^p_{ch}(\underline{r}-\underline{r}_i)$ and $f^n_{ch}(\underline{r}-\underline{r}_i)$ are the nucleon spatial distributions of the charge densities about the centres of the nucleons, or alternatively, they may be thought of as the three-dimensional Fourier transforms of the normalised nucleon electromagnetic form factors. The Fourier transform is therefore

$$ZF_{ch} = \iint \exp(i\ \underline{q}\cdot\underline{r})\Psi^*\ \rho_c(\underline{r},\underline{r}_i)\Psi\ d^3r\ d^3r_i \qquad (182)$$

The integration over $\underline{r}$ is performed by changing variables from $\underline{r}$ to $\underline{r}-\underline{r}_i$ which causes the nucleon form factors to appear as multiplying factors. The isospin sums are carried out by means of Table 1 and evaluating the integrals involving $\exp(i\ \underline{q}\cdot\underline{r}_i)$, we get the expressions

$$2F_{ch}(^3He) = (F^n_{ch}+2F^p_{ch})F_1 + (F^n_{ch}+F^p_{ch})F_2 \qquad (183)$$

$$F_{ch}(^3H) = (F^p_{ch}+2F^n_{ch})F_1 + (F^p_{ch}-F^n_{ch})F_2 \qquad (184)$$

where

$$F_1 = F_1(q^2) \qquad (185)$$

$$F_2 = \tfrac{2}{3}\,F_2(q^2)$$

and

$$F_1(q^2) = \int |\psi^S|^2 \exp(i\ \underline{q}\cdot\underline{x}_1)\ d\tau \qquad (187)$$

$$F_2(q^2) = \frac{1}{\sqrt{6}} \int \left[\exp(i\ \underline{q}\cdot\underline{x}_1)-\exp(i\ \underline{q}\cdot\underline{x}_3)\right]\psi^S g(12,3)\ d\tau \qquad (188)$$

$\underline{x}_i$ is the distance from the centre of mass of the trinucleon to the nucleon i. The evaluation of

these integrals is easily accomplished with the help of the universal function

$$F(q^2) = \int \exp(-\alpha_1 r_1 - \alpha_2 r_2 - \alpha_3 r_3) \exp(i\, \underline{q} \cdot \underline{x}_1) r_1 r_2 r_3 \, dr_1 dr_2 dr_3$$

$$= \frac{1024\ \alpha_1 \alpha_2 \alpha_3}{\Pi} \int_0^\infty \left[\frac{A(k_1)}{B(k_1)} \ln \left\{ C(k_1) \times D(k_1) \right\} + \frac{2}{E(k_1)} + \frac{1}{2G(k_1)} \right] dk_1 \qquad (189)$$

with

$$A(k_1) = \frac{k_1^2}{(k_1^2+\alpha_2^2)^2\ (2k_1^2+4\alpha_3^2-2\alpha_1^2-\frac{4}{9}\,q^2)^2}$$

$$B(k_1) = -\frac{4}{3}\, q k_1 (2k_1^2 + 4\alpha_3^2 - 2\alpha_1^2 - \frac{4}{9}\, q^2)^2$$

$$C(k_1) = \frac{k_1^2 + \alpha_1^2 + \frac{4}{9}\, q^2 - \frac{4}{3}\, q k_1}{k_1^2 + \alpha_1^2 + \frac{4}{9}\, q^2 + \frac{4}{3}\, q k_1} \qquad (190)$$

$$D(k_1) = \frac{4k_1^2 + 4\alpha_3^2 + \frac{4}{9}\, q^2 + \frac{8}{3}\, q k_1}{4k_1^2 + 4\alpha_3^2 + \frac{4}{9}\, q^2 - \frac{8}{3}\, q k_1}$$

$$E(k_1) = (4k_1^2 + 4\alpha_3^2 + \frac{4}{9}\, q^2)^2 - (\frac{8}{3}\, q k_1)^2$$

$$G(k_1) = (k_1^2 + \alpha_1^2 - \frac{4}{9}\, q^2)^2 - (\frac{4}{3}\, q k_1)^2$$

The derivation of this function is given explicitly in the appendix. Our computed values of $F_1(q^2)$ and $F_2(q^2)$ for various square momentum transfer q^2 can be compared directly with the experimental quantities

$$\left[2F_{ch}(^3He) + F_{ch}(^3H)\right] \Big/ \left[3F^p_{ch} + 3F^n_{ch}\right]$$

and

$$\left[F_{ch}(^3He)(2F^n_{ch}+F^p_{ch}) - \tfrac{1}{2}F_{ch}(^3H)(F^n_{ch}+2F^p_{ch})\right] \Big/ \left[3(F^n_{ch})^2 - 3(F^p_{ch})^2\right]$$

respectively. The first checks on the validity of ψ^S as a flexible function while the second determines the correct magnitude of $P_{S'}$.

6.3 Muon Capture Rate

In the reaction $^3He + \mu^- \rightarrow {}^3H + \nu$, the recoil ^3H is easily identified by its unique energy, 1.9 MeV. The capture rate has been carefully measured and two observed capture rates are (accurate to 3%)

$$\Lambda_\mu = \begin{matrix} 1485 \pm 40 \\ 1468 \pm 40 \end{matrix} \text{ sec}^{-1} \quad \begin{matrix} \text{(Ref. 81)} \\ \text{(Ref. 82)} \end{matrix} \qquad (191)$$

If we consider ^{3}He to be an agglomerate of three nucleons which may capture a muon as if they were free, neglect relativistic terms as a first approximation and assume the V-A theory for weak interactions, the capture rate for the transition ^{3}He to ^{3}H can be shown to be (see ref. 106)

$$\Lambda_\mu = \frac{1}{(2\Pi)^3}\left[\frac{Zm_\mu}{137}\right]^3 \frac{\nu^2}{1+\frac{|\underline{\nu}|}{M_{^3H}}} \int \frac{d\underline{\nu}}{|\underline{\nu}|} \sum_{m_{^3H}=\pm\frac{1}{2}} \sum_{m_{^3He}=\pm\frac{1}{2}} \tag{192}$$

$$\times \left\{ G_V^2 \left|\int 1\right|^2 + G_A^2 \left|\int \underline{\sigma}\right|^2 + (G_p^2 - 2G_pG_A) \left| \frac{\underline{\nu}}{|\underline{\nu}|} \int \underline{\sigma} \right|^2 \right\}$$

with

$$G_V \equiv g_V\left(1 + \frac{|\underline{\nu}|}{2M}\right) + g_s$$

$$G_A \equiv g_A - (g_V + g_M)\frac{|\underline{\nu}|}{2M}$$

$$G_P \equiv \frac{|\underline{\nu}|}{2M}(g_P - g_A - g_V - g_M + g_T) \tag{193}$$

$$\int 1 = \langle \Psi_{^3H} | \sum_{i=1}^{3} \exp[-i\,\underline{\nu}\cdot\underline{x}_i]\, \phi_\mu(\underline{x}_i)\, \tau_i^{(-)} | \Psi_{^3He} \rangle$$

$$\int \underline{\sigma} = \langle \Psi_{^3H} | \sum_{i=1}^{3} \exp[-i\,\underline{\nu}\cdot\underline{x}_i]\, \phi_\mu(\underline{x}_i)\, \tau_i^{(-)}\, \underline{\sigma}_i | \Psi_{^3He} \rangle$$

where $|\underline{\nu}|$ is the neutrino energy (taken to be 102.5 MeV), $\underline{\nu}/|\underline{\nu}|$ is the unit vector in the direction of the emitted neutrino, M the nucleon mass, $M_{^3H}$ the triton mass, m_μ the reduced mass of the muon and $\phi_\mu(\underline{x}_i)$ the muon wavefunction. In obtaining the above expression for the capture rate, one has to average over the initial muon polarisation and nuclear orientation and sum over the final neutrino polarisation and nuclear orientation. $\phi_\mu(\underline{x}_i)$ is very close to 1.0 over nuclear dimensions and as an approximation is assumed to have that value in our calculations. Using the relations

$$\langle\Psi_{^3H}|\sum_{i=1}^{3}\exp[-i\,\underline{\nu}.\underline{x}_i]\;O_i|\;\Psi_{^3He}\rangle = 3\langle\Psi_{^3H}|\exp[-i\,\underline{\nu}.\underline{x}_i]O_i|\Psi_{^3He}\rangle \qquad (194)$$

where O_i is 1 or $\underline{\sigma}_i$, and

$$\int \frac{d\underline{\nu}}{|\underline{\nu}|}\left|\frac{\underline{\nu}}{|\underline{\nu}|}\int\underline{\sigma}\right|^2 = \frac{1}{3}\int\frac{d\underline{\nu}}{|\underline{\nu}|}\left|\int\underline{\sigma}\right|^2 \qquad (195)$$

which follow since our transition does not lead to any change in the total spin and parity, and introducing the Gamow-Teller constant $\Gamma^2 = G_A^2 + \frac{1}{3}(G_p^2 - 2G_pG_A)$ we

find that the expression for the capture rate becomes

$$\Lambda_\mu = \frac{1}{(2\Pi)^3}\left[\frac{Zm_\mu}{137}\right]^3 \frac{\underline{\nu}^2}{1+\frac{|\underline{\nu}|}{M_{^3H}}} \frac{9}{2}\int \frac{d\underline{\nu}}{|\underline{\nu}|} \sum_{m_{^3H}=\pm\frac{1}{2}} \sum_{m_{^3He}=\pm\frac{1}{2}} \tag{196}$$

$$\left\{G_V^2 \left|\int 1\right|^2 + \Gamma^2 \left|\int \underline{\sigma}\right|^2\right\}$$

Rewriting $\left|\int \underline{\sigma}\right|^2$ as

$$\left|\int \underline{\sigma}\right|^2 = \left|\int \underline{\sigma}_z\right|^2 + 2\left|\int \underline{\sigma}_+\right|^2 + 2\left|\int \underline{\sigma}_-\right|^2 \tag{197}$$

where

$$\underline{\sigma}_+ = \tfrac{1}{2}(\underline{\sigma}_x + i\,\underline{\sigma}_y) \quad , \quad \underline{\sigma}_- = \tfrac{1}{2}(\underline{\sigma}_x - i\,\underline{\sigma}_y) \tag{198}$$

and evaluating the spin summation explicitly, we reduce Λ_μ to

$$\Lambda_\mu = \frac{1}{(2\Pi)^3}\left[\frac{Zm_\mu}{137}\right]^3 \frac{\nu^2}{1+\frac{|\underline{\nu}|}{M_{^3H}}} \frac{2}{2}\,4\Pi\left[G_V^2\,F_F^2 + \frac{3}{2}\,\Gamma^2 F_{GT}^2\right] \tag{199}$$

where

$$F_F = -\frac{2}{3}P_S\langle\psi^s|\exp[-i\,\underline{\nu}\cdot\underline{x}_1]|\psi^s\rangle - \frac{1}{3}P_{S'}\left(\langle\psi'|\exp[-i\,\underline{\nu}\cdot\underline{x}_1]\right.$$

$$\left.|\psi'\rangle + \langle\psi''|\exp[-i\,\underline{\nu}\cdot\underline{x}_1]|\psi''\rangle\right) - \frac{4}{3\sqrt{2}}\sqrt{P_S P_{S'}} \tag{200}$$

$$\langle\psi^s|\exp[-i\,\underline{\nu}\cdot\underline{x}_1]\,\psi'\rangle - \frac{4}{\sqrt{6}}\sqrt{P_S P_{S'}}\,\langle\psi^s|\exp[-i\,\underline{\nu}\cdot\underline{x}_1]|\psi''\rangle$$

and

$$F_{GT} = -\frac{2}{3} P_S \langle \psi^s | \exp[-i \underline{\nu}.\underline{x}_1] | \psi^s \rangle + \frac{1}{3} P_{S'} \langle \psi' | \exp[-i \underline{\nu}.\underline{x}_1] | \psi' \rangle - \frac{4}{3\sqrt{3}} P_{S'} \langle \psi' | \exp[-i \underline{\nu}.\underline{x}_1] | \psi'' \rangle - \frac{1}{9} P_{S'} \langle \psi'' | \exp[-i \underline{\nu}.\underline{x}_1] | \psi'' \rangle \quad (201)$$

These expectation values are easily obtained by using $F(q^2)$.

6.4 Results and Discussion

Our function (ii) has been shown in the last chapter to be the best of the correlated functions used and gives an excellent fit to the body form factor $F_1(q^2)$. It also yields good agreement with the static properties of the trinucleon and appears to be a good representation of the three-body system. Our use of these exponential wavefunctions, suitably modified by short-range correlations, should therefore give an improved estimate of $P_{S'}$ over that of Gibson and Mitra. We have checked $F_2(q^2)$ against experiment for three values of $P_{S'}$. These curves are plotted in Fig. 3. It is worthwhile mentioning here that for our calculated $F_2(q^2)$ to be positive i.e. in agreement

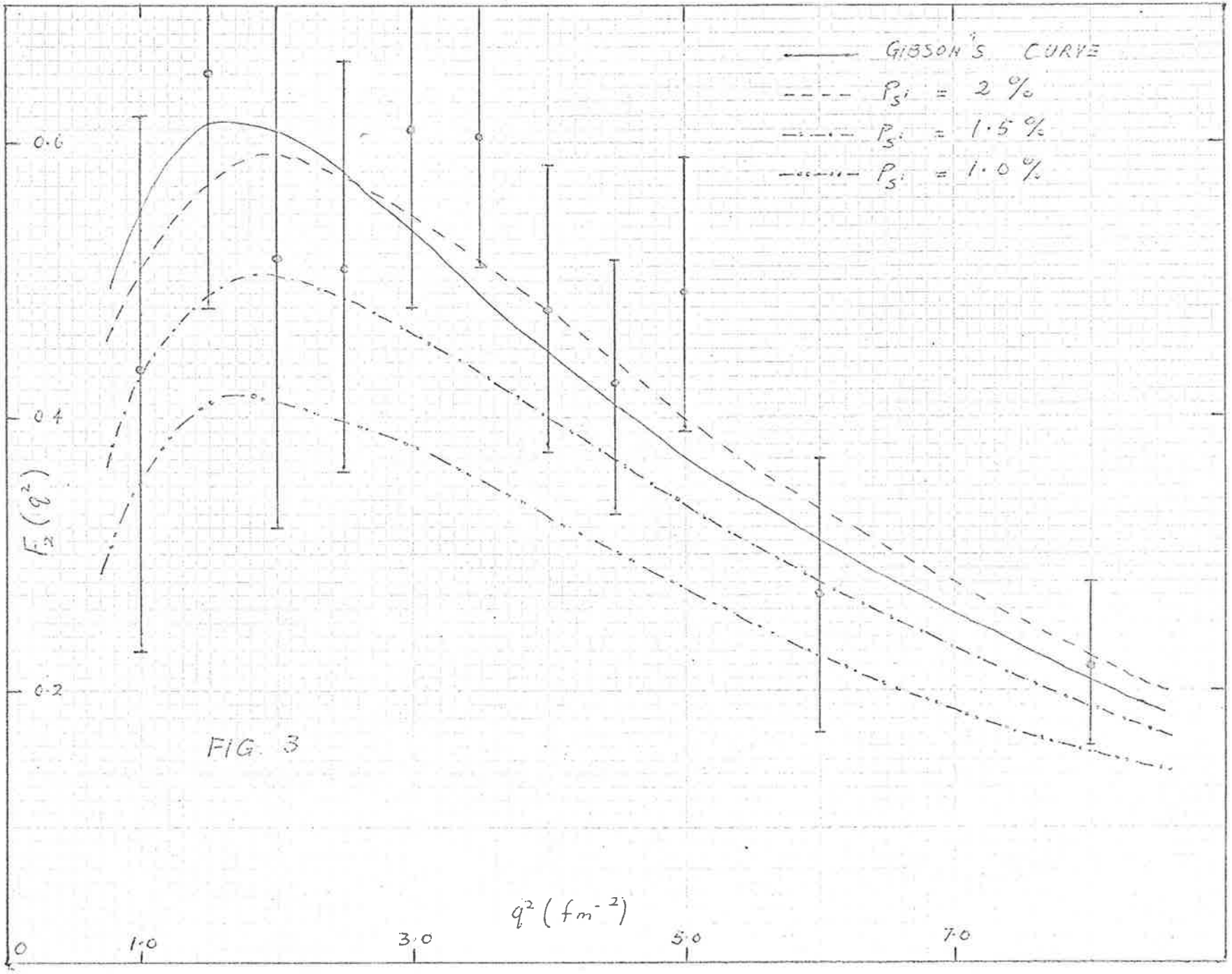

GIBSON'S CURVE
P_{Si} = 2 %
P_{Si} = 1·5 %
P_{Si} = 1·0 %
0·6
0·4
0·2
$F_2(q^2)$
0
1·0
3·0
5·0
7·0
q^2 (fm^{-2})

FIG. 3

with experiment, β must be larger than α. This means that the amplitude of the S and S′ states must have opposite signs, a situation which occurs in the variational calculations of Delves and Blatt [11] and Davies [109] too. From Fig. 3 it is plainly obvious that the experimental data of Collard et al. [95] is given by P_S, equal to 2%. This agrees completely with the estimate of Borysowicz and Dabrowski [124] from a binding energy calculation of the triton using non-local potentials with hard shell repulsion. It is also in substantial agreement with the $P_{S'}$ inferred from the work of Rosati and Barbi [92]. In their notation case A and case B correspond to taking the ground states of the trinucleon as pure S and S plus S′ respectively. The values of the Coulomb energy for point protons from their variational calculations with hard-core potentials are smaller in the case of B. In fact C.E.(B) are always lower than the values of C.E.(A) by 6-7%, irrespective of the core radius. This is the effect of the SS′ interference, as discussed by Okamoto [125]. In their calculations, Okamoto and Lucas [126] found this reduction to be 4-5% for $P_{S'}$

equal to 1.2%, and a simple argument suggests that Rosati and Barbi's $P_{S'}$ must be 2-2.5%. That Okamoto and Lucas do not obtain such a value of $P_{S'}$ must be ascribed to their 'poor' trial function, poor at least in comparison with the 'best' variational functions of Rosati and Barbi. Our estimate of $P_{S'}$ as with that of Borysowicz and Rosati, has not included the effect of the D state. For $F_2(q^2)$, it is known from the work of Gibson[105] that the D-D contribution increases the values obtained from assuming an S plus S' state only, at small momentum transfer q. Since there are no cross-terms between the S and D states in $F_2(q^2)$ we can use Gibson's result as a reasonable estimate of the contribution of the D state. Therefore assuming P_D to be about 6-8% (suggested by references 11 and 109), the increment in $F_2(q^2)$ due to the presence of the D state will be of the order of 0.01 for q^2 less than 4.0 fm^{-2}. This will spoil the fit for $P_{S'}$ equal to 2% but correspondingly will improve that for the other two values of $P_{S'}$. On this evidence, we can quite safely conclude that the values of $F_2(q^2)$ derived from our correlated exponential suggest that $P_{S'}$ is near 1.5%.

Using the second set of form factor coupling constants from reference 127

$$G_A = -1.39 g_V^{\beta}$$

$$G_V = 1.02 g_V^{\beta}$$

$$G_P = -0.59 g_V^{\beta}$$

$$g_V^{\beta} = 1.415 \times 10^{-49} \text{ erg.cm}^3$$

we have calculated Λ_μ for the same three values of $P_{S'}$.

TABLE 10

VALUES OF THE MUON CAPTURE RATE OF ^{3}He(sec^{-1})

Our Research	100%S	1%S'	1.5%S'	2.0%S'
	1464	1446	1434	1420
Experimental				
Ref. 81		1485 ± 40		
Ref. 82		1468 ± 40		

On comparison with the experimental values quoted by Rood and Pascual and Pascual, it is clear

that for our theoretical estimates of A_μ to agree with experiment, $P_{s'}$ cannot be more than 1.5%, in complete accord with the findings of $F_2(q^2)$. This statement may be open to question since relativistic corrections[128] will tend to increase A_μ. However this is more than offset by the inclusion of the D state which will decrease it[82]. Uncertainties in the coupling constants are present of course but on latest evidence, large variations away from the set we have used are most unlikely. In spite of the approximations made in our calculations, all these considerations suggest that $P_{s'}$ cannot be large and should probably be less than 1.5%.

Our analyses support the calculations of Davies who found in his extensive work on the binding energy of the three-nucleon system that $P_{s'}$ is about 1.2%.

CHAPTER SEVEN

THE PANOFSKY RATIO, R.M.S. RADII OF ^{3}He AND ^{3}H AND $^{16}O(^3H,p)\,^{18}O$(g.s.) REACTION

In the last two chapters we have made concerted efforts to establish the ground state wavefunctions of the trinucleon and the nucleon-nucleon interaction. It is reasonable as the next step in our investigation of the properties of the three-body system to discuss and explain other data using our results. Three quantities of interest are considered here. These are the Panofsky ratio for ^{3}He (recently determined by Zaimidoroga et al.[107]), the charge radii difference between ^{3}He and ^{3}H and the absolute value of the differential cross-section for the $^{16}O(^3H,p)^{18}O$(g.s.) reaction.

7.1 The Panofsky Ratio for ^{3}He

The Panofsky ratio for ^{3}He $P_{^3He}$ is the ratio of the transition probabilities of the processes:

$$\pi^- + {}^3He \rightarrow {}^3H + \pi^0 \text{ and } \pi^- + {}^3He \rightarrow {}^3H + \gamma$$

The theoretical analysis of Struminsky[84] is based on the assumption that the Π^--meson capture is considered in terms of the impulse approximation and that the ground state wavefunctions of the tri-nucleons can be approximated by the S state. Then if we follow Struminsky the Π^--meson capture interaction Hamiltonians are

$$H_{\Pi o} = a \sum_{i=1}^{3} \delta(\underline{r} - \underline{r}_i)\underline{\tau}_i^- \qquad (202)$$

and

$$H_{\gamma} = \frac{b}{\sqrt{W}} \sum_{i=1}^{3} \delta(\underline{r} - \underline{r}_i)\ (\underline{\sigma}_i\ \underline{e})\underline{\tau}_i \qquad (203)$$

where $\underline{r}_i$ is the spatial coordinate of the ith nucleon, $\underline{r}$ the Π^--meson coordinate, W the γ quantum energy, $\underline{e}$ the unit vector of the γ quantum polarisation, $\underline{\sigma}_i$ and $\underline{\tau}_i^-$ are the spin and isospin matrices, and a and b are constants. The ratio a^2/b^2 is easily expressed in terms of the Panofsky ratio for hydrogen which has been measured with great accuracy by Cocconi et al.[108]. After computing the matrix elements, we can reduce $P_{^3He}$ to

$$P_{^3He} = \frac{P_H\ k}{F_1^2(0.47)} \qquad (204)$$

where F_1 is the form factor function, and k the kinematic factor is

$$k = \frac{W_H}{W} \frac{(M_{^3H} + W)}{(M + W_{^3H})} \sqrt{\frac{E_{^3H}(m_{\Pi^o}+M)^3 \, M_{^3H}}{E_H(m_{\Pi^o}+M_{^3H})^3 \, M}} \qquad (205)$$

where m_{Π^o} is the Π^o-meson mass, $E_{^3H} = 4.06$ MeV is the kinetic energy of ^3H and Π^o, $E_H = 3.3$ MeV is the kinetic energy of the neutron and Π^o in the reaction $\Pi^- + p \rightarrow n + \Pi^o$, $W = 135.80$ MeV is the γ quantum energy in the process $\Pi^- + {}^3He \rightarrow {}^3H + \gamma$ ($q^2 = 0.47$ corresponds to this energy) and $W_H = 129.46$ MeV is the γ quantum energy in the process $\Pi^- + p \rightarrow n + \gamma$. The experimental values of the two Panofsky ratios are

$$P_H = 1.53 \pm 0.02 \qquad (206)$$

$$P_{^3He} = 2.28 \pm 0.18 \qquad (207)$$

while our calculations with function (ii) of chapter five yield

$$P_{^3He} = 2.6 \qquad (208)$$

We note that although our theoretical $P_{^3He}$ is somewhat larger than the experimentally determined value, there

is nevertheless no disagreement. Our computations have been carried out in the impulse approximation which is probably unsuitable for small q^2, and besides, further work is necessary to pinpoint the true experimental magnitude of $P_{^3He}$.

7.2 The Charge Radii of ^3He and ^3H

The experimental work of the group at Stanford on the elastic scattering of electrons from ^3He and ^3H has revealed a difference in the electromagnetic form factors of the two nuclei and hence a difference in the charge radii. The observed charge form factors have been explained by the inclusion of the S′ state in the last chapter. In this section we seek to account for the size difference in the charge root mean square radius. If we write the complete wave-function of ^3H as eqn. (174), the r.m.s. radius is

$$R_c^2(^3H) = \frac{1}{Z}\int \Psi(^3H)^* \left\{\frac{1}{2}\sum_i (1+\tau_{iz})(\underline{r}_i-\underline{R})^2\right\}\Psi(^3H)\, d\tau \qquad (209)$$

where the summation is taken over all the protons. The spin-isospin summation reduces the expression for $R_c^2(^3H)$ to

$$R_c^2(^3H) = \frac{2}{9} I_1 - \frac{2}{9}(I_2 + I_3) \qquad (210)$$

where

$$I_1 = \int \frac{1}{2} \left(\sum_{i<j}^{3} r_{ij}^2 \right) (\psi^{S_2} + \psi'^2 + \psi''^2) \, d\tau \qquad (211)$$

$$I_2 = \int \frac{1}{2} (r_{12}^2 + r_{13}^2 - r_{23}^2) \, \psi^S \psi' \quad d\tau \qquad (212)$$

$$I_3 = \sqrt{3} \int \psi^S \psi'' \, \frac{1}{2} (r_{12}^2 - r_{13}^2) \, d\tau = I_2 \qquad (213)$$

Similarly the mean square charge radius of ^{3}He is given by

$$R_c^2(^3\mathrm{He}) = \frac{2}{9} I_1 + \frac{1}{9} (I_2 + I_3) \qquad (214)$$

The integrals I_1, I_2 and I_3 are evaluated with the help of the appendix and our calculated values of the charge radii are tabulated on Table 11.

TABLE 11

R.M.S. CHARGE RADIUS OF ^{3}He AND ^{3}H

	$P_{S'}$ 0%	1%	1.5%	2%	Expt.(95)
$R_c(^3\mathrm{He})$ (fm.)	1.70	1.753	1.767	1.783	1.87 ± 0.05
$R_c(^3\mathrm{H})$ (fm.)	1.70	1.636	1.627	1.618	1.70 ± 0.05
ΔR_c (fm.)	0.0	0.117	0.140	0.164	0.17

It is obvious from the Table that our computed charge radii are close to those obtained from the experimental form factors, and that a 2% S′ state probability in the trinucleon ground state reproduces the observed radius difference. Davies[12,109] has shown however that the presence of 8.9% probability of the D state, which is not unreasonable, increases the size disagreement by 20%. Since our calculations have been done without including the D state, a correction for its presence in the ground state of the trinucleon indicates that the experimental ΔR_c can be explained by a value of $P_{S'}$ which is slightly less than 1.5%. This concurs with our calculations in chapter six. It should be noted that the D state contributions arise from D-D interference so that although our ψ^S differs from Davies's, we can still use his D state estimate as a satisfactory approximation for our calculations.

7.3 The Absolute Cross-Sections of the $^{16}O(^3H,p)$ $^{18}O(g.s.)$ Reaction

The mechanism of the $(^3H,p)$ reactions was first studied by Rook and Mitra[110], who were able to

calculate the angular distributions of the outgoing protons in the process through the Distorted Wave Born Approximation. The absolute value of the differential cross-section of the $^{16}O(^{3}H,p)^{18}O(g.s.)$ reaction was subsequently found by Mathur and Rook[111] to depend sensitively on the structure of the triton and the two-body nuclear forces responsible for the reaction. Their calculations were made using the hard-core exponential wavefunction and interaction potentials of Ohmura et al.[21], the parameters of the triton optical potential that gave the best fit with the experimental angular distribution, and wavefunctions for the initial and final states of the oxygen nuclei that were based on the Saxon-Woods potential. If we allow for the approximations that were assumed, Mathur and Rook's results are in substantial agreement with experiment. The velocity dependent potential is in many respects similar to the hard-core so we report here the evaluation of the differential cross-section of the stripping reaction using our correlated exponential wavefunction and our velocity dependent potential.

In the notation of Mathur and Rook, the differential cross-section of the stripping process $A(a,b)B$ in the Distorted Wave Born Approximation is given by

$$\frac{d\sigma}{d\Omega} = \frac{m_a\, m_b}{(2\pi\hbar^2)^2} \frac{k_b}{k_a} |M|^2 \qquad (215)$$

where m_a and m_b are the reduced masses of the incident and outgoing particles a and b and $\underline{k}_a$ and $\underline{k}_b$ are their momenta. It is understood that the right hand side of equation (215) is to be summed on the final magnetic quantum numbers and averaged over the initial magnetic quantum numbers.

The matrix element M for the process is given by

$$M = J\int d\underline{r}_a \int d\underline{r}_b \; U_b^{(-)}(\underline{k}_b, \underline{r}_b) \; \langle B, b|V|A, a\rangle \; U_a^{(+)}(\underline{k}_a, \underline{r}_a) \qquad (216)$$

Here $\underline{r}_a$ and $\underline{r}_b$ are the relative coordinates of a and b with respect to the target and the residual nuclei respectively, and J the Jacobian of the transformation to these relative coordinates from the natural coordinates $\underline{r}_{xA}$ and $\underline{r}_{bx}$, where x is the particle transferred in the reaction. From simple geometry one obtains

$$\underline{r}_{xA} = \frac{Ba}{x(A+a)} \left(\underline{r}_a - \frac{b}{a}\, \underline{r}_b\right) \qquad (217)$$

$$\underline{r}_{bx} = \frac{Ba}{x(A+a)} \left(\underline{r}_b - \frac{A}{B}\, \underline{r}_a\right) \qquad (218)$$

$$J = \left[\frac{aB}{x(A+a)} \right]^3 \qquad (219)$$

where the letters refer to the masses of the corresponding particles. The factor $\langle B,b|V|A,a\rangle$ is the matrix element of the interaction causing the stripping, taken between the internal states of the colliding pairs; it plays the role of an effective interaction causing the transition between the elastic scattering states U_a and U_b. After some manipulation, Mathur and Rook have succeeded in reducing M to

$$M = g\, M_{zr} \qquad (220)$$

where the factor g is a constant which depends on the strength of the interaction and the structure of the particles a and b and M_{zr} is the usual 'zero range' DWBA result. Consequently the value of the differential cross-section obtained from a zero range calculation need only be multiplied by a factor g^2 to give the absolute value.

For $(^3H,p)$ reactions and specifically for the $^{16}O(^3H,p)^{18}O$(g.s.) reaction

$$g(^3H) = g(^3He) = \tfrac{1}{2}\int \left[(3X_{static} + 1)V_{static}(r_{12}) + 4V_{vel.dep.}(r_{12})\right] \times \left[\psi^S - \psi'\right] d\tau \qquad (221)$$

if we assume that the transferred spin is zero, and T = 1. We compute g^2 and thus the maximum cross-section (corresponding to the first peak in the angular distribution) of the $^{16}O(^3H,p)^{18}O$(g.s.) reaction with 10 MeV tritons using the optical parameters of Mathur and Rook. Table 12 shows the calculated values of g^2, the maximum cross-section, and other three-body quantities obtained by us together with the relevant hard-core results of Mathur and Rook and the experimental values.

TABLE 12

$^{16}O(^3H,p)^{18}O$(g.s.) REACTION CROSS-SECTION

Reference	Core Radius	B.E.(^{3}H)	C.E. (Point nucleon)	R_c	g^2	Max. X-sect.
	(fm)	(MeV)	(Mev)	(fm)	(MeV^2.fm^6)	(mb/sr)
Mathur-Rook	0.40	7.52	0.746	1.74	50.4 10^6	52.2
This work (0% S′)	-	6.76	0.74	1.70	40.2 10^6	41.7
This work (1.5% S′)	-	-	-	1.63(^{3}H) 1.77(^{3}He) 1.70(^{3}H)	32.9 10^6	34.1
Experimental	-	8.49	0.764	1.87(^{3}He)		25 ±3[a])

[a]) From reference 129.

We find that results close to those of the hard-core are obtained when we use our soft-core wave-functions and potentials. This serves to confirm what we had discovered in our work in the last two chapters, namely that the effects of the hard-core and the soft repulsive core are very similar. The absolute cross-section of the $^{16}O(^{3}H,p)^{18}O(g.s.)$ reaction is reasonably well explained by the DWBA and our correlated exponential functions and velocity dependent potentials. We have extended the calculations of Mathur and Rook by including the S′ state in the ground state of the trinucleon. The improvement in the magnitude of the maximum cross-section is in the right direction; inclusion of the S′ state reduces the value of g. The presence of the D state would further complement this decrease in g. A rough estimate where we assume an 8% D state, indicates that g^2 (from 100% S)is diminished by nearly 30% thus indicating a maximum cross-section of about 29 mb/sr. Since the approximations of Mathur and Rook give an over-estimate, our results will then be in very good correspondence with the experimental results. This line of argument suggests conclusively that $P_{S'}$ is near 1.5%.

CHAPTER EIGHT

THE VELOCITY-DEPENDENT POTENTIAL AND ^{4}He

Calculations on the binding energy of the alpha particle are usually carried out with four types of wave functions. There are the Gaussian, Gunn-Irving, exponential and hard-core wavefunctions which, in the ground state, involve only the six internucleon distances. For the first two types, when hard-core potentials are not involved, all the relevant matrix elements for two-body forces can be evaluated in closed form by a transformation of the coordinates of the four nucleons. With the exponential and hard-core functions however, these integrals have only been computed by the Monte-Carlo method of six-dimensional numerical integration. Although the latter integrals can be reduced to three-dimensional form, a considerable amount of computer time will still be needed to evaluate a possible 4096 such three-dimensional integrations for the expectation value of each operator. In this work[76] therefore, we have overlooked the use of correlated exponential wavefunctions for the ground state of the alpha particle and have instead taken a two-parameter

Irving wavefunction as a reliable substitute. This function is expected to be inadequate for our velocity-dependent potential in binding energy calculations if only on the basis of the work of the previous chapters where short-range two-body correlations were found to be necessary. It should, nevertheless, give a fair estimate of the correct binding energy, and in calculations which do not depend critically on the close-in behaviour of the wavefunction, results that are very near those obtainable from a correlated function.

8.1 Binding Energy Formulae and Wavefunction of ^{4}He

We approximate the ground state of ^{4}He by the predominant spatially symmetric $^1S_0$ state i.e. Φ_1^α of eqn. (17) with

$$\psi^S = N\left\{\exp\left[-b_1\left(\sum_{i<j=1}^{4} r_{ij}^2\right)^{\frac{1}{2}}\right] - \exp\left[-b_2\left(\sum_{i<j=1}^{4} r_{ij}^2\right)^{\frac{1}{2}}\right]\right\},$$

$$b_1 < b_2 \qquad (222)$$

If we use the transformation of eqn. (83) and the expressions

$$\int_0^\infty \int_0^\infty \int_0^\infty e^{-4b_0(\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}}} \eta_1^2\eta_2^2\eta_3^2 \, d\eta_1 d\eta_2 d\eta_3 = \frac{3\Pi}{2^{12} b_0^9} \qquad (223)$$

$$d\underline{\tau} = d\underline{\eta}_1 d\underline{\eta}_2 d\underline{\eta}_3 = \eta_1^2\eta_2^2\eta_3^2 (4\Pi)^3 d\eta_1 d\eta_2 d\eta_3 \qquad (224)$$

from Irving's[16] work, the normalisation constant N is given by

$$N^2 = \frac{2^6}{3\Pi^4}\left[\frac{1}{b_1^9} + \frac{1}{b_2^9} - \frac{2}{\left(\frac{b_1+b_2}{2}\right)^9}\right] \qquad (225)$$

The parameters b_1 and b_2 are found by the method of chapter five, which allows us to have a trial function that is at least good at large interparticle distances. Thus b_1 and b_2 are evaluated by fitting the form factor of ^{4}He given by our wavefunction

$$F_B(q^2) = \frac{3\Pi^4 N^2}{2^6}\left[\left(\frac{16}{16+\frac{3q^2}{4b_1^2}}\right)^5 \frac{1}{b_1^9} + \left(\frac{16}{16+\frac{3q^2}{4b_2^2}}\right)^5 \frac{1}{b_2^9}\right. \qquad (226)$$

$$\left. - 2\left(\frac{16}{16+\frac{3q^2}{4\left(\frac{b_1+b_2}{2}\right)^2}}\right)^5 \frac{1}{\left(\frac{b_1+b_2}{2}\right)^9}\right]$$

to that obtained from the expression

$$F_B(q^2) = \frac{F_{ch}(^4He)}{F_{ch}(n) + F_{ch}(p)} \qquad (227)$$

In this calculation, the charge form factor $F_{ch}(^4He)$ is taken from the experimental work of Frosch et al.[112] and Repellin et al.[97]. For the neutron and proton charge form factors, we use the values of De Vries[113] et al. and Janssens et al.[114]. At the time this work was carried out there was uncertainty about the correct value of the r.m.s. radius of the alpha particle so three values of $R_{r.m.s.}$ were selected. These are 1.40, 1.45 and 1.50 fm.

The expression for the r.m.s. radius is

$$R^2_{r.m.s.} = \frac{135\Pi^4 N^2}{2^{11}} \left[\frac{1}{b_1^{11}} + \frac{1}{b_2^{11}} - \frac{2}{\left(\frac{b_1+b_2}{2}\right)^{11}} \right] \qquad (228)$$

since

$$\int \exp\left[-2b_c \left(\sum_{i<j=1}^{4} r_{ij}^2\right)^{\frac{1}{2}}\right] \eta_1^2 \, d\underline{\tau} = \frac{45\Pi^4}{2^9 b_c^{11}} \qquad (229)$$

The binding energy is found from

$$E = \langle T \rangle + \langle V \rangle_{static} + \langle V \rangle_{vel.dep.} \tag{230}$$

where

$$\langle T \rangle = -\frac{\hbar^2}{2M} \langle \nabla^2_{\eta_1} + \nabla^2_{\eta_2} + \nabla^2_{\eta_3} \rangle = \frac{3\hbar^2 \Pi^4 N^2}{2^5 M} \left[\frac{1}{b_1^7} + \frac{1}{b_2^7} - \frac{2\, b_1 b_2}{\left(\frac{b_1+b_2}{2}\right)^9} \right] \tag{231}$$

$$\langle V \rangle_{static} = -6(V_o)_{eff} \langle \exp(-k_s r_{12}) \rangle = -\frac{24(V_o)_{eff} 8! \Pi^4 N^2}{(\sqrt{2}k_s)^9}$$

$$\times \left[F(b_1, k_s) + F(b_2, k_s) - 2F\left(\frac{b_1+b_2}{2}, k_s\right) \right] \tag{232}$$

and

$$F(\alpha, k_s) = \frac{64\beta^3 + 69\beta^2 + 30\beta + 5}{840\beta^6 (1+\beta)^6} \tag{233}$$

$$\beta = \frac{2\sqrt{2}\,\alpha}{k_s} \tag{234}$$

The velocity-dependent contribution is

$$\langle V \rangle_{vel.dep.} = 3(V_o)_{vel.dep.} [G(b_1, b_1, k'_s) + G(b_2, b_2, k'_s)$$

$$- G(b_1, b_2, k_s) - G(b_2, b_1 k'_s)] + 3X_{vel}(V_o)_{vel.dep.}$$

$$\times [G(b_1, b_1, k'_t) + G(b_2, b_2, k'_t) - G(b_1, b_2, k'_t)$$

$$- G(b_2, b_1 k'_t)] \tag{235}$$

$$k' = \frac{2}{\beta'} \tag{236}$$

where

$$G(y,\delta,k) = -\int \left[4\,\delta^2 \eta_3^2 (\eta_1^2+\eta_2^2+\eta_3^2)^{-1} - 6\delta(\eta_1^2+\eta_2^2+\eta_3^2)^{-\frac{1}{2}} + 2\delta\eta_3^2(\eta_1^2+\eta_2^2+\eta_3^2)^{-\frac{3}{2}} + k^2 + 2\sqrt{2}\,k\delta\eta_3(\eta_1^2+\eta_2^2+\eta_3^2)^{-\frac{1}{2}} - \frac{\sqrt{2}k}{\eta_3} \right] \times \exp\left\{ -4y(\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}} - \sqrt{2}k\eta_3 \right\} d\underline{\eta}_1\, d\underline{\eta}_2\, d\underline{\eta}_3 \tag{237}$$

and can be evaluated explicitly. The full derivation of these relations and other useful integrals are presented in the appendix.

8.2 Photodisintegration Cross-Section and Coulomb Energy of ^{4}He

The bremsstrahlung-weighted cross-section $\sigma_b(^4\mathrm{He})$ is given by equation (159) whilst in the derivation of the integrated cross-section $\sigma_{int}(^4\mathrm{He})$, the term D_z in the determination of the summed oscillator strength is

$$D_z = \tfrac{1}{2}(z_{12} + z_{34}) \tag{238}$$

When we use the general two-body force given by equation (115) and evaluate Σf_{on} explicitly, the expression for σ_{int} becomes

$$\sigma_{int} = \frac{2\Pi^2 e^2 \hbar}{Mc}\left[\frac{N' Z}{A} - \frac{m}{3\hbar^2} < \Sigma_i \Sigma_j r_{ij}^2 (V)_{static}(r_{ij}) \right.$$

$$\left. (hP_{ij}^H + mP_{ij}^M) > \right]$$

where i and j stand for protons and neutrons respectively. The velocity-dependent contribution to σ_{int} has been absorbed into the term $\frac{N' Z}{A}$ ($N' = N$ if $V_{vel.dep.}$ is absent). When we apply our ground state wavefunction and the analysis of Srivastava and Jain[79], σ_{int} reduces to

$$\sigma_{int} = \frac{2\Pi^2 e^2 \hbar}{Mc}\left[1 + (V_0)_{vel.dep.}(<\omega(r_{12})> + X_{vel.}<\omega_t(r_{12})>)\right.$$

$$\left. - \frac{4M(m+\frac{1}{2}h)}{3\hbar^2} <(V)_{static}(r_{12})\ r_{12}^2 > \right] \quad (240)$$

The Coulomb energy of the alpha particle is given by the expression

$$C.E. = \int \left|\Psi_s\right|^2 \frac{e^2}{\sqrt{2}\eta_3} d^3\eta_1 d^3\eta_2 d^3\eta_3 \quad (241)$$

which on transformation to the R, θ, and ϕ coordinates, is easily evaluated and reduces to

$$C.E. = \frac{7!\Pi^4 e^2 N^2}{3.2^{\frac{31}{2}}} \left[\frac{1}{b_1^8} + \frac{1}{b_2^8} - \frac{2}{\left(\frac{b_1+b_2}{2}\right)^8} \right] \qquad (242)$$

We are of course assuming point protons; it is unnecessary to consider the Schneider-Thaler potential for finite nucleons as there are no experimental values of C.E. to compare with.

8.3 Numerical Results and Discussion

The parameters b_1 and b_2 are varied from 0.50 to 1.40 fm^{-1} and 0.52 to 5.0 fm^{-1} respectively. In Table 13 we record the three sets of parameters together with the binding energies obtained using the Rarita-Present and our own velocity-dependent potentials. Our values are compared with those of Irving[16], Borysowicz and Zielinska[43] and the hard-core results of Tang and Herndon. Fig. 4 and Table 14 show the close fits to the experimental body form factor given by the three wavefunctions. It is obvious from Table 15 that the disagreement between the experimental measurements by

TABLE 13

THE BINDING ENERGY, COULOMB ENERGY AND THE R.M.S. RADIUS OF ^{4}He

Wave-function	Parameters b_1 (fm^{-1})	Parameters b_2 (fm^{-1})	r.m.s. Radius (fm)	Potential	B.E. (MeV)	C.E. (MeV)
(i)	0.99	1.06	1.40	Rarita-Present	30.39	0.910
(ii)				Our Potential	19.14	
(iii)	0.95	1.04	1.45	R-P	29.67	0.881
(iv)				Our Potential	20.38	
(v)	0.91	1.01	1.50	R-P	28.72	0.849
(vi)				Our Potential	21.40	
Irv	0.92	-	1.29	R-P	31.90	1.00
B-Z	0.69	-	1.72	Herndon-Tang	17.00	0.749
T-H	Hard-core	(0.50fm)	1.48	Tang-Herndon	26.85	

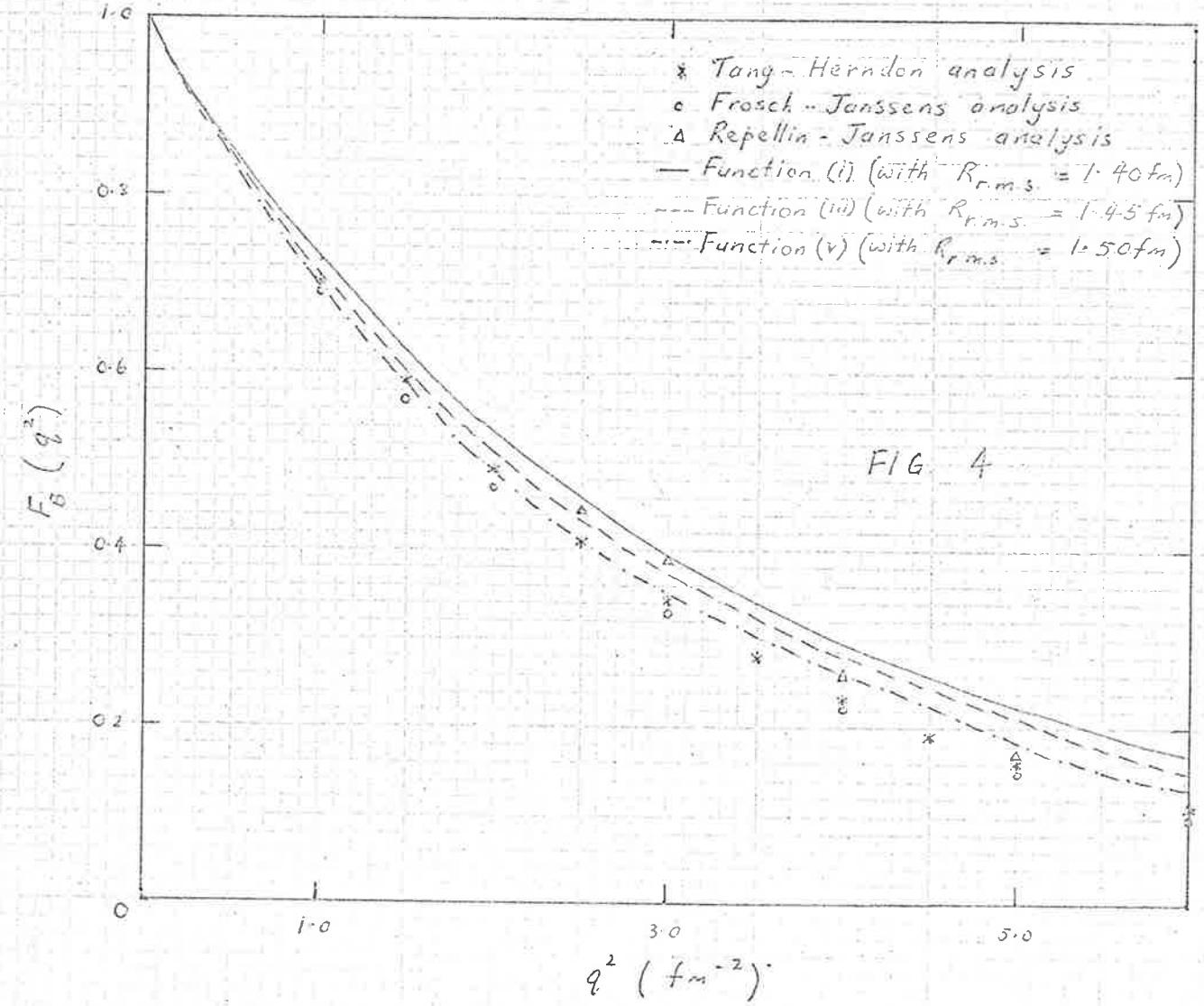

* Tang - Herndon analysis
o Frosch - Janssens analysis
Δ Repellin - Janssens analysis
— Function (i) (with $R_{r.m.s.}$ = 1·40 fm)
--- Function (iii) (with $R_{r.m.s.}$ = 1·45 fm)
-·-·- Function (v) (with $R_{r.m.s.}$ = 1·50 fm)
1·0
0·8
0·6
0·4
0·2
0
1·0
3·0
5·0
$F_D(q^2)$
q^2 (fm^{-2})

FIG 4

TABLE 14

THE BODY FORM FACTOR OF THE ALPHA PARTICLE

	Experimental			Calculated			
q^2	T-H[a])	F-J[b])	R-J[c])	(i)	(iii)	(v)	Hard-core (0.5fm)
1.0	0.703	0.690		0.725	0.710	0.692	0.702
1.5	0.590	0.569					0.591
2.0	0.490	0.471		0.532	0.511	0.487	0.499
2.5	0.409		0.454				
3.0	0.340	0.335	0.387	0.395	0.373	0.348	0.357
4.0	0.234	0.226	0.260	0.296	0.276	0.252	0.256
5.0	0.160	0.149	0.170	0.225	0.206	0.185	0.184
6.0	0.110	0.099	0.113	0.172	0.155	0.137	0.131

[a]) The analysis by Tang and Herndon of data collected by Frosch and De Vries.

[b]) The results of Frosch and Janssens.

[c]) The results of Repellin and Janssens.

TABLE 15

CALCULATED σ_{int} VALUES OF ^4He

Reference	σ_b (mb)	$\sigma_{int}(S)$ (Mev.mb)	$\sigma_{int}(B)$ (MeV.mb)	$\sigma_{int}(I)$ (MeV.mb)	$\sigma_{int}(R)$ (Mev.mb)
Rustgi-Levinger	0.8	89.0	-	-	-
Rustgi-Mukherjee	2.70	83.2	-	97.1	96.0
Srivastava-Jain (corrected to our potential)	2.40	103.3	113.0	132.7	132.7
Srivastava-Jain	2.40	105.0	-	127.0	127.0
(i)	2.51	94.9	103.9	121.9	121.9
(ii)		102.4	112.1	131.8	131.8
(iii)	2.69	94.5	103.5	121.3	121.3
(iv)		101.5	111.1	130.4	130.4
(v)	2.89	94.1	102.9	120.5	120.5
(vi)		100.4	109.8	128.9	128.9
Gorbunov-Spridinov (experimental)	2.40 ±0.15		95 ± 7		

Gorbunov and Spiridonov[115] of the integrated photo-disintegration cross-section and the theoretical work of Rustgi and Mukherjee[78] and Srivastava and Jain cannot be attributed to the failure of the sum-rules of Levinger and Bethe but to the particular choices of interaction potentials and ^{4}He ground state wavefunctions. Our calculated values of σ_{int} taken with those from our three-body calculations suggest that the Serber and Biel exchange mixtures are favoured over the Inglis and Rosenfeld. The good agreement with the experimental data indicates that σ_{int} is sensitive to the asymptotic behaviour of the wavefunction only, thus confirming the results obtained in our trinucleon calculations and showing that although our trial functions do not have the correct close-in structure they are still able to provide adequate descriptions of the ground state of the alpha particle in all other regions of configurational space. It is now known that the r.m.s. radius of ^{4}He is about 1.50 fm instead of the 1.42 fm suggested by old electron scattering data, which is not surprising since our fit to the body form factor with r.m.s. radius 1.50 fm is the best of the three curves plotted. The binding energy for the velocity-dependent potential and

σ_{int} with $R_{r.m.s.}$ equal to 1.50 fm are also better than those obtained using the other two values of the radius. Our velocity-dependent potential yields a slightly underbound alpha particle, but this result should properly be ascribed to our interpretation of $\underline{p}$ and the large value of ψ^s for small interparticle separations. In this area, short-range correlations between pairs of nucleons must be included so as to mirror the repulsive character of our potential. Borysowicz and Zielinska's[43] work and our experience with the trinucleon are suggestive of the importance of such correlations for improving the values of the binding energy. Our success in these investigations makes it obvious that the properties of the three- and four-nucleon systems can be adequately explained with the one form of the nucleon-nucleon interaction. It is also obvious that our study of the alpha-particle ratifies the general conclusions arrived at in our own work of the last three chapters.

CHAPTER NINE

APPLICATIONS TO OTHER LIGHT NUCLEI

The procedures for determining the wavefunctions of the very light nuclei ^{3}H and ^{4}He developed in the preceding chapters can be extended and applied to other light nuclei. Since our facility in these calculations is mainly with the three-body system, nuclei which offer themselves for consideration are genuine three-nucleon systems or are those that can be reduced to three-particle models. In this chapter we turn our attention to the trineutron, the alpha-particle model of ^{12}C and the quark model of the baryon.

9.1 The Trineutron

Early in 1965, Adjacic et al.[116] reported the possible existence of a bound state of three neutrons with about 1 MeV binding energy. This startling announcement that there was a stable nucleus ^{3}n, apparent through the observation of a peak in the proton distribution in the reaction $^{3}H(n,p)3n$, led Mitra and Bhasin[63] to carry out an analysis of the trineutron with non-local separable potentials. Their calculations

indicated that such a state was likely as the forces required to bind the three neutrons together were much less than those that gave a good fit to the ^3P phase shifts in nucleon-nucleon scattering. So far there has been no experimental support for the conclusions of Adjacic et al. and Mitra and Bhasin; as well, two variational calculations have also discredited the work of the latter. It is interesting to note that our exact calculations can offer us an explicit solution of an approximate trineutron wave-equation. The results obtainable are strongly in favour of an unbound ^3n.

The most likely quantum assignments (LSJT) for the trineutron in descending order of preference are $(1\ \frac{3}{2}\ \frac{1}{2}\ \frac{3}{2})$, $(1\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2})$, $(0\ \frac{1}{2}\ \frac{1}{2}\ \frac{3}{2})$ and $(2\ \frac{1}{2}\ \frac{3}{2}\ \frac{3}{2})$. However these states should be near together in the energy level scheme; in fact Mitra and Bhasin found that the S-state should be nearly as bound as the P-states, and this may be the only valid conclusion one can make from their work. It appears then that the determination of the S-state binding energy with a plausible local potential should offer a clear indication of the actual binding energy of the trineutron. We cannot

assume the spin-dependent potential of chapter three since this would give to ^3n the binding energy of the triton. We must take as the two-nucleon interaction the isospin generalisation of equation (23). Thus

$$V_{ij} = - V_o (\omega + bP^B_{ij} + hP^H_{ij} + mP^M_{ij}) \qquad (243)$$

where the force constants are given by $\omega = 0.4$, $b = 0.1$, $h = 0.1$ and $m = 0.4$. For the S-state of the triton the two internucleon potentials, given in equations (23) and (243), yield the same effective interaction. For the configuration $(0\ \frac{1}{2}\ \frac{1}{2}\ \frac{3}{2})$ the totally antisymmetric wavefunction describing the system is

$$\Psi(^3\eta) = (\psi'\chi'' + \psi''\chi')\eta^S \qquad (244)$$

where ψ' and ψ'', and χ' and χ'' have symmetries defined in chapter two; η^S is a completely symmetric isospin function. It follows then that

$$P^H_{ij}\Psi = - \Psi \qquad (245)$$

and

$$P^M_{ij}\Psi = P^B_{ij}\ P^H_{ij}\Psi = -P^B_{ij}\Psi \qquad (246)$$

whence

$$V_{ij}\,\Psi = -V_o[(\omega-h) + (b-m)\,P^B_{ij}]\,\Psi \qquad (247)$$

Substituting (244) and (247) into the Schrodinger equation for the trineutron, we derive coupled differential equations identical to those for the triton in chapter three. Here

$$\alpha = (\omega-h) \quad , \quad \beta = (b-m) \qquad (248)$$

We can then carry out the solution of these equations exactly as in chapter three.

The calculated binding energy is 20.9 MeV above that of the ground state of the triton. It suggests that the more favoured P-states should be unbound by between 8 to 10 MeV which is in substantial agreement with the variational calculation of Okamoto and Davies.[64] There are too many approximations involved in our calculations, however, for us to be able to claim this circumstance as a signal achievement of our method but we are, nevertheless, certain that it confirms that any local potential that yields the correct binding energy of the triton should give an unbound trineutron.

9.2 Three-Alpha Model of ^{12}C

The cluster model of the nucleus, which has had a long history of success, is based on the intuitive observation that nucleons in a large nucleus exhibit a loose but effective correlation. The correlation, where there exists an enhanced probability of finding four nucleons close together and with properly aligned spins and isospins, is termed an α-cluster. This weakly correlated substructure should not be naively considered as an α-particle within the nucleus although an exception occurs in configurations where the four nucleons are somewhat separated from the others, that is, at the surface. On the other hand, to assume that the clusters are rigid entities without any internal structure and interacting through a potential determined by α-α scattering experiments is to reduce the treatment of the nuclear system to its simplest form. Thus ^{12}C in this model becomes a bound structure of three spinless α-particles for which a non-relativistic theory should be adequate. The validity of the model may be judged by the accuracy of its predictions so the encouraging results of Harrington[117], derived from non-local separable potentials, indicate that the rigid model

may reflect some of the physical properties of the real ^{12}C nucleus. Consequently, we use this model of ^{12}C as an illustrative example of an application of our variational methods.

^{12}C is a closed-shell nucleus whose ground state has zero spin and no magnetic moments. In accordance with the Bose statistics of the particles, the total and therefore the spatial wavefunction of the ground state must be symmetric in the three particles, which makes the problem all the more favourable for analysis. The most successful phenomenological α-α potential, which has been used in the investigations of Bodmer and Ali[28] on the hypernucleus $^{9}Be_{\Lambda}$, has the form

$$V_{\alpha\alpha}(r) = V_R \exp(-\mu_R^2 r^2) - V_\Lambda \exp(-\mu_\Lambda^2 r^2) + \frac{4e^2}{r} \qquad (249)$$

where

$$V_R = 400\text{MeV},\ \mu_R = 0.635\text{fm}^{-1}\ V_\Lambda = 160\text{MeV and } \mu_\Lambda = 0.475\text{fm}^{-1} \qquad (250)$$

Our trial function should allow for the possibility of strong spatial correlations between the alpha particles. These correlations are expected to be important as a consequence of the strong short-range repulsive character

of the α-α interaction. We assume the symmetric spatial function to be of the form of the correlated Gaussian used in chapter five. The two most reliable measured physical constants of the ^{12}C ground state are the r.m.s. radius and the binding energy relative to the three α break-up threshhold. The latest experimental data of Engfer and Turck[118] indicate that the radius is 2.42 fm. while the binding energy is -12.7 MeV from the most recent tables of nuclear masses. (The energy of Coulomb repulsion among the alphas is determined from the formula for a uniform spherical charge distribution, $E_c(^AZ) = 0.584\ Z(Z-1)A^{-\frac{1}{3}}$ MeV. This gives a value of $E_c(^{12}C) - 3E_c(^4He) = 5.44$ MeV for the Coulomb energy of repulsion of the α's). The experimental value of the r.m.s. radius corresponds to an inter-alpha separation of about 4.2 fm which is considerably larger than twice the r.m.s. radius of the alpha particle. As there is little overlap between the alpha particles, it seems unnecessary to introduce any distortion of the wavefunction. In our radial function we have taken c equal to 1.0 while a and b, found from a fit to the r.m.s. radius and the binding energy, were 0.072 and 0.1015 fm^{-2} respectively.

To test the model and the wavefunction, we evaluate the form factor and Coulomb energy of ^{12}C. The body form factor is obtained from the expression

$$F(q^2) = Z^{-1} \quad \langle\psi| \Sigma_p \exp(i\, \underline{q}.\underline{r}) \,|\psi\rangle \qquad (251)$$

where Σ_p indicates summation over the Z protons and the vectors $\underline{r}$ their locations. In this model the expression reduces simply to

$$F(q^2) = \int \psi^*\psi \exp(i\, \tfrac{2}{3}\, \underline{q}.\underline{\rho})\, d\tau \qquad (252)$$

which is easily evaluated, since $\underline{\rho}$ is the distance from one α-particle to the c.m. of the other two and the relation now resembles that for the trinucleon. The Coulomb energy is derived as in the calculations of chapter five. Fig. 5 shows our calculated form factor together with the experimental data of Crannell et al.[119]. There is little resemblance between the experimental and theoretical form factors although a prejudiced eye may detect a correspondence in the behaviour of the two curves for small momentum transfers. The first predicted diffraction minimum occurs not far from the experimentally determined one but

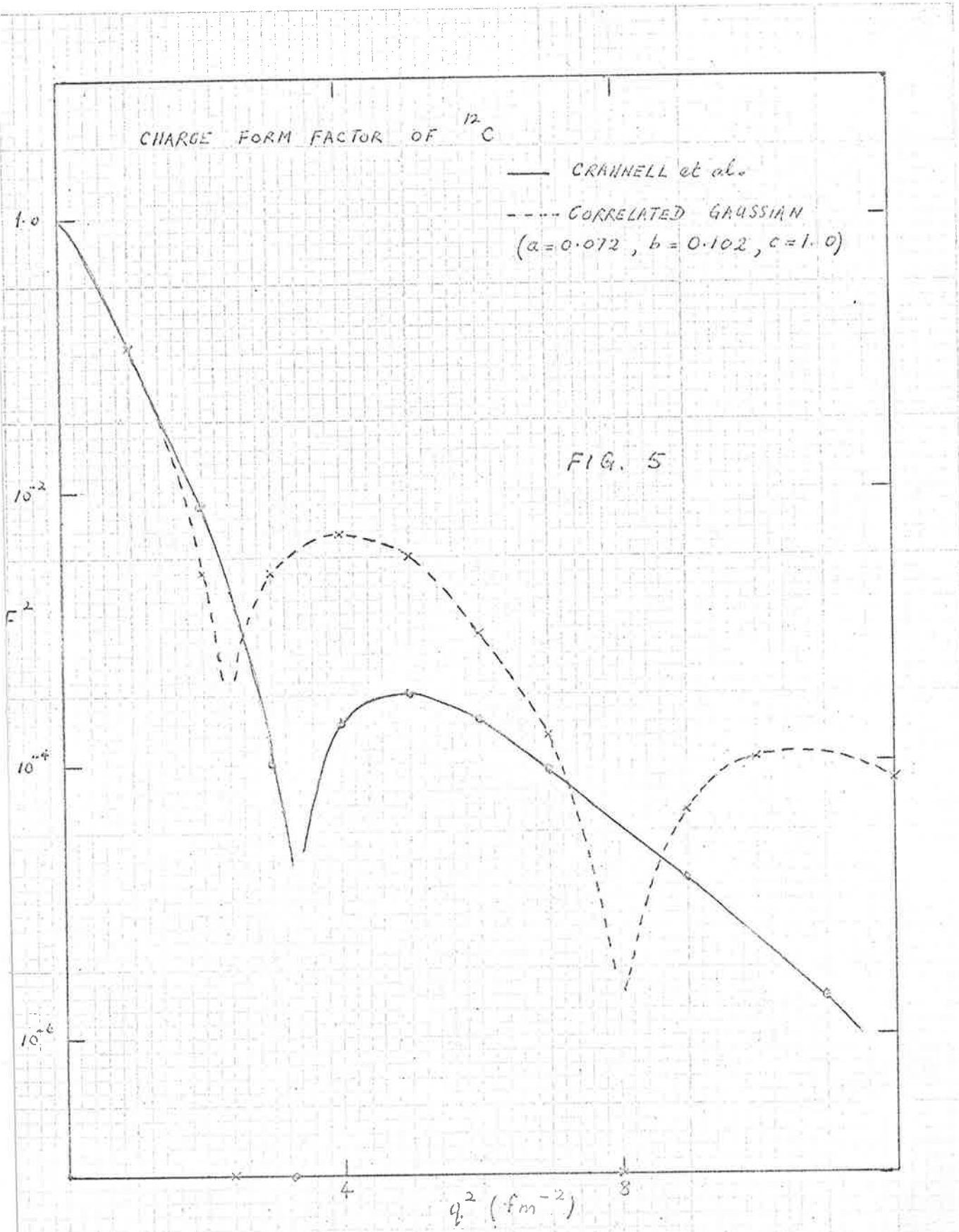

CHARGE FORM FACTOR OF ^{12}C
CRANNELL et al.
CORRELATED GAUSSIAN
(a = 0·072 , b = 0·102 , c = 1·0)
1·0
10^{-2}
F^2
10^{-4}
10^{-6}
4
8
q^2 (fm^{-2})

FIG. 5

there is no suggestion of a second minimum below 12 fm^{-2} in the work of Crannell. Our Coulomb energy is 4.65 MeV compared with the expected 5.44 MeV. If we are optimistic, we can say that our results are only moderately acceptable but this by no means suggests that the model is unrealistic. The Gaussian waveform can obviously be improved by removal of the restriction on the value of c. However, until a satisfactory three body model of ^{12}C is attained it appears that the wide acceptance of the alpha particle model in hypernuclear calculations should be reappraised.

9.3 The Non-Relativistic Quark Model

Ever since the quark model of baryons and mesons was proposed by Gell-Mann[120] and independently by Zweig[121], few topics in elementary particle and high energy physics have generated as much interest as the prospect of explaining phenomena in these fields through the use of these mysterious objects. Irrespective of their physical existence, the quarks have had a surprising degree of success, but they have thrown up a number of problems as well. We have neither the space nor the competence to discuss in

great detail all the achievements and consequences of the static quark model. In its essentials, therefore, the main assumptions of the model are

a) that there is a basic triplet of quarks, whose quantum numbers are

for the p quark $Q = \frac{2e}{3}$, $B = \frac{1}{3}$, and $Y = \frac{1}{3}$,

for the n quark $Q = \frac{-e}{3}$, $B = \frac{1}{3}$, and $Y = \frac{1}{3}$,

for the λ quark $Q = \frac{-e}{3}$, $B = \frac{1}{3}$, and $Y = \frac{-2}{3}$,

b) that the anti-quarks are denoted by $\bar{p}$, $\bar{n}$ and $\bar{\lambda}$, and have quantum numbers opposite those for the corresponding quark,

c) that the quarks have nearly equal and large masses ($M_Q \geq 10$ GeV),

d) that in the same way that nuclei are composed of a fixed number of nucleons, elementary particle states observed are interpreted as composite systems of quarks and anti-quarks.

This model by itself predicts, in simple fashion, a large number of SU(6) results such as the ratio of the proton-neutron magnetic moments, the ω-ϕ mixing angle, the Schwinger quadratic mass formula and the

hadron level classification scheme. Other more sophisticated SU(6) results like the electromagnetic decay of vector mesons and the high energy elastic scattering of mesons and baryons follow from very simple assumptions on the quark model.

It would appear then, in view of these impressive successes, that the next stage of the theory would be the direct comparison with experiment of results which depend on the dynamical features allotted to the model. In this respect, an important question concerns the precise role and form of the quark-quark interaction. Present evidence suggests that the $\underline{Q}$-$\underline{Q}$ potential is spin and unitary spin independent and that if it is smooth and very deep the motions of the bound quarks are non-relativistic. Thus the Schrodinger equation could conceivably describe accurately the quark motions. Even if the functional relationship between the energy and the potential is not that which is characteristic of the Schrodinger equation, it is expected that the description of the internal motions of the quark system by the Schrodinger equation could still give the correct

level ordering.

In the quark model of the baryon, the baryonic states are considered to be bound states of three quarks. The low-lying states are known to correspond to the $\underline{56}$ representation of SU(6) consisting of the $(\frac{1}{2}+)$ octet and the $(\frac{3}{2}+)$ decuplet. With the natural assumption of Fermi statistics for the quarks, the spatial wavefunctions are required to be antisymmetric under the interchange of any two quark coordinates. This development is unwelcome and although Mitra and Majumdar[122], and Kreps and de Swart[123] have argued that Fermi statistics need not be abandoned from nucleon form factor calculations, it may be more attractive and perhaps more advisable that quarks obey parastatistics. If these parastatistics are of order three, the three-quark wavefunctions will be totally symmetric. In fact with other modifications of the simple quark model other symmetries may even be used to accommodate $\underline{66}$.

Our interest in the problem arises from the discussions in the last two paragraphs, namely that the Schrodinger equation could be the operative relation

for the internal quark motions and that symmetric and mixed symmetric wavefunctions of three particles occur naturally as ground state configurations in the quark model of the baryon. We are hopeful that our exact and variational methods for the nuclear problem can be transferred with minimal alteration to the investigation of the systematics of the baryon states. We have conducted a pilot calculation with the $\underline{Q}$-$\underline{Q}$ force taken as a spin-dependent Gaussian of depth 10 GeV and range 0.1 fm. Our results indicate a small binding energy for the representative three-quark system which leads to a 'baryon mass' that is an order of magnitude too large. In a small way this is encouraging as, in spite of the obvious simplicity of the model and the defects in our exact calculation, the results are still sensible.

CHAPTER TEN

CONCLUSION

In detailing the main conclusions to be drawn from this study of the three- and four-nucleon systems, we are struck by one significant feature common to both methods. Our exact and variational calculations allow agreement with experiment that is good enough to warrant continued work with two-body forces alone. The introduction of many-body forces appears unnecessary for work of such accuracy although it would be erroneous to conclude from this evidence that these forces are absent altogether from complex nuclear systems. The two-body interaction model of the light nuclei has therefore sufficient physical reality to permit us to derive these general conclusions.

We have succeeded in placing Green's model of the nucleus on a firmer foundation by the introduction of spin into the nucleon-nucleon potential. Our exact calculations produce wavefunctions which possess functional forms resembling those used successfully

by Aranoff and Irving and which are directly obtainable from the assumed nuclear potential. The physical features of the trinucleon are accurately reproduced but the properties of the alpha particle as given by the model are deficient. The major drawback of the method and one that puts a stop to any ideas for extending the calculations to other nuclear systems is the complete reliance on shallow and long-ranged potentials for reasonable accuracy. It appears the methods of the model may be better used elsewhere.

Our variational calculations have had more success. It is evident from the work in chapters five to eight that the velocity dependent potential is equivalent in most respects to the hard-core interaction, and in view of the advantages it brings with its use, could be a reliable substitute for the latter in the modern potentials. The validity of our exponential velocity dependent potential seems confirmed so it can be employed in a fuller determination of the properties of the four-, five- and six-nucleon systems. Our product form wavefunctions with short-range two-body correlations are good and versatile

representations of the ground state wavefunctions of the trinucleon. The correlations are essential especially when velocity dependent potentials or other soft-core interactions are assumed to describe the two-particle interaction. Our calculations suggest that charge asymmetry is present to the order of 0.1 MeV in the trinucleon. The S' state of the trinucleon is likely to be present in the ground state but its percentage probability is almost certainly less than 1.5%. Our analysis of the photodisintegration cross-sections of the three- and four-nucleon systems indicate that the Levinger-Bethe sum-rules are correct and that σ_{int} depends sensitively on the asymptotic behaviour of the wavefunction. It is also apparent that the Serber and Biel force mixtures are favoured in σ_{int} calculations. That we are able to fit form factors and σ_{int} values simultaneously is highly suggestive of the similarity of the charge distributions in electron scattering and electric dipole absorption.

The application of our methods to other light nuclei indicates that the trineutron is unbound and serves to caution against too ready acceptance of the rigid alpha particle model.

The work in this thesis can be improved in a number of ways without having to resort to intractable numerical methods. The proper variational calculation can be introduced using our velocity dependent potential or an improvement of it. The singlet potential can be altered to give a better fit to the scattering length and effective range while the restriction on the ranges of the triplet static and velocity dependent potentials can be removed. As the S' state may be important in the trinucleon calculations it can be introduced into the method explicitly or by removing the symmetric form of the spatial function. For complete acceptability of some of our results the D states will have to be considered quantitatively and the product form wavefunction must be used for the alpha particle calculations.

In retrospect, we are convinced that the main achievement of this research has been the demonstration of the equivalence in nuclear structure calculations for the light nuclei of two seemingly different potentials which fit the two-nucleon data. If this result is true in all nuclear systems, then it suggests that too much time and effort should not be unnecessarily wasted in frenzied attempts to determine a unique two-body interaction.

APPENDIX

Evaluation of Integrals

As is usual in quantum-mechanical calculations, one of the unwelcome tasks is the calculation of integrals. However, for the functions which we have used in this thesis, all the integrals that arise can be evaluated explicitly. It is our purpose to collect here the three- and four-nucleon integrals and the methods for treating them that have not been included in the text.

A. For the Three-Nucleon System

The integrals which we encounter are of the form

$$\int f(r_1)\, g(r_2)\, h(r_{12})\, dv_1\, dv_2 \qquad (A1)$$

where functions f and g are spherically symmetric. A simple method for carrying out the integration is to transform such an integral over a product of functions to a product of integrals. To do this we first consider $\int f(r)F(R)\, dv$ where the notation is indicated in Fig. A1. We get directly

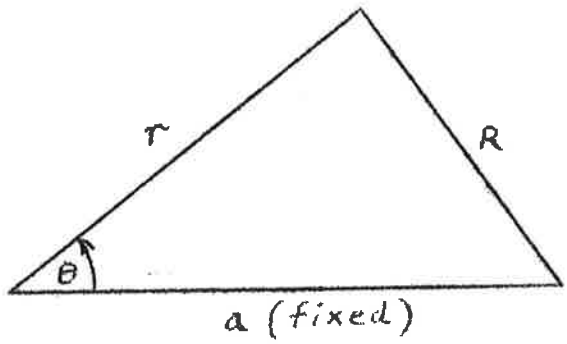


FIG. A1

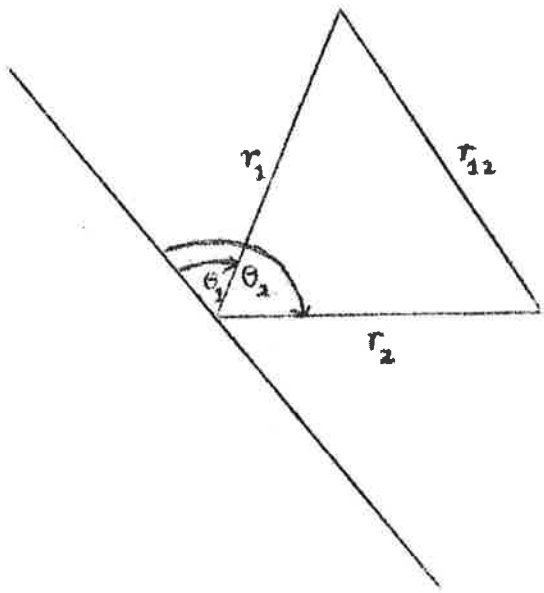


FIG. A2

$$dv = r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$R^2 = a^2 + r^2 - 2ar\cos\theta$$

$$\int f(r)\, F(R)\, dv = \int_0^{\infty} f(r)\, r^2\, dr \int_0^{\Pi} \sin\theta\, d\theta \int_0^{2\Pi} d\phi\, F(R)$$

$$= \left(\frac{2\Pi}{a}\right) \int_0^{\infty} f(r) r\, dr \int_{|a-r|}^{a+r} F(R) R dR$$

$$= \left(\frac{2\Pi}{a}\right) \left\{ \int_0^{a} f(r) r dr \int_{a-r}^{a+r} F(R) R dr \right.$$

$$\left. + \int_a^{\infty} f(r) r dr \int_{r-a}^{r+a} F(R) R dR \right\} \qquad (A2)$$

For the more complicated integral (A1), we get from the correspondence r_1---a, r_2---r, r_{12}---R (see Fig. A2) and formula (A2)

$$\int f(r_1)\, g(r_2)\, h(r_{12})\, dv_1 dv_2 = \int f(r_1) dv_1 \int g(r_2) h(r_{12})\, dv_2$$

$$= \int f(r_1) dv_1 \, \frac{2\Pi}{r_1} \int_0^{\infty} g(r_2) r_2\, dr_2 \int_{|r_1-r_2|}^{r_1+r_2} h(r_{12}) r_{12}\, dr_{12}$$

$$= 8\Pi^2 \int_0^{\infty} f(r_1) r_1 dr_1 \int_0^{\infty} g(r_2) r_2\, dr_2 \int_{|r_1-r_2|}^{r_1+r_2} h(r_{12}) r_{12}\, dr_{12}$$

(A3)

The final form of the integral expresses the condition referred to as the 'triangular inequality'. Using this formula, we can write the integration over all space of the function $G(r_1, r_2, r_{12})$

$$\int G(r_1, r_2, r_{12})\,d\tau = 8\Pi^2 \int_0^\infty r_1\,dr_1 \int_0^\infty r_2\,dr_2 \int_{|r_1-r_2|}^{r_1+r_2} G(r_1, r_2 r_{12})\, r_{12}\,dr_{12} \qquad (A4)$$

It is sometimes convenient, as with exponential functions, to transform our natural coordinates to the perimetric coordinates

$$\begin{aligned} \xi &= r_1+r_2-r_{12} & r_1 &= \tfrac{1}{2}(\xi+\eta) \\ \eta &= r_1-r_2+r_{12} \quad \text{i.e.} & r_2 &= \tfrac{1}{2}(\xi+\nu) \\ \nu &= -r_1+r_2+r_{12} & r_{12} &= \tfrac{1}{2}(\eta+\nu) \end{aligned} \qquad (A5)$$

The volume element $dr_1\,dr_2\,dr_{12}$ then becomes

$$dr_1\,dr_2\,dr_{12} = \frac{1}{4}\,d\xi\,d\eta\,d\nu$$

and relabelling r_{12} as r_3, r_{13} as r_2 and r_{23} as r_1 (the interparticle separations), we find

$$\int G(r_1, r_2, r_3)\,d\tau = \frac{\Pi^2}{4} \int_0^\infty d\xi \int_0^\infty d\eta \int_0^\infty d\nu\,(\xi+\eta)(\xi+\nu)(\eta+\nu)G(\xi,\eta,\nu) \qquad (A6)$$

Another transformation, which is useful with Gaussian functions, is

$$\underline{r} = \underline{x}_2 - \underline{x}_3$$
$$\underline{\rho} = x_1 - \tfrac{1}{2}(\underline{x}_2 + \underline{x}_3) \qquad (A7)$$

i.e. $r_3 = r_{12} = |\underline{\rho}+\tfrac{1}{2}\underline{r}|$, $r_2 = r_{13} = |\underline{\rho}-\tfrac{1}{2}\underline{r}|$, $r_1 = r_{23} = r$

so that

$$d\tau = d^3r\ d^3\rho$$

and

$$\int G(r_1, r_2, r_3)\, d\tau = \iint G(r_1, r_2, r_3)\ d^3r d^3\rho \qquad (A8)$$

where each rectangular component of $\underline{\rho}$ and $\underline{r}$ ranges from $-\infty$ to ∞.

Gaussian Functions

(i) For the potential energy evaluation and normalisation of any general Gaussian we need the integral

$$G_1 = \int \exp(-\alpha_1 r_1^2 - \alpha_2 r_2^2 - \alpha_3 r_3^2)\ d^3\rho\, d^3r$$

If we express each of the factors in the integral in terms of its Fourier transform

$$\exp(-\mu r^2) = \left(\frac{1}{2\Pi}\right)^3 \int \left(\frac{\Pi}{\mu}\right)^{3/2} e^{-k^2/4\mu} \; e^{i\underline{k}\cdot\underline{r}} \; d^3k \;,$$

$$G_1 = \left(\frac{1}{2\Pi}\right)^9 \frac{(\Pi)^{9/2}}{(\alpha_1\alpha_2\alpha_3)^{3/2}} \int \exp\Big\{ - \frac{k^2_{\alpha_1}}{4\alpha_1} - \frac{k^2_{\alpha_2}}{4\alpha_2} - \frac{k^2_{\alpha_3}}{4\alpha_3}$$

$$+ i\underline{k}_{\alpha_1}\cdot\underline{r}_1 + i\underline{k}_{\alpha_2}\cdot(\underline{\rho}-\tfrac{1}{2}\underline{r}) + i\underline{k}_{\alpha_3}\cdot(\underline{\rho}+\tfrac{1}{2}\underline{r})\Big\}$$

$$d^3k_{\alpha_1} d^3k_{\alpha_2} d^3k_{\alpha_3} d^3\rho\, d^3r$$

Integrating over $\underline{\rho}$ and $\underline{r}$ to obtain δ functions in the three $\underline{k}$ variables and then integrating over these, we are left with

$$G_1 = \frac{\Pi^3}{(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3)^{3/2}} \qquad (A9)$$

G_1 can also be derived by transforming the integrand to $\underline{\rho}$ and $\underline{r}$ coordinates. We then use

$$\int_0^\infty \tfrac{1}{2}\exp(-A\underline{\rho}^2 - 2C\underline{\rho}\cdot\underline{r} - B\underline{r}^2)\; d^3\rho\, d^3r = \frac{\Pi^3}{2}\,(AB-C^2)^{-3/2}$$

(ii) In the evaluation of the r.m.s. radius we use

$$G_2 = \int \exp(-\alpha_1 r_1^2 - \alpha_2 r_2^2 - \alpha_3 r_3^2) r_3^2 d^3\rho\, d^3r$$

which is easily derived from G_1 by differentiating it with respect to α_3

$$G_2 = -\frac{\partial G_1}{\partial \alpha_3} = \frac{3}{2}\frac{\Pi^3(\alpha_1+\alpha_2)}{(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)^{5/2}} \qquad (A10)$$

(iii)The <u>form factor integral</u> for the general Gaussian is

$$G_3 = \int \exp(-\alpha_1 r_1^2 - \alpha_2 r_2^2 - \alpha_3 r_3^2)\exp(i\tfrac{2}{3}\underline{q}\cdot\underline{\rho})\,d\tau$$

where $\underline{q}$ is the momentum transfer.

By transforming to the $(\underline{\rho},\underline{r})$ coordinates, we obtain

$$G_3 = \int \exp(-a\rho^2 - br^2 - c\underline{\rho}\cdot\underline{r} + i\tfrac{2}{3}\,\underline{q}\cdot\underline{\rho})\,d^3\rho\,d^3r$$

where $a = \alpha_2+\alpha_3$, $b = \alpha_1 + \overbrace{\tfrac{1}{4}\alpha_3}^{+\frac{1}{4}\alpha_2}$ and $c = \alpha_3-\alpha_2$. This is simplified by completing the square and integrating over the components of $\underline{r}$.

$$G_3 = \left(\frac{\Pi}{b}\right)^{3/2}\int \exp\left(-a\rho^2 + \frac{c^2}{4b}\rho^2 + i\tfrac{2}{3}\underline{q}\cdot\underline{\rho}\right)d^3\rho$$

To evaluate this we resort to the spherical coordinates of Sommerfeld[130]. The corresponding rectangular

coordinates are

$$\rho_x = \rho \cos\theta$$

$$\rho_y = \rho \sin\theta \cos\phi$$

$$\rho_z = \rho \sin\theta \sin\phi$$

Each of the components ranges from $-\infty$ to $+\infty$, ρ ranges from 0 to ∞, θ from 0 to Π and ϕ from $-\Pi$ to $+\Pi$. The volume element becomes

$$d^3\rho = \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi$$

If we choose the ρ_x axis along the direction of the vector $\underline{q}$, so that $\underline{q}\cdot\underline{\rho} = q\rho\cos\theta$, then the ϕ integration can be performed, leaving

$$G_3 = \left(\frac{\Pi}{b}\right)^{3/2} \int_0^\infty \exp\left\{-\left(a - \frac{c^2}{4b}\right)\rho^2\right\} \rho^2 \, d\rho \int_0^\Pi \exp\left(i\frac{2}{3}q\rho\cos\theta\right) 2\Pi \sin\theta \, d\theta$$

From Magnus and Oberhettinger[131]

$$\int_0^\Pi \exp(iZ\cos\theta) C_n^\nu(\cos\theta) \sin^{2\nu}\theta \, d\theta = \frac{2^\nu \Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(2\nu+n)}{n!\,\Gamma(2\nu)} \frac{J_{\nu+n}(z)}{z^\nu}$$

where $C_n^\nu(t)$ are the Gegenbauer functions and $C_0^\nu(t) = 1$;

$J_{\nu+n}(z)$ are Bessel functions. Thus

$$\int_0^{\Pi} \exp(i\tfrac{2}{3}q\rho\cos\theta)\ \sin\theta\, d\theta = \left(\frac{3\Pi}{q\rho}\right)^{\frac{1}{2}} J_{\frac{1}{2}}(\tfrac{2}{3}\ q\rho)$$

Finally, using the formula

$$\int_0^{\infty} \exp(-a^2 t^2)\, t^{\nu+1} J_{\nu}(bt)\, dt = \frac{b^{\nu}}{(2a^2)^{\nu+1}} \exp\left(-\frac{b^2}{4a^2}\right),$$

$$\text{Re } \nu > -1,\ \text{Re } a^2 > 0$$

We get

$$G_3 = \frac{8\Pi^3}{\left\{4(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)\right\}^{3/2}} \exp\left\{-\frac{q^2(4\alpha_1+\alpha_2+\alpha_3)}{36(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)}\right\} \quad (A11)$$

(iv) The integral required for the Coulomb energy calculations is

$$G_4 = \int \exp(-\alpha_1 r_1^2 - \alpha_2 r_2^2 - \alpha_3 r_3^2)\left(\frac{1}{r_3}\right) d\tau$$

$$= \left(\frac{\Pi}{\alpha_1+\alpha_2}\right)^{\frac{1}{2}} \frac{1}{4(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)} \quad (A12)$$

(v) The kinetic energy function $T(\phi,\xi)$ where

$$\phi = \exp(-\alpha_1 r_1^2 - \alpha_2 r_2^2 - \alpha_3 r_3^2) \quad \text{and}$$

$$\xi = \exp(-\alpha_1^1 r_1^2 - \alpha_2^1 r_2^2 - \alpha_3^1 r_3^2)$$

is obtained by straightforward differentiation and use of formula (A10)

$$T(\phi,\xi) = \frac{3\Pi^3\hbar^2}{M}\left\{\sum_c 2\alpha_1\alpha_1^1(\alpha_2+\alpha_2^1+\alpha_3+\alpha_3^1) + \sum_c(\alpha_1+\alpha_1^1)(\alpha_2\alpha_3^1+\alpha_3\alpha_2^1)\right\}\left\{\sum_c(\alpha_1+\alpha_1^1)(\alpha_2+\alpha_2^1)\right\}^{-5/2} \tag{A13}$$

where $\sum_c$ denotes sum over cyclic permutation.

Exponential Functions

(i) The corresponding potential energy and normalisation integral is, in the perimetric coordinates

$$E_1 = \frac{\Pi^2}{4}\int_0^\infty d\xi\int_0^\infty d\eta\int_0^\infty d\nu\,(\xi+\eta)\,(\xi+\nu)\,(\eta+\nu)\,E(\xi,\eta,\nu)$$

where

$$E(\xi,\eta,\nu) = 8\exp\left\{ -\frac{\alpha_1}{2}(\xi+\eta) - \frac{\alpha_2}{2}(\xi+\nu) - \frac{\alpha_3}{2}(\eta+\nu) \right\}$$

$$\left\{ (\xi+\eta)(\xi+\nu)(\eta+\nu) \right\}^{-1} = \exp(-\alpha_1 r_1 - \alpha_2 r_2 - \alpha_3 r_3)$$

$$(r_1 r_2 r_3)^{-1}$$

Therefore

$$E_1 = 2\Pi^2 \int_0^\infty d\xi \ \exp\left\{ -\tfrac{1}{2}(\alpha_1+\alpha_2)\xi \right\} \int_0^\infty d\eta \ \exp\left\{ -\tfrac{1}{2}(\alpha_1+\alpha_3)\eta \right\}$$

$$\int_0^\infty d\nu \ \exp\left\{ -\tfrac{1}{2}(\alpha_2+\alpha_3)\nu \right.$$

$$= \frac{16\Pi^2}{(\alpha_1+\alpha_2)\ (\alpha_2+\alpha_3)\ (\alpha_1+\alpha_3)} \qquad (A14)$$

More general formulae containing powers of r_1, r_2 and r_3 are easily derived by taking partial derivatives of both sides with respect to the parameters α_1, α_2 and α_3. If we use the notation

$$[k\ \ell\ m] = 8\Pi^2 \int_0^\infty r_1^k \exp(-\alpha_1 r_1)\,dr_1 \int_0^\infty r_2^\ell \exp(-\alpha_2 r_2)\,dr_2$$

$$\int_{|r_1-r_2|}^{r_1+r_2} r_3^m \exp(-\alpha_3 r_3)\,dr_3$$

then

$$[0\ 0\ 0] = E_1 = \frac{16\Pi^2}{(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_1+\alpha_3)}$$

By differentiating this k times with respect to α_1, ℓ times with respect to α_2 and m times with respect to α_3, we get a general expression for $[k\ \ell\ m]$

$$[k\ \ell\ m] = 16\Pi^2 k!\ell!m! \sum_{p=0}^{k}\sum_{q=0}^{\ell}\sum_{r=0}^{m} \binom{p+q}{q}\binom{k-p+r}{r}\binom{\ell-q+m-r}{m-r}$$

$$\times (\alpha_1+\alpha_2)^{-(p+q+1)} (\alpha_1+\alpha_3)^{-(k+r+1-p)}$$

$$(\alpha_2+\alpha_3)^{-(\ell+m+1-q-r)} \qquad (A15)$$

(ii) The formulae we use in our calculations for the r.m.s. radius and the Coulomb energy for the inverse square root and true expoentials are

$$E_2 = [110] = 32\Pi^2 \frac{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3) + (\alpha_1+\alpha_3)(\alpha_2+\alpha_3)}{(\alpha_1+\alpha_2)^3\ (\alpha_2+\alpha_3)^2\ (\alpha_1+\alpha_3)^2}$$

(A16)

$$E_3=[111]=64\Pi^2\ \frac{\Sigma_c\alpha_1(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)+2(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_1+\alpha_3)}{\left\{(\alpha_1+\alpha_2)\ (\alpha_2+\alpha_3)\ (\alpha_1+\alpha_3)\right\}^3} \tag{A17}$$

$$E_4=[002]=\frac{32\Pi^2}{(\alpha_1+\alpha_2)}\left\{\frac{(2\alpha_3+\alpha_1+\alpha_2)^2}{\left\{(\alpha_2+\alpha_3)(\alpha_1+\alpha_3)\right\}^3}-\frac{1}{\left\{(\alpha_2+\alpha_3)(\alpha_1+\alpha_3)\right\}^2}\right\} \tag{A18}$$

$$\begin{aligned}E_5=[113]=64\Pi^2\ \Bigg\{&\frac{12\ \Sigma_c\ \alpha_1}{(\alpha_1+\alpha_2)^2(\alpha_2+\alpha_3)^5(\alpha_1+\alpha_3)^2}\\&+\frac{12\ \Sigma_c\ \alpha_1}{(\alpha_1+\alpha_2)^2(\alpha_2+\alpha_3)^2(\alpha_1+\alpha_3)^5}\\&+\frac{9(2\alpha_3+\alpha_1+\alpha_2)}{(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)^4(\alpha_1+\alpha_3)^4}+\frac{3}{(\alpha_1+\alpha_2)^3(\alpha_2+\alpha_3)^4(\alpha_1+\alpha_3)}\\&+\frac{3}{(\alpha_1+\alpha_2)^3(\alpha_2+\alpha_3)(\alpha_1+\alpha_3)^4}+\frac{3(3\alpha_1+3\alpha_2+2\alpha_3)}{(\alpha_1+\alpha_2)^3(\alpha_2+\alpha_3)^3(\alpha_1+\alpha_3)^3}\Bigg\}\end{aligned} \tag{A19}$$

(iii) For the <u>body form factor</u>, we have the integral

$$E_6\ =\int\exp(-\alpha_1 r_1-\alpha_2 r_2-\alpha_3 r_3+i\tfrac{2}{3}q\cdot\underline{\rho})\ (r_1 r_2 r_3)^{-1}\,d^3\rho\,d^3r$$

We carry the integration out by expressing each of the factors in the integrand in terms of its Fourier transform. By using

$$r^{-1}\exp(-\alpha r) = (2\Pi^2)^{-1}\int (k_\alpha^2+\alpha^2)^{-1}\ \exp(i\underline{k}_\alpha\cdot\underline{r})\ d^3k_\alpha$$

$$E_6 = \int (2\Pi^2)^{-3}\left\{(k_{\alpha_1}^2+\alpha_1^2)(k_{\alpha_2}^2+\alpha_2^2)(k_{\alpha_3}^2+\alpha_3^2)\right\}^{-1}\exp\Big\{i\underline{k}_{\alpha_1}\cdot\underline{r} + i\underline{k}_{\alpha_2}(\underline{\rho}-\tfrac{1}{2}\underline{r}) + i\underline{k}_{\alpha_3}\cdot(\underline{\rho}+\tfrac{1}{2}\underline{r}) + i\tfrac{2}{3}\underline{q}\cdot\underline{\rho}\Big\}d^3k_{\alpha_1}\,d^3k_{\alpha_2}\,d^3k_{\alpha_3}\,d^3\rho\,d^3r$$

By integrating over $\underline{\rho}$ and $\underline{r}$ and subsequently over k_{α_2} and k_{α_3} we obtain

$$E_6 = 8\int\left[\left\{(\underline{k}_{\alpha_1}-\tfrac{1}{3}\underline{q})^2+\alpha_2^2\right\}\left\{(\underline{k}_{\alpha_1}+\tfrac{1}{3}\underline{q})^2+\alpha_3^2\right\}\left\{(k_{\alpha_1}^2+\alpha_1^2)\right\}\right]^{-1}d^3k_{\alpha_1}$$

The integration is performed over the magnitude of $\underline{k}_{\alpha_1}$, first, by residues and then over the angle between $\underline{k}_{\alpha_1}$ and $\underline{q}$. The calculation is lengthy and yields the expression

$$E_6 = \frac{8\Pi^2}{Q(2Q^2+\alpha_2^2+\alpha_3^2-2\alpha_1^2)}\left\{\tan^{-1}\left[\frac{Q(4Q^2+\alpha_2^2+3\alpha_3^2)}{\alpha_3(\alpha_2^2-\alpha_3^2)}\right]\right.$$

$$- \tan^{-1}\left[\frac{Q(4Q^2+3\alpha_2^2+\alpha_3^2)}{\alpha_2(\alpha_2^2-\alpha_3^2)}\right] + \tan^{-1}\left[\frac{Q(Q^2+\alpha_2^2+\alpha_1^2)}{\alpha_2(Q^2+\alpha_2^2-\alpha_1^2)}\right]$$

$$- \tan^{-1}\left(\frac{2Q\alpha_1}{Q^2+\alpha_2^2-\alpha_1^2}\right) + \tan^{-1}\left[\frac{Q(Q^2+\alpha_3^2+\alpha_1^2)}{\alpha_3(Q^2+\alpha_3^2-\alpha_1^2)}\right] - \tan^{-1}\left(\frac{2Q\alpha_1}{Q^2+\alpha_3^2-\alpha_1^2}\right)\Bigg\} \qquad (A20)$$

where $Q = \frac{q}{3}$

(iv) The <u>body form factor for the true exponential wave-function</u> can be obtained by differentiating E_6 explicitly

$$E_7 = \int \exp(-\alpha_1 r_1 - \alpha_2 r_2 - \alpha_3 r_3 + i\tfrac{2}{3}\underline{q}\cdot\underline{\rho})\, d^3\rho\, d^3r$$

$$= - \frac{\partial^3}{\partial\alpha_1\partial\alpha_2\partial\alpha_3} E_6$$

The labour involved is, however, substantial. We have consequently evaluated E_7 by a different method. This solution was developed with a different definition of the Fourier transform and to maintain consistency with the expression given in our papers we have decided not to alter the form of the transform. Thus

$$r^{-1}\exp(-\mu r) = (2\Pi^2)^{-1} \int (k_\mu^2+\mu^2)^{-1} \exp(-i\underline{k}_\mu\cdot\underline{r})\, d^3k_\mu$$

and E_7, after integration over $\underline{\rho}$, $\underline{r}$, and two of the k's in E_6 (remembering that with our definition of the Fourier transform, E_6 differs from E_6 in (iii)), is given by

$$E_7 = -32 \frac{\partial^3}{\partial\alpha_1\partial\alpha_2\partial\alpha_3} \int (k_1^2+\alpha_1^2)^{-1}\left\{(\tfrac{2}{3}\underline{q}-\underline{k}_1)^2 + 4\alpha_2^2\right\}^{-1} \left\{(\tfrac{2}{3}\underline{q}-\underline{k}_1)^2 + \alpha_3^2\right\}^{-1} d^3k_1$$

$$= 2^{10}\alpha_1\alpha_2\alpha_3 \iiint \frac{k_1^2 \sin\theta\, dk_1\, d\theta\, d\phi}{(k_1^2+\alpha_1^2)^2\,(\frac{4}{9}q^2+4k_1^2-\frac{8}{3}qk_1\cos\theta+4\alpha_2^2)^2\,(\frac{4}{9}q^2+k_1^2-\frac{4}{3}qk_1\cos\theta+\alpha_3^2)^2}$$

$$= 2^{11}\pi\alpha_1\alpha_2\alpha_3 \int_0^\infty \int_{-1}^{1} \frac{k_1^2\, dk_1\, dx}{(k_1^2+\alpha_1^2)^2\,(\frac{4}{9}q^2+4k_1^2-\frac{8}{3}qk_1 x+4\alpha_2^2)^2\,(\frac{4}{9}q^2+k_1^2-\frac{4}{3}qk_1 x+\alpha_3^2)^2}$$

where

$$x = \cos\theta$$

If

$$A = \frac{4}{9}q^2 + 4k_1^2 + 4\alpha_2^2$$

$$B = -\frac{8}{3}qk_1$$

$$C = \frac{4}{9} q^2 + k_1^2 + \alpha_3^2$$

and $D = -\frac{4}{3} qk_1$, then

$$E_7 = 2^{11} \Pi \alpha_1 \alpha_2 \alpha_3 \int_0^{\infty} \frac{k_1^2 \, dk_1}{(k_1^2+\alpha_1^2)^2} \int_{-1}^{1} \frac{dx}{(A+Bx)^2 (C+Dx)^2}$$

Since

$$\int_{-1}^{1} \frac{dx}{(a+x)^2 (c+x)^2} = \frac{-1}{(a-c)^2} \left(\frac{1}{a+x} + \frac{1}{c+x} \right) + \frac{2}{(a-c)^3} \log_e \left| \frac{a+x}{c+x} \right| \Bigg]_{-1}^{+1}$$

E_7 reduces to

$$E_7 = 2^{13} \Pi \alpha_1 \alpha_2 \alpha_3 \int_0^{\infty} \frac{k_1^2 \, dk_1}{(k_1^2+\alpha_1^2)^2} \left\{ 2k_1^2 + 4\alpha_2^2 - 2\alpha_3^2 - \frac{4}{9} q^2 \right\}^{-2} \left\{ \left[-\frac{4}{3} qk_1 (2k_1^2 + 4\alpha_2^2 - 2\alpha_3^2 - \frac{4}{9} q^2) \right]^{-1} \log_e \left[\left(\frac{k_1^2 + \alpha_3^2 + \frac{4}{9} q^2 + \frac{4}{3} qk_1}{k_1^2 + \alpha_3^2 + \frac{4}{9} q^2 - \frac{4}{3} qk_1} \right) \left(\frac{4k_1^2 + 4\alpha_2^2 + \frac{4}{9} q^2 - \frac{8}{3} qk_1}{4k_1^2 + 4\alpha_2^2 + \frac{4}{9} q^2 + \frac{8}{3} qk_1} \right) \right] + 2 \left\{ (4k_1^2 + 4\alpha_2^2 + \frac{4}{9} q^2)^2 - (\frac{8}{3} qk_1)^2 \right\}^{-1} + \frac{1}{2} \left\{ (k_1^2 + \alpha_3^2 + \frac{4}{9} q^2)^2 - (\frac{4}{3} qk_1)^2 \right\}^{-1} \right\} \qquad (A21)$$

(v) For the determination of the Coulomb Energy and kinetic energy of the inverse square root exponential

function, we use

$$E_8 = \int_0^\infty \exp(-\alpha_1 r_1)\,dr_1 \int_0^\infty \exp(-\alpha_2 r_2)\,dr_2 \int_{|r_1-r_2|}^{r_1+r_2} \exp(-\alpha_3 r_3)\,r_3^{-1}\,dr_3$$

$$= -\int d\alpha_3\; E_1 = \frac{8\Pi^2}{\alpha_2\,(\alpha_2+\alpha_3)} \qquad \text{if } \alpha_1 = \alpha_2 \qquad (A22)$$

$$= \frac{-16\Pi^2}{(\alpha_2+\alpha_1)(\alpha_2-\alpha_1)} \log_e \left|\frac{\alpha_1+\alpha_3}{\alpha_2+\alpha_3}\right| \qquad \text{if } \alpha_1 \neq \alpha_2$$

$$E_9 = \int_0^\infty \exp(-\alpha_1 r_1)\,dr_1 \int_0^\infty \exp(-\alpha_2 r_2)\,dr_2 \int_{|r_1-r_2|}^{r_1+r_2} \exp(-\alpha_3 r_3)\,(r_3)^{-2}\,dr_3$$

$$= -\int d\alpha_3\; E_8 = -\frac{8\Pi^2}{\alpha_2} \log_e(\alpha_2+\alpha_3) \qquad \text{if } \alpha_1 = \alpha_2 \qquad (A23)$$

$$= \frac{16\Pi^2}{(\alpha_2+\alpha_1)(\alpha_2-\alpha_1)} \left\{(\alpha_1+\alpha_3)\log_e(\alpha_1+\alpha_3) - (\alpha_2+\alpha_3)\log_e(\alpha_2+\alpha_3)\right\}$$

$$\text{if } \alpha_1 \neq \alpha_2$$

(vi) The evaluation of the kinetic energy expectation value of the true exponential requires the function $T(\phi,\xi)$.

If $\phi = \exp(-\alpha_1 r_1 - \alpha_2 r_2 - \alpha_3 r_3)$ and

$\xi = \exp(-\alpha_1' r_2 - \alpha_2' r_2 - \alpha_3' r_3)$

then

$$T(\phi,\xi) = \frac{\hbar^2}{M}(\alpha_1\alpha_1'+\alpha_2\alpha_2'+\alpha_3\alpha_3')I(\alpha_1+\alpha_1',\alpha_2+\alpha_2',\alpha_3+\alpha_3')$$

$$= \frac{\hbar^2}{4M}(\alpha_2\alpha_3' + \alpha_3\alpha_2')\ J\ (\alpha_2+\alpha_2',\ \alpha_3+\alpha_3',\ \alpha_1+\alpha_1')$$

$$+ \frac{\hbar^2}{4M}(\alpha_3\alpha_1' + \alpha_1\alpha_3')\ J\ (\alpha_3+\alpha_3',\ \alpha_1+\alpha_1',\ \alpha_2+\alpha_2')$$

$$+ \frac{\hbar^2}{4M}(\alpha_1\alpha_2' + \alpha_2\alpha_1')\ J\ (\alpha_1+\alpha_1',\ \alpha_2+\alpha_2',\ \alpha_3+\alpha_3') \qquad (A24)$$

where

$$I(\alpha,\beta,\gamma) \text{ is } \frac{E_3}{8\Pi^2}$$

and

$$J(\alpha,\beta,\gamma) = 16[(\alpha^3+\beta^3)\gamma^2 + (\alpha^2+\beta^2)(4\gamma^2+\alpha\beta\gamma)$$

$$+(\alpha+\beta)(4\gamma^4+7\alpha\beta\gamma^2) + \gamma^5 + 3\alpha^2\beta^2\gamma + 10\alpha\beta\gamma^3]$$

$$\times [(\alpha+\beta)^3\ (\beta+\gamma)^4\ (\alpha+\gamma)^4]^{-1}$$

B. Integrals for the Alpha Particle

With our choice of wavefunctions, all the integrals appearing in the four-nucleon system are exactly evaluated by transformation to the η coordinates.

Gaussian Functions

(i) Our general normalisation integral

$$\int \exp(-a_1 r_{12}^2 - b r_{13}^2 - c r_{14}^2 - d r_{23}^2 - e r_{24}^2 - f r_{34}^2)\, d\tau$$

becomes

$$\int \exp(-\alpha_1 \eta_1^2 - \beta \underline{\eta}_1 \cdot \underline{\eta}_2 - \gamma \eta_2^2 - \delta \eta_3^2 - \epsilon \underline{\eta}_1 \cdot \underline{\eta}_3 - \phi \underline{\eta}_2 \cdot \underline{\eta}_3)\, d^3\eta_1\, d^3\eta_2\, d^3\eta_3$$

in our transformed coordinates. Noting that

$$\alpha_i x^2 + \beta_i x = (\sqrt{\alpha_i} + \frac{\beta_i}{2\sqrt{\alpha_i}})^2 - \frac{\beta_i^2}{4\alpha_i}$$

we find

$$Ga_1 = \alpha_1^{-\frac{3}{2}} \int \exp(-\eta_1^{12} + \frac{\beta_1^2}{4\alpha_1^2} - \gamma \eta_2^2 - \delta \eta_3^2 - \phi \underline{\eta}_2 \cdot \underline{\eta}_3)\, d^3\eta_1^1\, d^3\eta_2\, d^3\eta_3$$

with $\alpha_1 = \alpha$, $\beta_1 = \beta \underline{\eta}_2 + \epsilon \underline{\eta}_3$, and finally

$$Ga_1 = (\alpha_1 \alpha_2)^{-\frac{3}{2}} \int \exp(-\eta_1^{12} - \eta_2^{12} + \frac{\beta_2^2}{4\alpha_2} - \eta_3^2 (\delta - \frac{\epsilon^2}{4\alpha}))\, d^3\eta_1^1\, d^3\eta_2^1\, d^3\eta_3$$

where $\alpha_2 = \gamma - \frac{\beta^2}{4\alpha}$, $\beta_2 = \phi \underline{\eta}_3 - \beta \epsilon \underline{\eta}_3 \Big/ 2\alpha$

Ga_1 then reduces to

$$Ga_1 = \pi^{\frac{9}{2}} 4^3 \left[(4\alpha\gamma-\beta^2)(4\delta-\frac{\epsilon^2}{\alpha}) - (2\alpha\phi-\beta\epsilon)(2\phi-\frac{\beta\epsilon}{\alpha}) \right]^{-\frac{3}{2}} \qquad (B1)$$

(ii) The r.m.s. radius calculation needs

$$Ga_2 = \int (B_1\eta_1^2+B_2\underline{\eta}_1\cdot\underline{\eta}_2+B_3\eta_2^2+B_4\eta_3^2+B_5\underline{\eta}_1\cdot\underline{\eta}_3+B_6\underline{\eta}_2\cdot\underline{\eta}_3)$$

$$\times \exp(-\alpha_1\eta_1^2-\beta_1\underline{\eta}_1\cdot\underline{\eta}_2-\gamma\eta_2^2-\delta\eta_3^2-\epsilon\underline{\eta}_1\cdot\underline{\eta}_3-\phi\underline{\eta}_2\cdot\underline{\eta}_3)$$

$$d^3\underline{\eta}_1 d^3\eta_2 d^3\eta_3$$

$$= -\sum_{i=1}^{6} B_i \frac{\partial}{\partial a_i} I_1(a_1, a_2, \ldots a_6) \qquad (B2)$$

where

$a_1 = \alpha$ $\qquad$ $a_4 = \delta$

$a_2 = \beta$ $\qquad$ $a_5 = \epsilon$

$a_3 = \gamma$ $\qquad$ $a_6 = \phi$

(iii) The body form factor integral with the Gaussian function is evaluated using the Gegenbauer-functions-method of (A11). Our most general integral is

$$Ga_3 = \int \exp[i\underline{q}_1\cdot\underline{\eta}_1+i\underline{q}_2\cdot\underline{\eta}_2+i\underline{q}_3\cdot\underline{\eta}_3-\alpha_1^2\eta_1^2-\alpha_2^2\eta_2^2-\alpha_3^2\eta_3^2]\, d^3\eta_1 d^3\eta_2 d^3\eta_3$$

We define the 9-dimensional vector $\underline{R}$ such that its first three components are equal to α_1 times the rectangular

components of $\underline{\eta}_1$, the next three are equal to α_2 times the components of $\underline{\eta}_2$ and the last three are α_3 times the components of $\underline{\eta}_3$. We also define $\underline{Q}$ such that

$$Q_{1,2,3} = \alpha_1^{-1} (q_1)_{x,y,z}, \quad Q_{4,5,6} = \alpha_2^{-1} (q_2)_{x,y,z}$$

$$Q_{7,8,9} = \alpha_3^{-1} (q_3)_{x,y,z}$$

Then $Ga_3 = (\alpha_1 \alpha_2 \alpha_3)^{-3} \int \exp(i\, \underline{Q}.\underline{R} - R^2) d^9R$

If we use Sommerfeld's expansion, our rectangular coordinates are

$x_1 = R \cos\theta$

$x_2 = R \sin\theta \cos\phi_1$

$x_3 = R \sin\theta \sin\phi_1 \cos\phi_2$

$x_4 = R \sin\theta \sin\phi_1 \sin\phi_2 \cos\phi_3$

$x_5 = R \sin\theta \sin\phi_1 \sin\phi_2 \sin\phi_3 \cos\phi_4$

$x_6 = R \sin\theta \sin\phi_1 \sin\phi_2 \sin\phi_3 \sin\phi_4 \cos\phi_5$

$x_7 = R \sin\theta \sin\phi_1 \sin\phi_2 \sin\phi_3 \sin\phi_4 \sin\phi_5 \cos\phi_6$

$x_8 = R \sin\theta \sin\phi_1 \sin\phi_2 \sin\phi_3 \sin\phi_4 \sin\phi_5 \sin\phi_6 \cos\phi_7$

$x_9 = R \sin\theta \sin\phi_1 \sin\phi_2 \sin\phi_3 \sin\phi_4 \sin\phi_5 \sin\phi_6 \sin\phi_7$

Each of the X's range from $-\infty$ to $+\infty$, R ranges from 0 to ∞, θ, ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5 and ϕ_6 range from 0 to Π and ϕ_7 ranges from $-\Pi$ to Π. The 9-dimensional volume element is

$$d^9R = R^8 \sin^7\theta \sin^6\phi_1 \sin^5\phi_2 \sin^4\phi_3 \sin^3\phi_4 \sin^2\phi_5 \sin\phi_6$$

$$dR\,d\theta\,d\phi_1\,d\phi_2\,d\phi_3\,d\phi_4\,d\phi_5\,d\phi_6\,d\phi_7$$

But $$\int_0^{\Pi} \sin^p\theta\,d\theta = \frac{\Pi}{p}\,\frac{2^{-p+2}\,\Gamma(p)}{\Gamma(p/2)\,\Gamma(p/2)}$$

Hence $$\int_0^{\Pi} \sin^p\theta_1\,d\theta_1 \int \sin^{p-1}\theta_2\,d\theta_2 \ldots \int_0^{\Pi} \sin\theta_p\,d\theta_p \int_{-\Pi}^{\Pi} d\theta_{p+1}$$

$$= \frac{2\Pi^{(1+\frac{p}{2})}}{\Gamma(1+\frac{p}{2})}$$

We choose the x_1 axis along the direction of the vector $\underline{Q}$, so $\underline{Q}\cdot\underline{R} = QR\cos\theta$ and after the ϕ integrations are made,

$$Ga_3 = (\alpha_1\alpha_2\alpha_3)^{-3}\left(\frac{\Pi^4}{3}\right)\int_0^{\infty}\int_0^{\Pi} \exp(iQR\cos\theta - R^2)R^8\sin^7\theta\,dR\,d\theta$$

The θ integration can be done by the method of (A11).

$$\int_0^{\Pi} \exp(iQR\cos\theta)\sin^7\theta\,d\theta = \frac{3.2^{9/2}\,\Pi^{\frac{1}{2}}J_{7/2}(QR)}{(QR)^{7/2}}$$

Evaluation of the R integral then gives

$$Ga_3 = (\alpha_1\alpha_2\alpha_3)^{-3}(\Pi)^{9/2}\exp(-\frac{Q^2}{4}) \qquad (B3)$$

where

$$Q^2 = \left(\frac{q_1}{\alpha_1}\right)^2 + \left(\frac{q_2}{\alpha_2}\right)^2 + \left(\frac{q_3}{\alpha_3}\right)^2$$

Irving Wavefunctions

(i) The normalisation of our Irving functions are carried out with the help of

$$I_1 = \int \exp\left\{-4\alpha_{\underline{r}}(\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}}\right\} d^3\eta_1\, d^3\eta_2\, d^3\eta_3$$

which is

$$I_1 = \int \exp\left\{-4\alpha_{\underline{r}}(\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}}\right\} (4\Pi)^3\ (\eta_1\eta_2\eta_3)^2\, d\eta_1\, d\eta_2\, d\eta_3\ ,$$

after integrating over the angles.

To evaluate I_1, we use the further transformation

$$\eta_1 = R\sin\theta\cos\phi,\ \eta_2 = R\sin\theta\sin\phi,\ \eta_3 = R\cos\theta$$

The Jacobian is easily found and the volume element

$$d^3\eta_1\, d^3\eta_2\, d^3\eta_3 = (4\Pi)^3\ R^8 \sin^5\theta\cos^2\theta\sin^2\phi\cos^2\phi\, dR d\theta d\phi$$

Now
$$I_1 = \int_0^{\Pi/2} \sin^5\theta\cos^2\theta\, d\theta \int_0^{\Pi/2} \sin^2\phi\cos^2\phi\, d\phi \int_0^{\infty} (4\Pi)^3 R^8 \exp(-4\alpha_{\underline{r}} R)\, dR$$

But $$\int_0^{\Pi/2} (\cos\theta)^{2m-1} \ (\sin\theta)^{2n-1} \ d\theta = \frac{\Gamma(m)\ \Gamma(n)}{2\Gamma(m+n)}$$

and $$\int_0^{\infty} R^n \exp(-kR)\,dR = \frac{n!}{k^{n+1}}$$

so, after some manipulation

$$I_1 = (4\Pi)^3 \ \frac{3\Pi}{2^{12}\,\alpha_{\underline{r}}^{\ 9}}$$

(ii) The computation of the body form factor integral is greatly simplified if we use the methods of (A11) and (B3), i.e. introduce the 9-dimensional integral $\underline{R}$ The most general integral we require is

$$I_2 = \int \exp\left\{-(\alpha_1^2\eta_1^2+\alpha_2^2\eta_2^2+\alpha_3^2\eta_3^2)^{\frac{1}{2}}+i\underline{q}_1\cdot\underline{\eta}_1+i\underline{q}_2\cdot\underline{\eta}_2+i\underline{q}_3\cdot\underline{\eta}_3\right\} d^3\eta_1\, d^3\eta_2\, d^3\eta_3$$

Introducing $\underline{R}$ and defining $\underline{Q}$ as in (B3) and integrating over θ and ϕ, we derive

$$I_2 = (\alpha_1\alpha_2\alpha_3)^{-3}\left(\frac{\Pi^4}{3}\right)\int_0^{\infty} \exp(-R)\,3.2^{9/2}\,\Pi^{\frac{1}{2}}\,J_{7/2}(QR)\,Q^{-7/2}\,R^{9/2}\,dR$$

But
$$\int_0^\infty \exp(-at) J_\nu(bt) t^{\nu+1} dt = \frac{2a(2b)^\nu \Gamma(\nu+\frac{3}{2})}{(a^2+b^2)^{\nu+\frac{3}{2}} \Pi^{\frac{1}{2}}}$$

so
$$I_2 = (\alpha_1 \alpha_2 \alpha_3)^{-3} \frac{3.2^{12} \Pi^4}{(1+Q^2)^5} \qquad (B5)$$

where
$$Q^2 = \left(\frac{q_1}{\alpha_1}\right)^2 + \left(\frac{q_2}{\alpha_2}\right)^2 + \left(\frac{q_3}{\alpha_3}\right)^2$$

(iii) By methods similar to (B4), we can evaluate
$$I_3 = \int \exp\left\{-4a_{\underline{r}} (\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}}\right\} (\eta_3)^2 \; d^3\eta_1 d^3\eta_2 d^3\eta_3$$

which we use in our r.m.s. radius calculations.
$$I_3 = \frac{45\Pi^4}{2^9 \alpha_{\underline{r}}^{11}} \qquad (B6)$$

(iv) For the Coulomb energy calculations we employ
$$I_4 = \int \exp\left\{-4\alpha_{\underline{r}} (\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}}\right\} \frac{e^2}{\sqrt{2}\eta_3} \; d^3\eta_1 d^3\eta_2 d^3\eta_3$$
$$= \frac{7!\, \Pi^4 e^2}{2^{\frac{31}{2}}\, 3\alpha_{\underline{r}}^8} \qquad (B7)$$

(v) In the evaluation of the kinetic energy values, the following integral arises

$$I_5 = -\frac{\hbar^2}{2M}\int \phi\,(\nabla^2_{\eta_1}+\nabla^2_{\eta_2}+\nabla^2_{\eta_3})\,\xi\; d^3\eta_1 d^3\eta_2 d^3\eta_3$$

where

$$\phi = \exp\left\{-2\alpha_{\underline{r}}(\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}}\right\}$$

and

$$\xi = \exp\left\{-2\alpha^1_{\underline{r}}(\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}}\right\}$$

Since $\nabla^2_u = \left(\frac{d^2}{du^2} + \frac{2}{u}\frac{d}{du}\right)$, we find after explicit differentiation

$$I_5 = -\frac{\hbar^2}{2M}\int \exp\left\{-(2\alpha_{\underline{r}}+2\alpha^1_{\underline{r}})(\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}}\right\}\left\{4\alpha'^2_{\underline{r}} - 16\alpha^1_{\underline{r}}(\eta_1^2+\eta_2^2+\eta_3^2)^{-\frac{1}{2}}\right\}$$

$$d^3\eta_1 d^3\eta_2 d^3\eta_3$$

A transformation to the R,θ,ϕ coordinates and a simple integration reveals

$$I_5 = \frac{\hbar^2}{M}\;\frac{3\Pi^4\; 2^{13}(\alpha_{\underline{r}}\alpha^1_{\underline{r}})}{(2\alpha_{\underline{r}}+2\alpha^1_{\underline{r}})^9} \qquad \text{(B8)}$$

(vi) For the static potential energy terms we come

across the integral

$$I_6 = \int \exp\left\{-4\alpha_{\underline{r}}(\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}}\right\} \exp\left(-k\sqrt{2}\eta_3\right) d^3\eta_1 d^3\eta_2 d^3\eta_3$$

This becomes

$$I_6 = (4\Pi)^3 \int \exp\left\{-(4\alpha_{\underline{r}}R+k\sqrt{2}R\cos\theta)\right\} R^8 \sin^5\theta\cos^2\theta\sin^2\phi\cos^2\phi \; dR d\theta d\phi$$

Integrating explicitly over R and ϕ we obtain

$$I_6 = (4\Pi)^3 \int_0^{\Pi/2} \frac{8!\sin^5\theta\cos^2\theta \; d\theta}{(4\alpha_{\underline{r}}+\sqrt{2}k\cos\theta)^9} \int_0^{\Pi/2} \sin^2\phi\cos^2\phi \; d\phi$$

$$= (4\Pi)^3 \int_0^{\Pi/2} \left(\frac{\Pi}{16}\right) \frac{8!\sin^5\theta\cos^2\theta d\theta}{(4\alpha_{\underline{r}}+\sqrt{2}k\cos\theta)^9}$$

If we let $u = \cos\theta$ and $\delta = \dfrac{2\sqrt{2}\alpha_{\underline{r}}}{k}$

$$I_6 = \frac{4\Pi^4 \; 8!}{(\sqrt{2}k)^9} \int_0^1 \frac{u^2(1-u^2)^2}{(\delta+u)^9} du$$

The general integral $\displaystyle\int_0^1 \frac{u^p(1-u^2)^q du}{(a+u)^r}$, can be evaluated by using the formula

$$\int \frac{u^m}{(a+u)^n}\,du = \sum_{s=0}^{m}\left[\frac{m!(-a)^s(a+u)^{m-n-s+1}}{(m-s)!\,s!\,(m-n-s+1)}\right]$$

When $m-n-s+1 = 0$, the corresponding term in the square brackets is

$$\frac{m!(-a)^{m-n+1}}{(m-n+1)!\,(n-1)!}\ \log_e|(a+u)|\ .$$

A simple expression for the general integral is obtained for the special case, $r-p-2q = 3$. Then

$$\int_0^1 \frac{u^p(1-u^2)^q}{(a+u)^n}\,du = \frac{a^{p-r+1}}{(a+1)^{p+q+2}}\sum_{s=0}^{q+1} C_s\left(\frac{a-1}{2}\right)^s$$

where

$$C_s = r\,\frac{2^q\,q!}{s!\,(q+2-s)!}\sum_{\ell=0}^{s}\frac{\binom{s}{\ell}(-1)^\ell}{\binom{p+q+2-s+2\ell}{q+2-s}}$$

and

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

(This expression was first derived by Abraham et al.[46] but there is a small error in C_s in their paper). The integral in u from I_6 satisfies the condition $r-p-2q = 3$

and using the formula above, we find

$$I_6 = \frac{4\Pi^4\ 8!}{(\sqrt{2}k)^9}\left[\frac{64\delta^3+69\delta^2+30\delta+5}{840\delta^6(1+\delta)^6}\right] \qquad (B9)$$

(vii) Other integrals which we encounter when we attempt to evaluate the photodisintegration and velocity dependent potential values include a couple which satisfy the condition referred to in (B9). These are

$$I_7 = \int \exp\left\{-4\alpha_{\underline{r}}(\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}}\right\}\exp(-\sqrt{2}k\eta_3)(\eta_3)^{-1}\,d^3\eta_1\,d^3\eta_2\,d^3\eta_3$$

$$= \frac{4\Pi^4\ 7!}{(\sqrt{2}k)^8}\int_0^1 \frac{u(1-u^2)^2}{(\delta+u)^8}\,du$$

$$= \frac{4\Pi^4\ 7!}{(\sqrt{2}k)^8}\left[\frac{35\delta^3+47\delta^2+25\delta+5}{210\,\delta^6\,(1+\delta)^5}\right] \qquad (B10)$$

and

$$I_8 = \int \exp\left\{-4\alpha_{\underline{r}}(\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}}\right\}\exp(-\sqrt{2}k\eta_3)(\eta_3)^2\,d^3\eta_1\,d^3\eta_2\,d^3\eta_3$$

$$= \frac{4\Pi^4\ 10!}{(\sqrt{2}k)^{11}}\int_0^1 \frac{u^4(1-u^2)^2}{(\delta+u)^{11}}\,du$$

$$= \frac{4\Pi^4\ 10!}{(\sqrt{2}k)^{11}} \left[\frac{32\delta^3 + 25\delta^2 + 8\delta + 1}{1260\ \delta^6\ (1+\delta)^6} \right] \qquad (B11)$$

In these calculations, we require too

$$I_9 = \int \exp\left\{ -4\alpha_{\underline{r}} (\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}} \right\} \exp(-\sqrt{2}k\eta_3)\,(\eta_1^2+\eta_2^2+\eta_3^2)^{-1}$$

$$d^3\eta_1\, d^3\eta_2\, d^3\eta_3$$

$$= \frac{4\Pi^4\ 7!}{(\sqrt{2}k)^8} \int_0^1 \frac{u^2\,(1-u^2)^2}{(\delta+u)^8}\ du$$

$$= \frac{4\Pi^4\ 7!}{(\sqrt{2}k)^8} \left[\frac{8\delta^2 + 5\delta + 1}{105\ \delta^5\,(1+\delta)^5} \right] \qquad (B12)$$

$$I_{10} = \int \exp\left\{ -4\alpha_{\underline{r}} (\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}} \right\} \exp(-\sqrt{2}k\eta_3)\,(\eta_3)^2\,(\eta_1^2+\eta_2^2+\eta_3^2)^{-1}$$

$$d^3\eta_1\, d^3\eta_2\, d^3\eta_3$$

$$= \frac{4\Pi^4\ 8!}{(\sqrt{2}k)^9} \int_0^1 \frac{u^4\,(1-u^2)^2}{(\delta+u)^9}\ du \qquad (B13)$$

$$I_{11} = \int \exp\left\{ -4\alpha_{\underline{r}} (\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}} \right\} \exp(-\sqrt{2}k\eta_3)\,(\eta_3)^2\,(\eta_1^2+\eta_2^2+\eta_3^2)^{-\frac{3}{2}}$$

$$d^3\eta_1\, d^3\eta_2\, d^3\eta_3$$

$$= \frac{4\Pi^4\ 7!}{(\sqrt{2}k)^8} \int_0^1 \frac{u^4\,(1-u^2)^2}{(\delta+u)^8}\ du \qquad (B14)$$

and

$$I_{12} = \int \exp\left\{-4\alpha_{\underline{r}}(\eta_1^2+\eta_2^2+\eta_3^2)^{\frac{1}{2}}\right\}\exp(-\sqrt{2k}\eta_3)\eta_3(\eta_1^2+\eta_2^2+\eta_3^2)^{-\frac{1}{2}}$$

$$d^3\eta_1 d^3\eta_2 d^3\eta_3$$

$$= \frac{4\Pi^4\ 8!}{(\sqrt{2k})^9}\int_0^1 \frac{u^3(1-u^2)^2}{(\delta+u)^9}\ du\ . \qquad (B15)$$

```
C     C  COMPUTER PROGRAMS USED IN THE EVALUATION OF THE PHYSICAL
C     PROPERTIES OF THE LIGHT NUCLEI
C     THESE PROGRAMS ARE WRITTEN IN FORTRAN LANGUAGE AND FOR CONVENIENCE
C     ARE BUILD-UPS OF MANY SMALLER ONES. THE VARIABLES USED ARE SELF
C     EXPLANATORY BUT WHERE CONFUSION MAY ARISE WE HAVE INCLUDED OUR
C     COMMENTS. TO PREVENT REPETITION WE HAVE INCLUDED FUNCTIONS ONLY
C     IN THE FIRST PROGRAMS IN WHICH THEY APPEAR. SUBSEQUENT PROGRAMS
C     USING SUCH FUNCTIONS DO NOT CONTAIN THEM
```

```
C     PROGRAM FOR DERIVING THE GAUSSIAN CENTRAL POTENTIALIN CHAPTER 3
C     OUR PARAMETERS VO AND MU ARE OBTAINED BY FITTING THE BINDING
C     ENERGY AND THE FORM FACTOR
      PROGRAM EXTRI(INPUT,OUTPUT)
      DIMENSION FF(8)
      VOMU=0.8
    1 ASQ=5.4*VOMU/41.47
      A=SQRT(ASQ)
      B=SQRT(2.1/2.7)*A
      E=41.47*3.0*(A+B)/2.0
      ANORM=AI(A/3.0,A/3.0,A/3.0+0.5*(B-A))
      VO=17.0
   39 AMU=VOMU/VO
      F1=AI(A/3.0,A/3.0,A/3.0+0.5*(B-A)+AMU)
      F1=F1/ANORM
      F2=AI(A/3.0,A/3.0+AMU,A/3.0+0.5*(B-A))
      F2=F2/ANORM
      F4=AR(A/3.0,A/3.0,A/3.0+0.5*(B-A))
      F4=F4/ANORM
      F5=AS(A/3.0,A/3.0,A/3.0+0.5*(B-A))
      F5=F5/ANORM
      BE=E-0.8*VO*(F1+F2+F2)-0.8*VOMU*(F4+F5+F5)+0.2*VO*F1+0.2*VOMU*F4
     1-0.5*0.2*VO*(F2+F2)-0.5*0.2*VOMU*(F5+F5)
      PRINT 20,F1,F2,F4,F5
   20 FORMAT(4F15.8)
      PRINT 21,E,BE
   21 FORMAT(2F15.8)
      PRINT 22,VO,AMU
   22 FORMAT(2F15.8)
      A=A/3.0
      B=B/3.0
      DO 11 I=1,8
      FF(I)=EXP(-I*(4.0*B+2.0*A)/(36.0*(2.0*A*B+A*A)))
      PRINT 12,FF(I)
   12 FORMAT(F15.8)
   11 CONTINUE
      RADSQ=2.0*(3.0*(A+B)-0.75*(A+A))/(9.0*(2.0*A*B+A*A))
      RAD=SQRT(RADSQ)
      PRINT13,RAD
   13 FORMAT(F15.8)
      A=A*3.0
      B=B*3.0
      VO=VO+0.5
      IF(VO.GT.22.0) GO TO 41
      GO TO 39
   41 VOMU=VOMU+0.1
      IF(VOMU.GT.1.3) GO TO 40
      GO TO 1
   40 STOP
      END
      FUNCTION AI(AL1,AL2,AL3)
      PI=3.14159
      AI=PI/(8.0*(AL1*AL2+AL1*AL3+AL2*AL3)*SQRT(AL1*AL2+AL1*AL3+AL2*AL3)
     1 )
       RETURN
      END
      FUNCTION AR(AL1,AL2,AL3)
      PI=3.14159
      AR=3.0*PI*(AL1+AL2)/(16.0*(AL1*AL2+AL1*AL3+AL2*AL3)**2.5)
      RETURN
```

```
END
FUNCTION AS(AL1,AL2,AL3)
PI=3.14159
AS=3.0*PI*(AL2+AL3)/(16.0*(AL1*AL2+AL1*AL3+AL2*AL3)**2.5)
RETURN
END
```

```
      PROGRAM TRION(INPUT,OUTPUT)
C     TRITON AND TRINEUTRON BINDING ENERGIES BY OUR EXACT METHODS
C     THE INTEGRALS ARE COMPUTED USING MILLERS QUADRATURE FORMULA
      DIMENSIONA(17,17),B(17,17),C(17,17),D(17,17),E(17,17),F(17,17,31),
     1G(17,17),H(17,17),P(17),Q(17,17),R(17),S(17,17),T(17,17),U(17,17),
     2X(17,17),Y(17,17),VALUE(31),PQ(17,17),H1(17),H2(17),UA(17,17),
     3UB(17,17)
      REAL MUAL
      REAL  MU
   70 READ80,W,BEE,VO,MU,N
      PRINT80,W,BEE,VO,MU,N
C     THE VALUE ASSUMED FOR N FIXES THE MESH SIZE
   80 FORMAT(4F10.5,I10)
      IF (VO) 40,40,65
   65 PLANCK=1.0543
      EQPT=1.0
      AM=1.670$Z=2.0/(N-1)$CONST=1.602
      PLANCK=2.0*PLANCK*PLANCK
      AMCON=AM*CONST
      TEM1=(3.0*W+1.5*BEE)*VO*MU*MU
      TEM2=(3.0*W-1.5*BEE)*VO*MU*MU
      EPSP=10.0*(SQRT(TEM1)+SQRT(TEM2))*SQRT(PLANCK/AMCON)
      PEE=EPSP*AMCON/(2.0*PLANCK*100)
      EPSQ=10.0*(SQRT(TEM1)-SQRT(TEM2))*SQRT(PLANCK/AMCON)
      QUE=EPSQ*AMCON/(2.0*100*PLANCK)
      PI=8.0*3.141592*3.141592
      DO35 I=1,N
      DO35 J=1,N
      H1(I)=((I-(N+1.0)/2.0)*Z+1.0)
      H2(J)=(J-(N+1.0)/2.0)*Z
      P(I)=PI*H1(I)*H1(I)*SQRT(1.0-H1(I)*H1(I)/4.0)/4.0
      R(I)=(0.5*H1(I)*H1(I)-1.0)
      S(I,J)=(H1(I)*H1(I)*H2(J)*H2(J)*(1.0-H1(I)*H1(I)/4.0))
      Q(I,J)=SQRT(R(I)*R(I)+S(I,J))
      T(I,J)=0.5+H1(I)*H1(I)/4.0-0.866*H1(I)*H2(J)*
     1SQRT(1.0-H1(I)*H1(I)/4.0)
      U(I,J)=0.5+H1(I)*H1(I)/4.0+0.866*H1(I)*H2(J)*
     1SQRT(1.0-H1(I)*H1(I)/4.0)
      UA(I,J)=0.5*R(I)+0.5*(T(I,J)-U(I,J))
      UB(I,J)=0.5*R(I)-0.5*(T(I,J)-U(I,J))
      PQ(I,J)=(2.0*PEE+2.0*QUE*Q(I,J))
      A(I,J)=P(I)*(Q(I,J)+R(I))
      B(I,J)=P(I)*(Q(I,J)-R(I))
      C(I,J)=P(I)*SQRT(S(I,J))
      D(I,J)=PQ(I,J)*PQ(I,J)*Q(I,J)*Q(I,J)
      E(I,J)=(1.0-R(I))/(PQ(I,J)*PQ(I,J)*PQ(I,J)*Q(I,J)*Q(I,J))
      G(I,J)=(PQ(I,J)+MU*MU*(1.0-R(I)))**2*Q(I,J)*Q(I,J)
      H(I,J)=(PQ(I,J)+MU*MU*T(I,J))**2*Q(I,J)*Q(I,J)
      X(I,J)=(PQ(I,J)+MU*MU*U(I,J))**2*Q(I,J)*Q(I,J)
      Y(I,J)=PQ(I,J)*PQ(I,J)*PQ(I,J)*Q(I,J)*Q(I,J)
    1 F(I,J,1)=0.25*A(I,J)/D(I,J)
    2 F(I,J,2)=0.25*B(I,J)/D(I,J)
    3 F(I,J,3)=0.25*C(I,J)/D(I,J)
    4 F(I,J,4)=MU*MU*0.5*A(I,J)*E(I,J)
      F(I,J,4)=F(I,J,4)/(EQPT*EQPT)
    5 F(I,J,5)=MU*MU*0.5*B(I,J)*E(I,J)
      F(I,J,5)=F(I,J,5)/(EQPT*EQPT)
    6 F(I,J,6)=MU*MU*0.5*C(I,J)*E(I,J)/1.0
      F(I,J,6)=F(I,J,6)/(EQPT*EQPT)
    7 F(I,J,7)=0.25*A(I,J)/G(I,J)
```

```
 8 F(I,J,8)=0.25*B(I,J)*1.0/G(I,J)
 9 F(I,J,9)=0.25*C(I,J)/G(I,J)
10 F(I,J,10)=0.25*A(I,J)/H(I,J)
11 F(I,J,11)=0.25*B(I,J)/H(I,J)
12 F(I,J,12)=0.25*C(I,J)/H(I,J)
13 F(I,J,13)=0.25*A(I,J)/X(I,J)
14 F(I,J,14)=0.25*B(I,J)/X(I,J)
15 F(I,J,15)=0.25*C(I,J)/X(I,J)
16 F(I,J,16)=MU*MU*0.5*A(I,J)*T(I,J)/(Y(I,J))
   F(I,J,16)=F(I,J,16)/(EQPT*EQPT)
17 F(I,J,17)=MU*MU*0.5*B(I,J)*T(I,J)/(Y(I,J))
   F(I,J,17)=F(I,J,17)/(EQPT*EQPT)
18 F(I,J,18)=MU*MU*0.5*C(I,J)*T(I,J)/(Y(I,J))
   F(I,J,18)=F(I,J,18)/(EQPT*EQPT)
19 F(I,J,19)=MU*MU*0.5*A(I,J)*U(I,J)/(Y(I,J))
   F(I,J,19)=F(I,J,19)/(EQPT*EQPT)
20 F(I,J,20)=MU*MU*0.5*B(I,J)*U(I,J)/(Y(I,J))
   F(I,J,20)=F(I,J,20)/(EQPT*EQPT)
21 F(I,J,21)=MU*MU*0.5*C(I,J)*U(I,J)/(Y(I,J))
   F(I,J,21)=F(I,J,21)/(EQPT*EQPT)
22 F(I,J,22)=(Q(I,J)-R(I))*H1(I)*H1(I)/(PQ(I,J)*SQRT(PQ(I,J)/3.142)*
  1Q(I,J)*Q(I,J)*32.0*1.414)
23 F(I,J,23)=(Q(I,J)+R(I))*H1(I)*H1(I)/(PQ(I,J)*SQRT(PQ(I,J)/3.142)*
  1Q(I,J)*Q(I,J)*32.0*1.414)
24 F(I,J,24)=SQRT(Q(I,J)+UA(I,J))*U(I,J)*P(I)*0.5*SQRT(Q(I,J)-R(I))
  1/(X(I,J)*(PQ(I,J)+MU*MU*U(I,J)))
25 F(I,J,25)=SQRT(Q(I,J)-UA(I,J))*U(I,J)*P(I)*0.5*SQRT(Q(I,J)+R(I))
  1/(X(I,J)*(PQ(I,J)+MU*MU*U(I,J)))
26 F(I,J,26)=SQRT(Q(I,J)+UB(I,J))*T(I,J)*P(I)*0.5*SQRT(Q(I,J)-R(I))
  1/(H(I,J)*(PQ(I,J)+MU*MU*T(I,J)))
27 F(I,J,27)=SQRT(Q(I,J)-UB(I,J))*T(I,J)*P(I)*0.5*SQRT(Q(I,J)+R(I))
  1/(H(I,J)*(PQ(I,J)+MU*MU*T(I,J)))
28 F(I,J,28)=SQRT(Q(I,J)-UA(I,J))*U(I,J)*P(I)*0.5*SQRT(Q(I,J)-R(I))
  1/(X(I,J)*(PQ(I,J)+MU*MU*U(I,J)))
29 F(I,J,29)=SQRT(Q(I,J)+UA(I,J))*U(I,J)*P(I)*0.5*SQRT(Q(I,J)+R(I))
  1/(X(I,J)*(PQ(I,J)+MU*MU*U(I,J)))
30 F(I,J,30)=SQRT(Q(I,J)-UB(I,J))*T(I,J)*P(I)*0.5*SQRT(Q(I,J)-R(I))
  1/(H(I,J)*(PQ(I,J)+MU*MU*T(I,J)))
31 F(I,J,31)=SQRT(Q(I,J)+UB(I,J))*T(I,J)*P(I)*0.5*SQRT(Q(I,J)+R(I))
  1/(H(I,J)*(PQ(I,J)+MU*MU*T(I,J)))
35 CONTINUE
   DO50 K=1,31
   VALUE(K)=0
   NM1=N-1
   NM2=N-2
   DO 100 I=2,NM1,2
100 VALUE(K)=VALUE(K)+F(I,1,K)+F(I,N,K)
   DO101 J=2,NM1,2
   DO102 I=2,NM1
102 VALUE(K)=VALUE(K)+2.0*F(I,J,K)
101 VALUE(K)=VALUE(K)+F(1,J,K)+F(N,J,K)
   DO 103 J=3,NM2,2
   DO103 I=2,NM1,2
103 VALUE(K)=VALUE(K)+2.0*F(I,J,K)
 50 CONTINUE
   WOO=VO*W*(VALUE(4)+VALUE(5)+VALUE(7)+VALUE(8)+VALUE(10)+VALUE(11)+
  1VALUE(13)+VALUE(14)+VALUE(16)+VALUE(17)+VALUE(19)+VALUE(20))
   ZOO=VO*BEE*(VALUE(8)+VALUE(5)+VALUE(12)*1.732+VALUE(18)*1.732-
  1VALUE(7)-VALUE(4)+0.5*(VALUE(11)+VALUE(14)+VALUE(17)+VALUE(20))
  2-0.5*(VALUE(10)+VALUE(13)+VALUE(16)+VALUE(19))
```

```
     3-VALUE(15)*1.732-VALUE(21)*1.732)
      YOO=VALUE(1)+VALUE(2)
      QOO=(2.0*(VALUE(16)+VALUE(17)+VALUE(19)+VALUE(20))-
     1(VALUE(4)+VALUE(5)))/(9.0*MU*MU)
      RAD=SQRT(QOO/YOO)
      BE=1.5*EPSP-(WOO+ZOO)/YOO
      EESQ=1.437
      COEN=1.437*80.0*(VALUE(22)+VALUE(23))/YOO
      PRINT60,BE,COEN
   60 FORMAT(2E20.10)
      PRINT 61, RAD
   61 FORMAT(F6.4)
      GO TO 70
   40 STOP
      END
```

```
C       THE DEUTERON FUNCTION OF CHAPTER FIVE
C       DETERMINATION OF THE WAVEFUNCTION PARAMETERS , XSTAT AND XVEL
        PROGRAM DEU(INPUT,OUTPUT)
        PI =3.141592
   40   READ 41,A,B,C
        PRINT 41,A,B,C
   41   FORMAT(F6.4,F5.3,F3.1)
        IF(A.LT.0.1) GO TO 60
        C=-C
C        C IS ALWAYS TAKEN TO BE 1.0
        D=2.0
        E=1.25
        ANORM=PI*(1.0/(2.0*A)**3+(C*C)/(2.0*B)**3+(2.0*C)/(A+B)**3)
        AKE=41.47*PI*0.5*(1.0/A+(C*C)/B+(8.0*C)/(A+B)-(8.0*C*(A*A+B*B))/
       1(A+B)**3)
        PRINT 3,ANORM,AKE
    3   FORMAT(2F15.6)
        XVEL=0.3
    9   VOVEL=82.94*XVEL
        VVEL=-VOVEL*4.0*PI*((2.0*(A+D)**2+2.0*A*A)/(2.0*A+D)**3
       1-2.0/(2.0*A+D)+2.0*C*(A*A+B*B+(B+D)**2+(A+D)**2)/(A+B+D)**3
       2-4.0*C/(A+B+D)-(2.0*C*C/(2.0*B+D))+(2.0*C*C)*((B+D)**2+B*B)/
       3(2.0*B+D)**3)
        PRINT 4,VVEL
    4   FORMAT(F15.6)
        XSTAT=1.0
   10   VOSTAT=100.0*XSTAT
        VSTAT=-VOSTAT*PI*(1.0/(A+E/2.0)**3+(2.0*C)/((A+B+E)/2.0)**3+
       1(C*C)/(B+E/2.0)**3)
        PRINT 5,VSTAT
    5   FORMAT(F15.6)
        BE=(AKE+VVEL+VSTAT)/ANORM
        PRINT 6, BE
    6   FORMAT(F10.5)
        XSTAT=XSTAT+0.1
        IF(XSTAT-2.5)10,10,11
   11   XVEL=XVEL+0.1
        IF(XVEL-0.8)9,9,4
C       OUR RESULTS ARE GIVEN IN FIGURE 1
   60   STOP
        END
```

```
C       CALCULATIONS WITH THE CORRELATED GAUSSIAN FUNCTION OF CHAPTER 5
        PROGRAM ROSGAU(INPUT,OUTPUT)
        DIMENSION FF(13,1),Q(13)
        COMMON PI
        N=1
      1 READ 2,A,B,C
      2 FORMAT(2F6.4,F3.1)
        PRINT 2,A,B,C
        IF (A) 40,40,3
      3 AB=A+B
        A=A+A
        B=B+B
        PI=3.14159
        C=-C
        C1=C
        C2=C*C
        C3=C2*C
        C4=C3*C
        C5=C4*C
        C6=C5*C
        ANORM=AI(A,A,A)+3.0*C2*AI(A,A,B)+6.0*C1*AI(A,A,AB)+3.0*C4*AI(B,B,A
       1)+C6*AI(B,B,B)+6.0*C5*AI(B,B,AB)+12.0*C2*AI(AB,AB,A)+12.0*C4*AI(
       2AB,AB,B)+8.0*C3*AI(AB,AB,AB)+12.0*C3*AI(A,B,AB)
        DO 21 I=1,13
        Q(I)=I
        FF(I,N)=F(A,A,A,Q(I))+C2*F(A,A,B,Q(I))+2.0*C1*F(A,A,AB,Q(I))
       1+2.0*C2*F(A,B,A,Q(I))+2.0*C4*F(A,B,B,Q(I))+4.0*C3*F(A,B,AB,Q(I))+
       24.0*C1*F(A,AB,A,Q(I))+4.0*C3*F(A,AB,B,Q(I))+8.0*C2*F(A,AB,AB,Q(I))
       3+C6*F(B,B,B,Q(I))+C4*F(B,B,A,Q(I))+2.0*C5*F(B,B,AB,Q(I))+4.0*C3*
       4F(B,AB,A,Q(I))+4.0*C5*F(B,AB,B,Q(I))+8.0*C4*F(B,AB,AB,Q(I))
       5+8.0*C3*F(AB,AB,AB,Q(I))+4.0*C2*F(AB,AB,A,Q(I))+4.0*C4*F(AB,AB,B,
       6Q(I))
        FF(I,N)=FF(I,N)/(ANORM*8.0*PI*PI)
        FF(I,N)=FF(I,N)*FF(I,N)
        PRINT6, FF(I,N),Q(I)
      6 FORMAT(2F15.10)
     21 CONTINUE
     11 AR23=2.0*(A2(A,A,A)+C2*A2(A,A,B)+2.0*C1*A2(A,A,AB)+2.0*C2*A2(A,B
       1,A)+2.0*C4*A2(A,B,B)+4.0*C3*A2(A,B,AB)+4.0*C1*A2(A,AB,A)+4.0*C3*
       2A2(A,AB,B)+8.0*C2*A2(A,AB,AB)+C6*A2(B,B,B)+C4*A2(B,B,A)+2.0*C5*
       3A2(B,B,AB)+4.0*C3*A2(B,AB,A)+4.0*C5*A2(B,AB,B)+8.0*C4*A2(B,AB,AB)
       4+8.0*C3*A2(AB,AB,AB)+4.0*C2*A2(AB,AB,A)+4.0*C4*A2(AB,AB,B))/9.0
        RADS=1.5*AR23/ANORM
        RAD=SQRT(RADS)
        PRINT 201, RAD
    201 FORMAT(F15.5)
        RADM=ABS(RAD-1.66)
        IF(RADM.GT.0.005) GO TO 1
        ACF=1.437*(A1(A,A,A)+2.0*C2*A1(A,A,B)+4.0*C1*A1(A,A,AB)+C4*A1(A,B
       1,B)+4.0*C3*A1(A,B,AB)+4.0*C2*A1(A,AB,AB)+C6*A1(B,B,B)+2.0*C4*A1
       2(B,B,A)+4.0*C5*A1(B,B,AB)+C2*A1(B,A,A)+4.0*C3*A1(B,A,AB)+4.0*C4*
       3A1(B,AB,AB)+8.0*C3*A1(AB,AB,AB)+8.0*C2*A1(AB,AB,A)+8.0*C4*A1(AB,
       4AB,B)+2.0*C1*A1(AB,A,A)+4.0*C3*A1(AB,A,B)+2.0*C5*A1(AB,B,B))
        ACF=ACF/ANORM
        PRINT 202, ACF
    202 FORMAT(F10.5)
     36 READ 37,VO,AMU,WO,BETA
     37 FORMAT(4F7.3)
     69 AL=A+AMU
        BE=B+AMU
```

```
      ABG=AB+AMU
      AAI=(A/2.0-B/2.0)**2
      AAII=A/2.0+B/2.0
      VSTAT=-VO*(AI(AL,A,A)+2.0*C2*AI(AL,A,B)+4.0*C1*AI(AL,A,AB)+4.0*
     1C3*AI(AL,B,AB)+C4*AI(AL,B,B)+4.0*C2*AI(AL,AB,AB)+C2*AI(BE,A,A)
     2+2.0*C4*AI(BE,A,B)+4.0*C3*AI(BE,A,AB)+4.0*C5*AI(BE,B,AB)+C6*AI(
     3BE,B,B)+4.0*C4*AI(BE,AB,AB)+2.0*C1*AI(ABG,A,A)+4.0*C3*AI(ABG,A,B)
     4+8.0*C2*AI(ABG,A,AB)+8.0*C4*AI(ABG,B,AB)+2.0*C5*AI(ABG,B,B)
     5+8.0*C3*AI(ABG,AB,AB))
      VSTAT=VSTAT/ANORM
      VSTAT=-VSTAT
      VSAR12=-VO*(A4(AL,A,A)+2.0*C2*A4(AL,A,B)+4.0*C1*A4(AL,A,AB)+C4*
     1A4(AL,B,B)+4.0*C3*A4(AL,B,AB)+4.0*C2*A4(AL,AB,AB)+C2*A4(BE,A,A)
     2+2.0*C4*A4(BE,A,B)+4.0*C3*A4(BE,A,AB)+C6*A4(BE,B,B)+4.0*C5*A4(BE,
     3B,AB)+4.0*C4*A4(BE,AB,AB)+2.0*C1*A4(ABG,A,A)+4.0*C3*A4(ABG,A,B)
     4+8.0*C2*A4(ABG,A,B)+2.0*C5*A4(ABG,B,B)+8.0*C4*A4(ABG,B,AB)+8.0*
     5C3*A4(ABG,AB,AB))
      VSAR12=VSAR12/ANORM
      VSAR12=-VSAR12/41.47
      PRINT 66,VSAR12,VSTAT
   66 FORMAT(2F15.8)
      AVO=ABS(VO)
      IF(AVO.LE.22.0) GO TO 39
      BL1=B+BETA
      AB1=AB+BETA
      WOOO=WO*(AI(AL1,A,A)+2.0*C2*AI(AL1,A,B)+4.0*C1*AI(AL1,A,AB)+4.0*
     1C3*AI(AL1,B,AB)+C4*AI(AL1,B,B)+4.0*C2*AI(AL1,AB,AB)+C2*AI(BL1,A,
     2A)+2.0*C4*AI(BL1,A,B)+4.0*C3*AI(BL1,A,AB)+4.0*C5*AI(BL1,B,AB)+C6*
     3AI(BL1,B,B)+4.0*C4*AI(BL1,AB,AB)+2.0*C1*AI(AB1,A,A)+4.0*C3*AI(AB1,
     4A,B)+8.0*C2*AI(AB1,A,AB)+8.0*C4*AI(AB1,B,AB)+2.0*C5*AI(AB1,B,B)
     5+8.0*C3*AI(AB1,AB,AB))
      WOOO=WOOO/ANORM
      VROS1=WO*(3.0*A*AI(A,A,AL1)+6.0*C1*AAII*AI(A,A,AB1)+3.0*B*C2*
     1AI(A,A,BL1)+6.0*A*C2*AI(A,B,AL1)+12.0*AAII*C3*AI(A,B,AB1)+6.0*B*
     2C4*AI(A,B,BL1)+12.0*A*C1*AI(A,AB,AL1)+24.0*C2*AAII*AI(A,AB,AB1)
     3+12.0*C3*B*AI(A,AB,BL1)+3.0*C4*A*AI(B,B,AL1)+12.0*C5*AAII*AI
     4(B,B,AB1)+3.0*C6*B*AI(B,B,BL1)+12.0*C3*A*AI(B,AB,AL1)+24.0*C4*
     5AAII*AI(B,AB,AB1)+12.0*C5*B*AI(B,AB,BL1)+12.0*A*C2*AI(AB,AB,AL1)
     6+24.0*C3*AAII*AI(AB,AB,AB1)+12.0*C4*B*AI(AB,AB,BL1)-4.0*C1*
     7AAI*A2(A,A,AB1)-16.0*C2*AAI*A2(A,AB,AB1)-8.0*C3*AAI*A2(A,B,AB1)
     8-16.0*C3*AAI*A2(AB,AB,AB1)-16.0*C4*AAI*A2(AB,B,AB1)-4.0*C5*
     9AAI*A2(B,B,AB1))
      VROS1=VROS1/ANORM
      VROS2=WO*(3.0*A*AI(A,AL1,A)+3.0*C2*B*AI(A,AL1,B)+9.0*A*C1*AI(
     1A,AL1,AB)+3.0*B*C1*AI(A,AL1,AB)+6.0*C1*A*AI(A,AB1,A)+12.0*C3*AA
     2II*AI(A,AB1,B)+12.0*C2*AAII*AI(A,AB1,AB)+3.0*C2*A*AI(A,BL1,A)
     3+6.0*C4*AAII*AI(A,BL1,B)+(B+AAII)*C3*6.0*AI(AB,AL1,B)+12.0*C2*AAII
     4*AI(AB,AL1,AB)+(AAII+A)*6.0*C3*AI(A,BL1,AB)+12.0*C2*A*AI(AB,AB1,A)
     5+12.0*C4*B*AI(AB,AB1,B)+24.0*C3*AAII*AI(AB,AB1,AB)+(B+AAII*2.0)
     6*6.0*C5*AI(AB,BL1,B)+12.0*C4*AAII*AI(AB,BL1,AB)+3.0*C2*A*AI(B,AL1,
     7A)+3.0*C4*B*AI(B,AL1,B)+6.0*C5*B*AI(B,AB1,B)+12.0*C4*AAII*AI(B,AB1
     8,AB)+3.0*C6*B*AI(B,BL1,B)-4.0*C1*AAI*A2(A,AL1,AB)-8.0*C2*AAI*A2
     9(A,AB1,AB)-4.0*C3*AAI*A2(A,BL1,AB)-8.0*C2*AAI*A2(AB,AL1,AB))
      VROS7=WO*(-16.0*C3*AAI*A2(AB,AB1,AB)-8.0*C4*AAI*A2(AB,BL1,AB)
     1-4.0*C3*AAI*A2(B,AL1,AB)-8.0*C4*AAI*A2(B,AB1,AB)-4.0*C5*AAI*
     2A2(B,BL1,AB))
      VROS2=VROS2+VROS7
      VROS2=VROS2/ANORM
      VROS3=-3.0*BETA*WOOO
      AL=AL1
```

```
      BE=BL1
      ABG=AB1
      VSAR21=-VO*(A4(AL,A,A)+2.0*C2*A4(AL,A,B)+4.0*C1*A4(AL,A,AB)+C4*
     1A4(AL,B,B)+4.0*C3*A4(AL,B,AB)+4.0*C2*A4(AL,AB,AB)+C2*A4(BE,A,A)
     2+2.0*C4*A4(BE,A,B)+4.0*C3*A4(BE,A,AB)+C6*A4(BE,B,B)+4.0*C5*A4(BE,
     3B,AB)+4.0*C4*A4(BE,AB,AB)+2.0*C1*A4(ABG,A,A)+4.0*C3*A4(ABG,A,B)
     4+8.0*C2*A4(ABG,A,B)+2.0*C5*A4(ABG,B,B)+8.0*C4*A4(ABG,B,AB)+8.0*
     5C3*A4(ABG,AB,AB))
      VROS4=2.0*BETA*BETA*WO*VSAR21/(-VO)
      VROS4=VROS4/ANORM
      VROS3=VROS3+VROS4
      VVEL=3.0*(VROS1+0.5*VROS2)-3.0*VROS3
       PRINT 63,VVEL
   63 FORMAT(F15.8)
      WOOO=WOOO/41.47
      PRINT 64, WOOO
   64 FORMAT(F10.6)
      GO TO 36
   39 A=A/2.0
      B=B/2.0
      AKE=T(A,A,A,A,A,A)+3.0*C2*T(A,A,B,A,A,B)+3.0*C4*T(A,B,B,A,B,B)
     1+C6*T(B,B,B,B,B,B)+6.0*C1*T(A,A,A,A,A,B)+12.0*C2*T(A,A,A,A,B,B)
     2+8.0*C3*T(A,A,A,B,B,B)+12.0*C3*T(A,B,A,A,B,B)+6.0*C5*T(B,B,B,B,B,
     3A)+12.0*C4*T(B,B,B,B,A,A)
      AKE=AKE/ANORM
      PRINT 61,VSTAT,AKE,VROS1,VROS2,VROS3,VO
   61 FORMAT(6F12.5)
      PRINT 62, A,B,C
   62 FORMAT(3F10.5)
      GO TO 1
C     THE B.E. OF THE TRITON IS OBTAINED BY ADDING 3/2 THE AVERAGE
C     VALUES OF VSTAT AND VVEL TO THE K.E.
C     THE CONTRIBUTIONS OF THE STATIC AND VELOCITY DEPENDENT PARTS OF
C     THE POTENTIAL TO THE INTEGRATED CROSS-SECTION IS GIVEN BY THE
C     AVERAGE OF THE TWO VALUES OF VSAR12 AND  WOOO
   40 STOP
      END
      FUNCTION AI(C,D,E)
      COMMON PI
      AI=PI/((C*D+D*E+E*C)*SQRT(C*D+D*E+E*C)*8.0)
      RETURN
      END
      FUNCTION F(X,Y,Z,QUE)
      COMMON PI
      FI=QUE*(X+Y+4.0*Z)
      FII=4.0*(X*Y+Y*Z+X*Z)
      F=8.0*PI*PI*PI*EXP(-FI/(9.0*FII))/(FII*SQRT(FII))
      RETURN
      END
      FUNCTION T(AO,BO,CO,DO,EO,FO)
      COMMON PI
      AOO=AO+DO
      BOO=BO+EO
      COO=CO+FO
      T1=AO*DO*(BOO+COO)*2.0
      T2=BO*EO*(AOO+COO)*2.0
      T3=CO*FO*(AOO+BOO)*2.0
      T4=AOO*(BO*FO+CO*EO)
      T5=BOO*(AO*FO+CO*DO)
      T6=COO*(AO*EO+BO*DO)
```

```
T7=A00*B00+A00*C00+B00*C00
T=3.0*41.47*(T1+T2+T3+T4+T5+T6)*PI/(T7*T7*SQRT(T7)*8.0)
RETURN
END
FUNCTION A1(C1,D1,E1)
COMMON PI
A11=4.0*(D1*E1+C1*D1+C1*E1)
A12=SQRT(D1+E1)
A1=SQRT(PI)/(A11*A12)
RETURN
END
FUNCTION A2(C2,D2,E2)
COMMON PI
A21=C2*D2+D2*E2+E2*C2
A22=C2+D2
A2=3.0*PI*A22/(16.0*A21*A21*SQRT(A21))
RETURN
END
FUNCTION A4(C4,D4,E4)
COMMON PI
A41=C4*D4+D4*E4+E4*C4
A42=D4+E4
A4=3.0*PI*A42/(16.0*A41*A41*SQRT(A41))
RETURN
END
```

```
C       CALCULATIONS WITH THE CORRELATED EXPONENTIAL FUNCTION (2) OF CHAP.
C       5
        PROGRAM TRION(INPUT,OUTPUT)
        COMMON AQ
      1 READ2,A,B,C
      2 FORMAT(F6.4,F5.3,F3.1)
        IF(A.LE.0.1) GO TO 40
      3 AB=A+B
        A=A+A
        B=B+B
        PI=3.14159
        C=-C
        C1=C
        C2=C*C
        C3=C2*C
        C4=C3*C
        C5=C4*C
        C6=C5*C
        ANORM=AI(A,A,A)+3.0*C2*AI(A,A,B)+6.0*C1*AI(A,A,AB)+3.0*C4*AI(B,B,A
       1)+C6*AI(B,B,B)+6.0*C5*AI(B,B,AB)+12.0*C2*AI(AB,AB,A)+12.0*C4*AI(
       2AB,AB,B)+8.0*C3*AI(AB,AB,AB)+12.0*C3*AI(A,B,AB)
        DO 21 I=1,10
        AQ=I
     17 FF=FI(A,A,A)+2.0*C2*FI(A,B,A)+4.0*C1*FI(A,A,AB)+C4*FI(A,B,B)
       1+4.0*C3*FI(A,B,AB)+4.0*C2*FI(A,AB,AB)+C2*FI(B,A,A)+2.0*C4*
       2FI(B,A,B)+4.0*C3*FI(B,A,AB)+C6*FI(B,B,B)+4.0*C5*FI(B,B,AB)
       3+4.0*C4*FI(B,AB,AB)+2.0*C1*FI(AB,A,A)+4.0*C3*FI(AB,A,B)
       4+8.0*C2*FI(AB,A,AB)+2.0*C5*FI(AB,B,B)+8.0*C4*FI(AB,B,AB)+8.0*C3
       5*FI(AB,AB,AB)
        FF=FF/(8.0*PI*PI*ANORM)
        PRINT 55,FF
     55 FORMAT(F10.6)
     21 CONTINUE
     11 AR23=2.0*(A2(A,A,A)+C2*A2(A,A,B)+2.0*C1*A2(A,A,AB)+2.0*C2*A2(A,B
       1,A)+2.0*C4*A2(A,B,B)+4.0*C3*A2(A,B,AB)+4.0*C1*A2(A,AB,A)+4.0*C3*
       2A2(A,AB,B)+8.0*C2*A2(A,AB,AB)+C6*A2(B,B,B)+C4*A2(B,B,A)+2.0*C5*
       3A2(B,B,AB)+4.0*C3*A2(B,AB,A)+4.0*C5*A2(B,AB,B)+8.0*C4*A2(B,AB,AB)
       4+8.0*C3*A2(AB,AB,AB)+4.0*C2*A2(AB,AB,A)+4.0*C4*A2(AB,AB,B))/9.0
        RADS=1.5*AR23/ANORM
        RAD=SQRT(RADS)
        PRINT 201,RAD
    201 FORMAT(F15.5)
        RADM=ABS(RAD-1.70)
        IF(RADM.GT.0.005) GO TO 1
        ACE=1.437*(A1(A,A,A)+2.0*C2*A1(A,A,B)+4.0*C1*A1(A,A,AB)+C4*A1(A,B
       1,B)+4.0*C3*A1(A,B,AB)+4.0*C2*A1(A,AB,AB)+C6*A1(B,B,B)+2.0*C4*A1
       2(B,B,A)+4.0*C5*A1(B,B,AB)+C2*A1(B,A,A)+4.0*C3*A1(B,A,AB)+4.0*C4*
       3A1(B,AB,AB)+8.0*C3*A1(AB,AB,AB)+8.0*C2*A1(AB,AB,A)+8.0*C4*A1(AB,
       4AB,B)+2.0*C1*A1(AB,A,A)+4.0*C3*A1(AB,A,B)+2.0*C5*A1(AB,B,B))
        ACE=ACE/ANORM
        PRINT 202, ACE
    202 FORMAT(F10.5)
     40 STOP
        END
        FUNCTION AI(C,D,E)
        AI1=8.0*C*(C+D)*(C+E)
        AI2=8.0*D*(D+E)*(D+C)
        AI3=8.0*E*(E+C)*(E+D)
        AI4=16.0*(C+D)*(D+E)*(E+C)
        AI5=((C+D)*(D+E)*(E+C))**3
```

```
      AI=(AI1+AI2+AI3+AI4)/AI5
      RETURN
      END
      FUNCTION A1(C1,D1,F1)
       A11=4.0*(D1+E1)*(C1+D1+E1)
      A12=4.0*(C1+D1)*(C1+E1)
      A13=(D1+E1)**3.0*(C1+D1)**2.0*(C1+E1)**2.0
      A1=(A11+A12)/A13
      RETURN
      END
       FUNCTION A2(C2,D2,E2)
      A21=96.0*(C2+D2+E2)/((D2+E2)**5.0*(C2+E2)**2.0*(C2+D2)**2.0)
      A22=96.0*(C2+D2+E2)/((C2+E2)**5.0*(C2+D2)**2.0*(D2+E2)**2.0)
      A23=72.0*(2.0*E2+D2+C2)/((D2+E2)**4.0*(C2+E2)**4.0*(C2+D2))
      A24=24.0*1.0/((D2+E2)**4.0*(C2+E2)*(C2+D2)**3.0)
      A25=24.0/((D2+E2)*(C2+E2)**4.0*(C2+D2)**3.0)
      A26=24.0*(3.0*D2+3.0*C2+2.0*E2)/((D2+E2)*(E2+C2)*(C2+D2))**3.0
      A2=A21+A22+A23+A24+A25+A26
      RETURN
      END
      FUNCTION FI(A,B,C)
      COMMON AQ
C     THE UNIVERSAL FUNCTION FI IS EVALUATED USING SIMPSONS RULE
    4 N=60
      V=0
      VI=0
      NM1=N-1
      NM2=N-2
      DO 105 I=1,NM1,2
  105 V=V+X(A,B,C,I,AQ)
      DO 106 I=2,NM2,2
  106 VI=VI+X(A,B,C,I,AQ)
      H=0.05
      FI=(H*(4.0*V+2.0*VI+X(A,B,C,N,AQ))/3.0)*3.14159*A*B*C*8192
      RETURN
      END
      FUNCTION X(C,D,E,I,G)
      N=60
      AQ=G
      AQS=SQRT(AQ)
      ASQ=C*C
      BSQ=D*D
      CSQ=E*E
      M=20
      AI=I
      SQI=(AI/M)*(AI/M)
      F1=SQI/(SQI+BSQ)**2
      F2=1.0/(2.0*SQI+4.0*ASQ-2.0*CSQ-4.0*AQ/9.0)**2
      F3=(4.0*AQS*AI*(2.0*SQI+4.0*ASQ-2.0*CSQ-4.0*AQ/9.0))/(3.0*M)
      F4=(SQI+CSQ+4.0*AQ/9.0-4.0*AQS*AI/(3.0*M))/(SQI+CSQ+4.0*AQ/9.0
     1+4.0*AQS*AI/(3.0*M))
      F5=(4.0*SQI+4.0*ASQ+4.0*AQ/9.0+8.0*AQS*AI
     1/(3.0*M))/(4.0*SQI+4.0*ASQ+4.0*AQ/9.0-8.0*AQS*AI/(3.0*M))
      F6=(4.0*SQI+4.0*ASQ+4.0*AQ/9.0)**2
      F7=(8.0*AQS*AI/(3.0*M))**2
      F8=(SQI+CSQ+4.0*AQ/9.0)**2
      F9=(4.0*AQS*AI/(3.0*M))**2
      Y=SQRT(F6/F7)-SQRT(F8/F9)
      Y=ABS(Y)
      IF(Y.LT.0.04) GO TO 11
```

```
      X=F1*F2*(ALOG(F4*F5)/F3+2.0/(F6-F7)+0.5/(F8-F9))
      RETURN
   11 H1SQ=F7*F9
      H2=-SQRT(F6/F7)
      H3=-SQRT(F8/F9)
      XA=-1.0/(3.0*(H3+1.0)**3)
      XB=-1.0/(3.0*(H3-1.0)**3)
      DO 51 J=2,15
      XJ=J
      JM1=J-1
      JA2=J+2
      XA=XA+(XJ/JA2)*(-1.0)**J*(H2-H3)**JM1/(H3+1.0)**JA2
      XB=XB+(XJ/JA2)*(-1.0)**J*(H2-H3)**JM1/(H3-1.0)**JA2
   51 CONTINUE
      X=F1*(XA-XB)/(4.0*H1SQ)
      RETURN
      END




C     THE B.E. AND PHOTODISINTEGRATION COMPUTATIONS
C     SRIVASTAVA'S VELOCITY DEPENDENT CALCULATIONS FOR B.E.WITH EXP S C
    1 READ 2,VO,AMU,WO,BETA
    2 FORMAT(4F7.3)
      IF (VO) 40,40,6
    6 READ 7, A,B,C
    7 FORMAT(F6.4,F5.3,F3.1)
      IF (A) 1,1,5
      AL1=A+BETA
      BL1=B+BETA
      AB1=AB+BETA
      ALO=(0.5*A+BETA)**2.0
      BLO=(0.5*B+BETA)**2.0
      AAA=AI(A,A,AL1)*(ALO+(A*A)/4.0)
      AAAB=-AI(A,A,AB1)*(ALO+BLO+(A*A+B*B)/4.0)      *(-C1)
      AAB=AI(A,A,BL1)*(BLO+(B*B)/4.0)       *C2
      ABA=AI(A,B,AL1)*(2.0*ALO+(A*A)/2.0)          *C2
      ABAB=-AI(A,B,AB1)*(2.0*(ALO+BLO)+(A*A+B*B)/2.0)        *(-C3)
      ABB=AI(A,B,BL1)*(2.0*BLO+(B*B)/2.0)         *C4
      AABA=-AI(A,AB,AL1)*(4.0*ALO+A*A)          *(-C1)
      AABAB=AI(A,AB,AB1)*(4.0*(ALO+BLO)+(A*A+A*B*B/A))       *C2
      AABB=-AI(A,AB,BL1)*(4.0*BLO+B*B)          *(-C3)
      BBA=AI(B,B,AL1)*(ALO+(A*A)/4.0)       *C4
      BBAB=-AI(B,B,AB1)*(ALO+BLO+(A*A+B*B)/4.0)       *(-C5)
      BBB=AI(B,B,BL1)*(BLO+(B*B)/4.0)           *C6
      BABA=-AI(B,AB,AL1)*(4.0*ALO+A*A)          *(C3*(-1.0))
      BABAB=AI(B,AB,AB1)*(4.0*(ALO+BLO)+A*A+B*B)       *C4
      BABB=-AI(B,AB,BL1)*(4.0*BLO+B*B)      *(-C5)
      ABABA=AI(AB,AB,AL1)*(4.0*ALO+A*A)         *C2
      ABABAB=-AI(AB,AB,AB1)*(4.0*(ALO+BLO)+A*A+B*B)        *(-C3)
      ABABB=AI(AB,AB,BL1)*(4.0*BLO+B*B)       *(C4)
      AAA1=-A1(A,A,AL1)*(2.0*SQRT(ALO)+A)
      AAAB1=A1(A,A,AB1)*2.0*(SQRT(ALO)+SQRT(BLO)+AB)          *(-C1)
      AAB1=-A1(A,A,BL1)*(2.0*SQRT(BLO)+B)       *C2
      ABA1=-A1(A,B,AL1)*(4.0*SQRT(ALO)+2.0*A)          *C2
      ABAB1=A1(A,B,AB1)*4.0*(SQRT(ALO)+SQRT(BLO)+AB)         *(-C3)
      ABB1=-A1(A,B,BL1)*(4.0*SQRT(BLO)+2.0*B)        *C4
      AABA1=A1(A,AB,AL1)*(8.0*SQRT(ALO)+4.0*A)        *(-C1)
```

```
      AABAB1=-A1(A,AB,AB1)*8.0*(SQRT(ALO)+SQRT(BLO)+AB)              *C2
      AABB1=A1(A,AB,BL1)*(8.0*SQRT(BLO)+4.0*B)      *(-C3)
      BBA1=-A1(B,B,AL1)*(2.0*SQRT(ALO)+A)        *C4
      BBAB1=A1(B,B,AB1)*2.0*(SQRT(ALO)+SQRT(BLO)+AB)      *(-C5)
      BBB1=-A1(B,B,BL1)*(2.0*SQRT(BLO)+B)        *C6
      BABA1=A1(B,AB,AL1)*(8.0*SQRT(ALO)+4.0*A)       *(-C3)
      BABAB1=-A1(B,AB,AB1)*8.0*(SQRT(ALO)+SQRT(BLO)+AB)     *C4
      BABB1=A1(B,AB,BL1)*(8.0*SQRT(BLO)+4.0*B)      *(-C5)
      ABABA1=-A1(AB,AB,AL1)*(8.0*SQRT(ALO)+4.0*A)      *C2
      ABABAB1=A1(AB,AB,AB1)*8.0*(SQRT(ALO)+SQRT(BLO)+AB)   *(-C3)
      ABABB1=-A1(AB,AB,BL1)*(8.0*SQRT(BLO)+4.0*B)      *C4
      VVEL =AAA+AAAB+AAB+ABA+ABAB+ABB+AABA+AABAB+AABB+BBA+BBAB
     1+BBB+BABA+BABAB+ABABA+ABABAB+ABABB+BABB
     2+AAA1+AAAB1+AAB1+ABA1+ABAB1+ABB1+AABA1+AABAB1+AABB1+BBA1+BBAB1+
     3BBB1+BABA1+BABAB1+BABB1+ABABA1+ABABAB1+ABABB1
      VVEL=VVEL/ANORM
      VVEL=-VVEL*WO
      VSAR12=-VO*(A4(AL,A,A)+2.0*C2*A4(AL,A,B)+4.0*C1*A4(AL,A,AB)+C4*
     1A4(AL,B,B)+4.0*C3*A4(AL,B,AB)+4.0*C2*A4(AL,AB,AB)+C2*A4(BE,A,A)
     2+2.0*C4*A4(BE,A,B)+4.0*C3*A4(BE,A,AB)+C6*A4(BE,B,B)+4.0*C5*A4(BE,
     3B,AB)+4.0*C4*A4(BE,AB,AB)+2.0*C1*A4(ABG,A,A)+4.0*C3*A4(ABG,A,B)
     4+8.0*C2*A4(ABG,A,B)+2.0*C5*A4(ABG,B,B)+8.0*C4*A4(ABG,B,AB)+8.0*
     5C3*A4(ABG,AB,AB))
      VSAR12=VSAR12/ANORM
      VSAR12=VSAR12/41.47
      WOOO=WO*(AI(AL1,A,A)+2.0*C2*AI(AL1,A,B)+4.0*C1*AI(AL1,A,AB)+4.0*
     1C3*AI(AL1,B,AB)+C4*AI(AL1,B,B)+4.0*C2*AI(AL1,AB,AB)+C2*AI(BL1,A,
     2A)+2.0*C4*AI(BL1,A,B)+4.0*C3*AI(BL1,A,AB)+4.0*C5*AI(BL1,B,AB)+C6*
     3AI(BL1,B,B)+4.0*C4*AI(BL1,AB,AB)+2.0*C1*AI(AB1,A,A)+4.0*C3*AI(AB1,
     4A,B)+8.0*C2*AI(AB1,A,AB)+8.0*C4*AI(AB1,B,AB)+2.0*C5*AI(AB1,B,B)
     5+8.0*C3*AI(AB1,AB,AB))
      WOOO=WOOO/ANORM
      WOOO=WOOO/41.47
      A=A/2.0
      B=B/2.0
      AKE=T(A,A,A,A,A,A)+3.0*C2*T(A,A,B,A,A,B)+3.0*C4*T(A,B,B,A,B,B)
     1+C6*T(B,B,B,B,B,B)+6.0*C1*T(A,A,A,A,A,B)+12.0*C2*T(A,A,A,A,B,B)
     2+8.0*C3*T(A,A,A,B,B,B)+12.0*C3*T(A,B,A,A,B,B)+6.0*C5*T(B,B,B,B,B,
     3A)+12.0*C4*T(B,B,B,B,A,A)
      AKE=AKE/ANORM
      PRINT 7,A,B,C
      PRINT300,VO,AMU,VSTAT,VVEL,AKE,WO,BETA
  300 FORMAT(F7.3,F6.2,F9.3,F15.3,F18.3,F9.3,F7.2)
      PRINT 31,VSAR12,WOOO
   31 FORMAT(F9.4,F12.4)



C     THE EXPECTATION VALUES OF THE POTENTIAL OPERATORS IN THE
C     TRITON ARE GIVEN BY 3 TIMES THE EFFECTIVE VALUES


C     THE VSTATIC AND VVEL CONTRIBUTIONS TO THE INTEGRATED CROSS-SECTION
C     ARE THEIR EFFECTIVE VALUES

      GO TO 6
      FUNCTION T(AO,BO,CO,DO,EO,FO)
      AOO=AO+DO
      BOO=BO+EO
      COO=CO+FO
```

```
      T1=41.47*(AO*DO+BO*EO+CO*FO)*AI(AOO,BOO,COO)
      T2=10.37*(BO*FO+CO*EO)*AJ(BOO,COO,AOO)
      T3=10.37*(CO*DO+AO*FO)*AJ(COO,AOO,BOO)
      T4=10.37*(AO*EO+BO*DO)*AJ(AOO,BOO,COO)
      T=T1+T2+T3+T4
      RETURN
      END
      FUNCTION AJ(B1,B2,B3)
      TJ1=16.0*(B1*B1*B1+B2*B2*B2)*B3*B3
      TJ2=16.0*(B1*B1+B2*B2)*(4.0*B3*B3*B3+B1*B2*B3)
      TJ3=16.0*(B1+B2)*(4.0*B3*B3*B3*B3+7.0*B1*B2*B3*B3)
      TJ4=B3*B3*B3*B3*B3+3.0*B1*B1*B2*B2*B3+10.0*B1*B2*B3*B3*B3
      TJ4=TJ4*16.0
      TJ5=(B1+B2)**3.0*(B2+B3)**4.0*(B3+B1)**4.0
      AJ=(TJ1+TJ2+TJ3+TJ4)/TJ5
      RETURN
      END



C     ROSATI'S VEL. DEPT. CALCULATIONS
      PROGRAM ROSATI(INPUT,OUTPUT)
    1 READ 2 ,VO,AMU,WO,BETA
    2 FORMAT(4F7.3)
      IF (VO) 40,40,6
    6 READ 7, A,B,C
    7 FORMAT(F6.4,F5.3,F3.1)
      PRINT 7,A,B,C
      IF (A) 1,1,5
    5 AB=A+B
      AB1=AB+BETA
      A=A+A
      AL1=A+BETA
      B=B+B
      BL1=B+BETA
      C=-C
      C1=C
      C2=C*C
      C3=C2*C
      C4=C3*C
      C5=C4*C
      C6=C5*C
      ANORM=AI(A,A,A)+3.0*C2*AI(A,A,B)+6.0*C1*AI(A,A,AB)+3.0*C4*AI(B,B,A
     1)+C6*AI(B,B,B)+6.0*C5*AI(B,B,AB)+12.0*C2*AI(AB,AB,A)+12.0*C4*AI(
     2AB,AB,B)+8.0*C3*AI(AB,AB,AB)+12.0*C3*AI(A,B,AB)
      AAI=(A/2.0-B/2.0)**2
      AAII=(A/2.0+B/2.0)
      X=-C1*AAI
      VROS1=WO*(X*AI(A,A,AB1)+2.0*X*C2*AI(A,B,AB1)+4.0*C1*X*AI(A,AB,AB1)
     1+4.0*X*C3*AI(B,AB,AB1)+C4*X*AI(B,B,AB1)+4.0*X*C2*AI(AB,AB,AB1)
     2+A*A1(A,A,AL1)+2.0*A*C2*A1(A,B,AL1)+4.0*A*C1*A1(A,AB,AL1)+4.0*A*
     3C3*A1(B,AB,AL1)+A*C4*A1(B,B,AL1)+4.0*A*C2*A1(AB,AB,AL1)+2.0*AAII
     4*C1*A1(A,A,AB1)+4.0*C3*AAII*A1(A,B,AB1)+4.0*AAII*C2*A1(A,AB,AB1)
     5+4.0*AAII*C4*A1(B,AB,AB1)+2.0*AAII*C5*A1(B,B,AB1)+8.0*AAII*C3*
     6A1(AB,AB,AB1)+B*C2*A1(A,A,BL1)+2.0*B*C4*A1(A,B,BL1)+4.0*B*C3*A1(
     7A,AB,BL1)+4.0*B*C5*A1(B,AB,BL1)+B*C6*A1(B,B,BL1)+4.0*B*C4*A1(A
     8B,AB,BL1))
      VROS1=VROS1/ANORM
      VROS2=WO*(X*AI(A,AL1,AB)+X*C2*AI(A,BL1,AB)+2.0*X*C1*AI(A,AB1,AB)
```

```
     1+X*C2*AI(B,AL1,AB)+X*C4*AI(B,BL1,AB)+2.0*X*C3*AI(B,AB1,AB)
     2+2.0*X*C1*AI(AB,AL1,AB)+2.0*X*C3*AI(AB,BL1,AB)+4.0*X*C2*AI(AB,
     3AB1,AB)+A*A1(A,AL1,A)+A*C2*A1(A,BL1,A)+2.0*A*C1*A1(A,AB1,A)+A*C2
     4*A1(B,AL1,A)+A*C4*A1(B,BL1,A)+2.0*A*C3*A1(B,AB1,A)+2.0*A*C1*A1(
     5AB,AL1,A)+2.0*C3*A*A1(AB,BL1,A)+4.0*A*C2*A1(AB,AB1,A)+2.0*C1*AAII*
     6A1(A,AL1,AB)+2.0*C3*AAII*A1(A,BL1,AB)+4.0*C2*AAII*A1(A,AB1,AB)
     7+2.0*C3*AAII*A1(B,AL1,AB)+2.0*C5*AAII*A1(B,BL1,AB)+4.0*C4*AAII*
     8A1(B,AB1,AB)+4.0*C2*AAII*A1(AB,AL1,AB)+4.0*C4*AAII*A1(AB,BL1,AB)
     9+8.0*C3*AAII*A1(AB,AB1,AB)+B*C2*A1(A,AL1,B)+B*C4*A1(A,BL1,B))
      VROS6=WO*(2.0*C3*B*A1(A,AB1,B)+B*C4*A1(B,AL1,B)+B*C6*A1(B,BL1,B)
     1+2.0*B*C5*A1(B,AB1,B)+2.0*B*C3*A1(AB,AL1,B)+2.0*B*C5*A1(AB,BL1,B)
     2+4.0*B*C4*A1(AB,AB1,B))
      VROS2=VROS2+VROS6
      VROS2=VROS2/ANORM
      VROS3=WO*BETA*(A1(A,A,AL1)+2.0*C1*A1(A,A,AB1)+C2*A1(A,A,BL1)
     1+2.0*C2*A1(A,B,AL1)+C3*4.0*A1(A,B,AB1)+2.0*C4*A1(A,B,BL1)
     2+4.0*C1*A1(A,AB,AL1)+8.0*C2*A1(A,AB,AB1)+4.0*C3*A1(A,AB,BL1)
     3+C4*A1(B,B,AL1)+C5*2.0*A1(B,B,AB1)+C6*A1(B,B,BL1)+4.0*C3*A1(B,
     4AB,AL1)+8.0*C4*A1(B,AB,AB1)+4.0*C5*A1(B,AB,BL1)+4.0*C2*A1(AB,AB,AL
     51)+8.0*C3*A1(AB,AB,AB1)+4.0*C4*A1(AB,AB,BL1))
      VROS4=-WO*BETA*BETA*0.5*(AI(AL1,A,A)+2.0*C2*AI(AL1,A,B)+4.0*C1*
     1AI(AL1,A,AB)+4.0*C3*AI(AL1,B,AB)+C4*AI(AL1,B,B)+4.0*C2*AI(AL1,AB,
     2AB)+C2*AI(BL1,A,A)+2.0*C4*AI(BL1,A,B)+4.0*C3*AI(BL1,A,AB)+4.0*C5*
     3AI(BL1,B,AB)+C6*AI(BL1,B,B)+4.0*C4*AI(BL1,AB,AB)+2.0*C1*AI(AB1,A,
     4A)+4.0*C3*AI(AB1,A,B)+8.0*C2*AI(AB1,A,AB)+8.0*C4*AI(AB1,B,AB)
     6+2.0*C5*AI(AB1,B,B)+8.0*C3*AI(AB1,AB,AB))
      VROS3=VROS3+VROS4
      VROS3=VROS3/ANORM
      PRINT 250,VROS1,VROS2,VROS3
  250 FORMAT(F8.3,2F12.3)
      VROS=VROS1+VROS3+0.5*VROS2
      PRINT 251, VROS
  251 FORMAT(F10.6)


C     THE VELOCITY DEPENDENT CONTRIBUTION TO THE B.E. OF THE TRITON
C     USING ROSATI S INTERPRETATION OF P IS 3 TIMES THE AVERAGE OF THE
C     VALUES OF VROS

      GO TO 6
   40 STOP
      END
```

```
C      CALCULATIONS WITH THE EXPONENTIAL FUNCTION (3) IN CHAPTER 5
       PROGRAM TRION(INPUT,OUTPUT)
        DIMENSION FF(8),Q(8),QS(8)
       READ 1,A
     1 FORMAT(F6.4)
    61 B=0.7
     3 AB=A+B
       C=1.0
       A=A+A
       B=B+B
       C=-C
       C1=C
       C2=C1*C
       C3=C2*C
       C4=C3*C
        C5=C4*C
       C6=C5*C
     5 ANORM=AI(A,A,A)+3.0*C2*AI(A,A,B)+6.0*C1*AI(A,A,AB)+3.0*C4*AI(B,B,A
      1)+C6*AI(B,B,B)+6.0*C5*AI(B,B,AB)+12.0*C2*AI(AB,AB,A)+12.0*C4*AI(
      2AB,AB,B)+8.0*C3*AI(AB,AB,AB)+12.0*C3*AI(A,B,AB)
    11 AR23=2.0*(A2(A,A,A)+C2*A2(A,A,B)+2.0*C1*A2(A,A,AB)+2.0*C2*A2(A,B
      1,A)+2.0*C4*A2(A,B,B)+4.0*C3*A2(A,B,AB)+4.0*C1*A2(A,AB,A)+4.0*C3*
      2A2(A,AB,B)+8.0*C2*A2(A,AB,AB)+C6*A2(B,B,B)+C4*A2(B,B,A)+2.0*C5*
      3A2(B,B,AB)+4.0*C3*A2(B,AB,A)+4.0*C5*A2(B,AB,B)+8.0*C4*A2(B,AB,AB)
      4+8.0*C3*A2(AB,AB,AB)+4.0*C2*A2(AB,AB,A)+4.0*C4*A2(AB,AB,B))/9.0
       RADS=1.5*AR23/ANORM
       IF(RADS.LT.0.0) GO TO 51
       RAD=SQRT(RADS)
       PRINT 201,RAD
   201 FORMAT(F15.5)
       RADM=ABS(RAD-1.70)
       IF(RADM.GT.0.005) GO TO 51
       DO 21 I=1,8
       TI=I
       QS(I)=SQRT(TI)/3.
       IF(2.0*QS(I)*QS(I)+A*A+A*A-2.0*B*B.EQ.0.0000) GO TO 51
       IF(2.0*QS(I)*QS(I)+AB*AB+AB*AB-2.0*B*B.EQ.0.000) GO TO 51
       IF(2.0*QS(I)*QS(I)+A*A+A*A-2.0*AB*AB.EQ.0.000) GO TO 51
       IF(2.0*QS(I)*QS(I)+A*A+B*B-2.0*B*B.EQ.0.000) GO TO 51
       IF(2.0*QS(I)*QS(I)+A*A+B*B-2.0*AB*AB.EQ.0.000) GO TO 51
       IF(2.0*QS(I)*QS(I)+A*A+AB*AB-2.0*AB*AB.EQ.0.000) GO TO 51
       IF(2.0*QS(I)*QS(I)+B*B+A*A-2.0*B*B.EQ.0.000) GO TO 51
       IF(2.0*QS(I)*QS(I)+AB*AB+A*A-2.0*B*B.EQ.0.000) GO TO 51
       IF(2.0*QS(I)*QS(I)+AB*AB+B*B-2.0*B*B.EQ.0.000) GO TO 51
       FF(I)=FI(A,A,A,QS(I))+C2*FI(A,A,B,QS(I))+2.0*C1*FI(A,A,AB,QS(I))
      1+2.0*C2*FI(A,B,A,QS(I))+2.0*C4*FI(A,B,B,QS(I))+4.0*C3*FI(A,B,AB,
      2QS(I))+4.0*C1*FI(A,AB,A,QS(I))+4.0*C3*FI(A,AB,B,QS(I))+8.0*C2
      3*FI(A,AB,AB,QS(I))+C6*FI(B,B,B,QS(I))+C4*FI(B,B,A,QS(I))+2.0*C5*
      4FI(B,B,AB,QS(I))+4.0*C3*FI(B,AB,A,QS(I))+4.0*C5*FI(B,AB,B,QS(I))
      5+8.0*C4*FI(B,AB,AB,QS(I))+8.0*C3*FI(AB,AB,AB,QS(I))+4.0*C2*FI(
      6AB,AB,A,QS(I))+4.0*C4*FI(AB,AB,B,QS(I))
       FF(I)=FF(I)/ANORM
       PRINT 99,FF(I)
    99 FORMAT(F10.6)
    21 CONTINUE
    51 B=B/2.0
       A=A/2.0
       B=B+0.025
       IF(B.LE.3.00) GO TO 3
    41 A=A+0.005
```

```
      IF(A.LE.0.29) GO TO 61
   40 STOP
      END
      FUNCTION AI(A,B,C)
      AI=2.0/((A+B)*(B+C)*(C+A))
      RETURN
      END
      FUNCTION A2(A,B,C)
      A2=2.0*(-2.0*1.0/(C**2+B*C+A*C+A*B)**2+2.0*(2.0*C+A+B)**2/(C*C+B
     1*C+A*C+A*B)**3)/(A+B)
      RETURN
      END
      FUNCTION FI(A,B,C,D)
      IF(A.EQ.B) 1,2
    1 QA1=1.571
      QB1=1.571
      GO TO 3
    2 QA1=ATAN(D*(4.0*D*D+A*A+3.0*B*B)/(B*(A*A-B*B)))
      QB1=ATAN(D*(4.0*D*D+3.0*A*A+B*B)/(A*(A*A-B*B)))
    3 IF(D*D+A*A-C*C.EQ.0.000) 4,5
    4 QA2=1.571
      QA3=1.571
      GO TO 6
    5 QA2=ATAN(D*(D*D+A*A+C*C)/(A*(D*D+A*A-C*C)))
      QA3=ATAN(2.0*D*C/(D*D+A*A-C*C))
    6 IF(D*D+B*B-C*C.EQ.0.000)7,8
    7 QB2=1.571
      QB3=1.571
      GO TO 9
    8 QB2=ATAN(D*(D*D+B*B+C*C)/(B*(D*D+B*B-C*C)))
      QB3=ATAN(2.0*D*C/(D*D+B*B-C*C))
    9 FI=(QA1+QA2+QB2-QB1-QB3-QA3)/(D*(2.0*D*D+A*A+B*B-2.0*C*C))
      RETURN
      END



C     THE BINDING ENERGY IS EVALUATED USING ROSATI S INTERPRETATION OF P
      PROGRAM TRIBE(INPUT,OUTPUT)
    1 READ 1, A,B,C
    3 AB=A+B
      C=1.0
      A=A+A
      B=B+B
      C=-C
      C1=C
      C2=C1*C
      C3=C2*C
      C4=C3*C
       C5=C4*C
      C6=C5*C
    5 ANORM=AI(A,A,A)+3.0*C2*AI(A,A,B)+6.0*C1*AI(A,A,AB)+3.0*C4*AI(B,B,A
     1)+C6*AI(B,B,B)+6.0*C5*AI(B,B,AB)+12.0*C2*AI(AB,AB,A)+12.0*C4*AI(
     2AB,AB,B)+8.0*C3*AI(AB,AB,AB)+12.0*C3*AI(A,B,AB)
       ACE=1.437*(A1(A,A,A)+2.0*C2*A1(A,A,B)+4.0*C1*A1(A,A,AB)+C4*A1(A,B
     1,B)+4.0*C3*A1(A,B,AB)+4.0*C2*A1(A,AB,AB)+C6*A1(B,B,B)+2.0*C4*A1
     2(B,B,A)+4.0*C5*A1(B,B,AB)+C2*A1(B,A,A)+4.0*C3*A1(B,A,AB)+4.0*C4*
     3A1(B,AB,AB)+8.0*C3*A1(AB,AB,AB)+8.0*C2*A1(AB,AB,A)+8.0*C4*A1(AB,
     4AB,B)+2.0*C1*A1(AB,A,A)+4.0*C3*A1(AB,A,B)+2.0*C5*A1(AB,B,B))
```

```
      ACE=ACE/ANORM
      PRINT 202, ACE
  202 FORMAT(F10.5)
      VO=0.5*(1.0+1.52)*100.0
      AMU=1.25
      AL=A+AMU
      BE=B+AMU
      ABG=AB+AMU
      VSTAT=-VO*(AI(AL,A,A)+2.0*C2*AI(AL,A,B)+4.0*C1*AI(AL,A,AB)+4.0*
     1C3*AI(AL,B,AB)+C4*AI(AL,B,B)+4.0*C2*AI(AL,AB,AB)+C2*AI(BE,A,A)
     2+2.0*C4*AI(BE,A,B)+4.0*C3*AI(BE,A,AB)+4.0*C5*AI(BE,B,AB)+C6*AI(
     3BE,B,B)+4.0*C4*AI(BE,AB,AB)+2.0*C1*AI(ABG,A,A)+4.0*C3*AI(ABG,A,B)
     4+8.0*C2*AI(ABG,A,AB)+8.0*C4*AI(ABG,B,AB)+2.0*C5*AI(ABG,B,B)
     5+8.0*C3*AI(ABG,AB,AB))
      VSTAT=VSTAT/ANORM
      VSTAT=3.0*VSTAT
      WO=82.94/2.0
      BBE=2.80
   39 AU=A+BBE
      BU=B+BBE
      ABU=AB+BBE
      AE=AU
      ABE=ABU
      BE=BU
      W1=((A-B)**2/4.0)*(AI(ABU,A,A)+AI(ABU,B,B)+4.0*AI(ABU,AB,AB)
     1+2.0*AI(ABU,A,B)-4.0*AI(ABU,A,AB)-4.0*AI(ABU,B,AB))
      W1=W1+A*(A1(AU,A,A)+A1(AU,B,B)+4.0*A1(AU,AB,AB)+2.0*A1(AU,A,B)
     1-4.0*A1(AU,A,AB)-4.0*A1(AU,B,AB))
      W1=W1+B*(A1(BU,A,A)+A1(BU,B,B)+4.0*A1(BU,AB,AB)+2.0*A1(BU,A,B)
     1-4.0*A1(BU,A,AB)-4.0*A1(BU,B,AB))
      W1=W1-2.0*AB*(A1(ABU,A,A)+A1(ABU,B,B)+4.0*A1(ABU,AB,AB)+2.0*A1(A
     1BU,A,B)-4.0*A1(ABU,A,AB)-4.0*A1(ABU,B,AB))
      W1=W1+0.5*(A5(AU,A,A)+A5(AU,B,B)+4.0*A5(AU,AB,AB)+2.0*A5(AU,A,B)
     1-4.0*A5(AU,A,AB)-4.0*A5(AU,B,AB))
      W1=W1-(A5(ABU,A,A)+A5(ABU,B,B)+4.0*A5(ABU,AB,AB)+2.0*A5(ABU,A,B)
     1-4.0*A5(ABU,A,AB)-4.0*A5(ABU,B,AB))
      W1=W1+0.5*(A5(BU,A,A)+A5(BU,B,B)+4.0*A5(BU,AB,AB)+2.0*A5(BU,A,B)
     1-4.0*A5(BU,A,AB)-4.0*A5(BU,B,AB))
      W2=((A-B)**2/4.0)*(AI(AB,AU,A)+AI(AB,BU,B)+4.0*AI(AB,ABU,AB)
     1+2.0*AI(AB,AU,B)-4.0*AI(AB,AU,AB)-4.0*AI(AB,BU,AB))
      W2=W2+A*(A1(A,AU,A)+A1(A,BU,B)+4.0*A1(A,ABU,AB)+2.0*A1(A,AU,B)
     1-4.0*A1(A,AU,AB)-4.0*A1(A,BU,AB))
      W2=W2+B*(A1(B,AU,A)+A1(B,BU,B)+4.0*A1(B,ABU,AB)+2.0*A1(B,AU,B)
     1-4.0*A1(B,AU,AB)-4.0*A1(B,BU,AB))
      W2=W2-2.0*AB*(A1(AB,AU,A)+A1(AB,BU,B)+4.0*A1(AB,ABU,AB)+2.0*A1(A
     1B,AU,B)-4.0*A1(AB,AU,AB)-4.0*A1(AB,BU,AB))
      W2=W2+0.5*(A5(A,AU,A)+A5(A,BU,B)+4.0*A5(A,ABU,AB)+2.0*A5(A,AU,B)
     1-4.0*A5(A,AU,AB)-4.0*A5(A,BU,AB))
      W2=W2-(A5(AB,AU,A)+A5(AB,BU,B)+4.0*A5(AB,ABU,AB)+2.0*A5(AB,AU,B)
     1-4.0*A5(AB,AU,AB)-4.0*A5(AB,BU,AB))
      W2=W2+0.5*(A5(B,AU,A)+A5(B,BU,B)+4.0*A5(B,ABU,AB)+2.0*A5(B,AU,B)
     1-4.0*A5(B,AU,AB)-4.0*A5(B,BU,AB))
      WD=AI(AE,A,A)+2.0*C2*AI(AE,A,B)+4.0*C1*AI(AE,A,AB)+4.0*
     1C3*AI(AE,B,AB)+C4*AI(AE,B,B)+4.0*C2*AI(AE,AB,AB)+C2*AI(BE,A,A)
     2+2.0*C4*AI(BE,A,B)+4.0*C3*AI(BE,A,AB)+4.0*C5*AI(BE,B,AB)+C6*AI(
     3BE,B,B)+4.0*C4*AI(BE,AB,AB)+2.0*C1*AI(ABE,A,A)+4.0*C3*AI(ABE,A,B)
     4+8.0*C2*AI(ABE,A,AB)+8.0*C4*AI(ABE,B,AB)+2.0*C5*AI(ABE,B,B)
     5+8.0*C3*AI(ABE,AB,AB)
      WS=A1(AE,A,A)+2.0*C2*A1(AE,A,B)+4.0*C1*A1(AE,A,AB)+C4*A1(AE,B,B)
     1+4.0*C3*A1(AE,B,AB)+4.0*C2*A1(AE,AB,AB)+C6*A1(BE,B,B)+2.0*C4*A1(BE
```

```
     2,B,A)+4.0*C5*A1(BE,B,AB)+C2*A1(BE,A,A)+4.0*C3*A1(BE,A,AB)
     3+4.0*C4*A1(BE,AB,AB)+8.0*C3*A1(ABE,AB,AB)+8.0*C2*A1(ABE,AB,A)+8.0*
     4C4*A1(ABE,AB,B)+2.0*C1*A1(ABE,A,A)+4.0*C3*A1(ABE,A,B)+2.0*C5*A1(
     5ABE,B,B)
      W1=W1*WO*3.0/ANORM
      W2=W2*1.5*WO/ANORM
      WD=1.5*WO*BBE*BBE*WD/ANORM
      WS=-3.0*WO*BBE*WS/ANORM
      VVEL=W1+W2-WD-WS
      AKE=((A-B)**2/4.0)*(AI(AB,A,A)+AI(AB,B,B)+4.0*AI(AB,AB,AB)
     1+2.0*AI(AB,A,B)-4.0*AI(AB,A,AB)-4.0*AI(AB,B,AB))
      AKE=AKE+A*(A1(A,A,A)+A1(A,B,B)+4.0*A1(A,AB,AB)+2.0*A1(A,A,B)
     1-4.0*A1(A,A,AB)-4.0*A1(A,B,AB))
      AKE=AKE+B*(A1(B,A,A)+A1(B,B,B)+4.0*A1(B,AB,AB)+2.0*A1(B,A,B)
     1-4.0*A1(B,A,AB)-4.0*A1(B,B,AB))
      AKE=AKE-2.0*AB*(A1(AB,A,A)+A1(AB,B,B)+4.0*A1(AB,AB,AB)+2.0*A1(A
     1B,A,B)-4.0*A1(AB,A,AB)-4.0*A1(AB,B,AB))
      AKE=AKE+0.5*(A5(A,A,A)+A5(A,B,B)+4.0*A5(A,AB,AB)+2.0*A5(A,A,B)
     1-4.0*A5(A,A,AB)-4.0*A5(A,B,AB))
      AKE=AKE-(A5(AB,A,A)+A5(AB,B,B)+4.0*A5(AB,AB,AB)+2.0*A5(AB,A,B)
     1-4.0*A5(AB,A,AB)-4.0*A5(AB,B,AB))
      AKE=AKE+0.5*(A5(B,A,A)+A5(B,B,B)+4.0*A5(B,AB,AB)+2.0*A5(B,A,B)
     1-4.0*A5(B,A,AB)-4.0*A5(B,B,AB))
      AKE=AKE*1.5*41.47
      AKE=AKE/ANORM
      BE=AKE+VSTAT
       PRINT 41,VSTAT,W1,W2,WD,WS
   41 FORMAT(5F14.6)
      PRINT 42,BE,VVEL
   42 FORMAT(2F20.6)
      WO=WO*0.415
      BBE=2.0
      IF(WO.LT.10.0) GO TO 40
      GO TO 39

C     THE BINDING ENERGY IS GIVEN BY THE SUM OF BE AND THE TWO VALUES OF
C     VVEL

   40 STOP
      END
      FUNCTION A1(A,B,C)
      IF(B-C.EQ.0.000) GO TO 31
      A1=-2.0*(ALOG(A+C)-ALOG(A+B))/((B+C)*(B-C))
      RETURN
   31 A1=2.0/((B+C)*(A+B))
      RETURN
      END
      FUNCTION A5(A,B,C)
      IF(B-C.EQ.0.000) GO TO 5
      A5=2.0*((C+A)*ALOG(C+A)-(B+A)*ALOG(B+A))/((B+C)*(B-C))
      RETURN
    5 A5=-2.0*ALOG(A+B)/(B+C)
      RETURN
      END



C     THE PHOTODISINTEGRATION CROSS-SECTION
      PROGRAM SIGIN(INPUT,OUTPUT)
```

```
    1 READ 1, A,B,C
    3 AB=A+B
      C=1.0
      A=A+A
      B=B+B
      C=-C
      C1=C
      C2=C1*C
      C3=C2*C
      C4=C3*C
       C5=C4*C
      C6=C5*C
    5 ANORM=AI(A,A,A)+3.0*C2*AI(A,A,B)+6.0*C1*AI(A,A,AB)+3.0*C4*AI(B,B,A
     1)+C6*AI(B,B,B)+6.0*C5*AI(B,B,AB)+12.0*C2*AI(AB,AB,A)+12.0*C4*AI(
     2AB,AB,B)+8.0*C3*AI(AB,AB,AB)+12.0*C3*AI(A,B,AB)
      VO=152.0
      AMU=1.25
   32 AL=A+AMU
      BE=B+AMU
      ABG=AB+AMU
      VSAR12=-VO*(A2(AL,A,A)+2.0*C2*A2(AL,A,B)+4.0*C1*A2(AL,A,AB)+C4*
     1A2(AL,B,B)+4.0*C3*A2(AL,B,AB)+4.0*C2*A2(AL,AB,AB)+C2*A2(BE,A,A)
     2+2.0*C4*A2(BE,A,B)+4.0*C3*A2(BE,A,AB)+C6*A2(BE,B,B)+4.0*C5*A2(BE,
     3B,AB)+4.0*C4*A2(BE,AB,AB)+2.0*C1*A2(ABG,A,A)+4.0*C3*A2(ABG ,A,B)
     4+8.0*C2*A2(ABG,A,B)+2.0*C5*A2(ABG,B,B)+8.0*C4*A2(ABG,B,AB)+8.0*
     5C3*A2(ABG,AB,AB))
      VSAR12=VSAR12/ANORM
      VSAR12=VSAR12/41.47
      PRINT 31,VSAR12
   31 FORMAT(F15.5)
      BETA=2.80
      WO=82.94
  121 AL1=A+BETA
      BL1=B+BETA
      AB1=AB+BETA
      WOOO=WO*(AI(AL1,A,A)+2.0*C2*AI(AL1,A,B)+4.0*C1*AI(AL1,A,AB)+4.0*
     1C3*AI(AL1,B,AB)+C4*AI(AL1,B,B)+4.0*C2*AI(AL1,AB,AB)+C2*AI(BL1,A,
     2A)+2.0*C4*AI(BL1,A,B)+4.0*C3*AI(BL1,A,AB)+4.0*C5*AI(BL1,B,AB)+C6*
     3AI(BL1,B,B)+4.0*C4*AI(BL1,AB,AB)+2.0*C1*AI(AB1,A,A)+4.0*C3*AI(AB1,
     4A,B)+8.0*C2*AI(AB1,A,AB)+8.0*C4*AI(AB1,B,AB)+2.0*C5*AI(AB1,B,B)
     5+8.0*C3*AI(AB1,AB,AB))
      WOOO=WOOO/ANORM
      WOOO=WOOO/41.47
      PRINT 122,WOOO
  122 FORMAT(F15.8)
      IF(BETA.LT.2.10) GO TO 40
      BETA=2.0
      WO=0.415*82.94
      GO TO 121

C     THE STATIC CONTRIBUTION TO SIG-IN IS GIVEN BY VSAR12 BUT THE VEL.
C     DEP. PART IS THE SUM OF THE TWO WOOO S

   40 STOP
      END
```

```
C      THE COULOMB ENERGY FOR FINITE NUCLEONS USING OUR CORRELATED
C       EXPONENTIAL FUNCTION
       PROGRAM COUL(INPUT,OUTPUT)
C      CORRECTIONS TO THE COULOMB ENERGY FROM ASSUMING A FINITE
C      NUCLEON
     1 READ 2,A,B,C
     2 FORMAT(F6.4,F5.3,F3.1)
       IF(A)40,40,3
     3 AB=A+B
       A=A+A
       B=B+B
       C=-C
       C1=C
       C2=C*C
       C3=C2*C
       C4=C3*C
       C5=C4*C
       C6=C5*C
     8 READ 7, AMU
     7 FORMAT(F5.3)
       IF(AMU)1,1,6
     6 ACE=1.0
       D=A+AMU
       E=B+AMU
       X=AB+AMU
       ACE=1.437*(A1(D,A,A)+2.0*C2*A1(D,A,B)+4.0*C1*A1(D,A,AB)+C4*A1(D,B
      1,B)+4.0*C3*A1(D,B,AB)+4.0*C2*A1(D,AB,AB)+C6*A1(E,B,B)+2.0*C4*A1
      2(E,B,A)+4.0*C5*A1(E,B,AB)+C2*A1(E,A,A)+4.0*C3*A1(E,A,AB)+4.0*C4*
      3A1(E,AB,AB)+8.0*C3*A1(X,AB,AB)+8.0*C2*A1(X,AB,A)+8.0*C4*A1(X,
      4AB,B)+2.0*C1*A1(X,A,A)+4.0*C3*A1(X,A,B)+2.0*C5*A1(X,B,B))
       ACE=ACE/ANORM
       IF(AMU.EQ.3.360) ACE=ACE*2.776
       IF(AMU.EQ.2.970) ACE=ACE*3.639
C      THE CORRECTION DUE TO THE TERMS IN R-1 IS GIVEN BY THE DIFFERENCE
C      BETWEEN ACE(AMU=2.97) AND ACE(AMU=3.36)
       PRINT 10,A,B,C,AMU,ACE
    10 FORMAT(F6.4,F5.3,F3.1,F5.3,F7.4)
       GO TO 8
    40 STOP
       END
```

```
       PROGRAM TRION(INPUT,OUTPUT)
     1 READ2,VO,AMU
     2 FORMAT(2F5.3)
       IF (VO) 40,40,6
     6 READ 7, A,B,C
     7 FORMAT(F6.4,F5.3,F3.1)
       IF (A) 1,1,5
     5 AB=A+B
       A=A+A
       B=B+B
       C=-C
       C1=C
       C2=C*C
```

```
      C3=C2*C
      C4=C3*C
      C5=C4*C
      C6=C5*C
      ANORM=AI(A,A,A)+3.0*C2*AI(A,A,B)+6.0*C1*AI(A,A,AB)+3.0*C4*AI(B,B,A
     1)+C6*AI(B,B,B)+6.0*C5*AI(B,B,AB)+12.0*C2*AI(AB,AB,A)+12.0*C4*AI(
     2AB,AB,B)+8.0*C3*AI(AB,AB,AB)+12.0*C3*AI(A,B,AB)
      AL=A+AMU
      BE=B+AMU
      ABG=AB+AMU
      VSTAT=-V0*(AI(AL,A,A)+2.0*C2*AI(AL,A,B)+4.0*C1*AI(AL,A,AB)+4.0*
     1C3*AI(AL,B,AB)+C4*AI(AL,B,B)+4.0*C2*AI(AL,AB,AB)+C2*AI(BE,A,A)
     2+2.0*C4*AI(BE,A,B)+4.0*C3*AI(BE,A,AB)+4.0*C5*AI(BE,B,AB)+C6*AI(
     3BE,B,B)+4.0*C4*AI(BE,AB,AB)+2.0*C1*AI(ABG,A,A)+4.0*C3*AI(ABG,A,B)
     4+8.0*C2*AI(ABG,A,AB)+8.0*C4*AI(ABG,B,AB)+2.0*C5*AI(ABG,B,B)
     5+8.0*C3*AI(ABG,AB,AB))
      VSTAT=VSTAT/ANORM
      IF(AMU.EQ.3.36) ACE=VSTAT*0.582
      IF(AMU.EQ.2.97) ACE=VSTAT*0.644
      ACE=-ACE
      PRINT 11,A,B,C,V0,AMU,VSTAT
   11 FORMAT(2F6.4,F4.1,2F5.3,F10.5)
      PRINT 202,ACE
  202 FORMAT(F10.5)
C     THE CONTRIBUTION OF THE TERMS INDEPENDENT OF R IS THE SUM OF THE
C     VALUES OF ACE
      GO TO 6
   40 STOP
      END



C     THE AMOUNT TO BE TAKEN OFF THE VALUE OF C.E. IS THE TOTAL SUM
C     OF THE TWO CORRECTIONS
```

```
C       THE EVALUATION OF BETA IN OUR S PRIME STATE WAVEFUNCTIONS
        PROGRAM SDASH(INPUT,OUTPUT)
        A=0.74
      2 B=0.76
      3 P1=(2.0*A)**3*(A+B)**2*(5.5*A*A+2.0*B*B+6.5*A*B)
        P2=(8.0*A*A+5.0*A*B+B*B)*(1.5*A+0.5*B)**5
        P=2.0*(1.0-P1/P2)
C       P IS THE PROBABILITY OF THE S PRIME STATE
        PRINT5,P,A,B
      5 FORMAT(F10.6,2F5.3)
        B =B+0.005
        IF(B-A.LE.0.40) GO TO 3
C        OUR RESULTS ARE
C       1.0 PERCENT P,        BETA =0.435
C       1.5 PERCENT P,       BETA =0.450
C       2.0 PERCENT P,       BETA =0.465
        STOP
        END
```

```
C       PROGRAM FOR CHAPTER SIX
C       EVALUATION OF F2(Q2) AND THE MUON CAPTURE RATE MATRIX ELEMENTS
        PROGRAM CORR(INPUT,OUTPUT)
        COMMON AQ
      1 READ 2,A,B,C
      2 FORMAT(3F5.3)
        IF(A.EQ.0) GO TO 40
        PI=3.14159
        AB=A+B
        A=A+A
        B=B+B
        PRINT 14,A,B,C
     14 FORMAT(3F10.6)
      5 C=-C
        C1=C
        C2=C*C
        C3=C2*C
        C4=C3*C
        C5=C4*C
        C6=C5*C
         BP=0.870
      3 AP=0.74
C       THE MOMENTUM TRANSFER IN THE MUON CAPTURE CORRESPONDS TO Q2
C       EQUAL TO 0.27 FM-2
        AQ=0.27
        ALA=(A+AP)/2.0
        BBE=(B+BP)/2.0
        ALB=(B+AP)/2.0
        BEA=(A+BP)/2.0
        BA=(AP+BP)/2.0
        ANORMG=(AP+BP)*(AP+BP)*SQRT(AP+BP)*AP*SQRT(AP)/(SQRT(8.0*PI*PI
       1*(8.0*AP*AP+5.0*AP*BP+BP*BP)))
        FV1V1=ANORMG*ANORMG*(FI(AP,AP,BP)+5.0*FI(BP,AP,AP)-2.0*FI(AP,
       1BA,BA)-4.0*FI(BA,BA,AP))/6.0
        FV1V2=ANORMG*ANORMG*(FI(AP,AP,BP)+2.0*FI(BA,BA,AP)-FI(BP,AP,AP)
       1-2.0*FI(AP,BA,BA))/SQRT(12.0)
        FV2V2=ANORMG*ANORMG*(FI(AP,AP,BP)+FI(BP,AP,AP)-2.0*FI(AP,BA,BA)
       1)/2.0
        FV1V1=FV1V1*8.0*PI*PI
        FV1V2=FV1V2*8.0*PI*PI
        FV2V2=FV2V2*8.0*PI*PI
        PRINT 10, FV1V1,FV1V2,FV2V2
     10 FORMAT(3F12.6)
      5 ANORM=AI(A,A,A)+3.0*C2*AI(A,A,B)+6.0*C1*AI(A,A,AB)+3.0*C4*AI(B,B,A
       1)+C6*AI(B,B,B)+6.0*C5*AI(B,B,AB)+12.0*C2*AI(AB,AB,A)+12.0*C4*AI(
       2AB,AB,B)+8.0*C3*AI(AB,AB,AB)+12.0*C3*AI(A,B,AB)
        ANN=SQRT(8.0*PI*PI*ANORM)
        FCOU1=ANN*ANORMG*(FI(ALA,ALA,BEA)+C1*FI(ALA,ALB,BEA)+C1*FI(ALB,A
       1LA,BEA)+C2*FI(ALB,ALB,BEA)+C1*FI(ALA,ALA,BBE)+C2*FI(ALA,ALB,BBE)
       2+C2*FI(ALB,ALA,BBE)+C3*FI(ALB,ALB,BBE))/SQRT(6.0)
        FCOU1=FCOU1+ANN*ANORMG*(FI(BEA,ALA,ALA)+C1*FI(BEA,ALB,ALA)+
       1C1*FI(BBE,ALA,ALA)+C2*FI(BBE,ALB,ALA)+C1*FI(BEA,ALA,ALB)+C2*FI(
       2BEA,ALB,ALB)+C2*FI(BBE,ALA,ALB)+C3*FI(BBE,ALB,ALB))/SQRT(6.0)
        FCOU1=FCOU1-2.0*ANN*ANORMG*(FI(ALA,BEA,ALA)+C1*FI(ALA,BBE,ALA)
       1+C1*FI(ALB,BEA,ALA)+C2*FI(ALB,BBE,ALA)+C1*FI(ALA,BEA,ALB)+C2*FI
       3(ALA,BBE,ALB)+C2*FI(ALB,BEA,ALB)+C3*FI(ALB,BBE,ALB))/SQRT(6.0)
        FCOU1=FCOU1/ANN**2
        FCOU2=ANN*ANORMG*(FI(ALA,ALA,BEA)+C1*FI(ALA,ALB,BEA)+C1*FI
       1(ALB,ALA,BEA)+C2*FI(ALB,ALB,BEA)+C1*FI(ALA,ALA,BBE)+C2*FI(ALA,A
       2LB,BBE)+C2*FI(ALB,ALA,BBE)+C3*FI(ALB,ALB,BBE))/SQRT(2.0)
```

```
      FCOU2=FCOU2-ANN*ANORMG*(FI(BEA,ALA,ALA)+C1*FI(BEA,ALB,ALA)+C1
     1*FI(BBE,ALA,ALA)+C2*FI(BBE,ALB,ALA)+C1*FI(BEA,ALA,ALB)+C2*FI(BEA,
     3ALB,ALB)+C2*FI(BBE,ALA,ALB)+C3*FI(BBE,ALB,ALB))/SQRT(2.0)
      FCOU2=FCOU2/ANN**2
      FCOU1=FCOU1*8.0*PI*PI
      FCOU2=FCOU2*8.0*PI*PI
      PRINT11,FCOU1,FCOU2
      DO21 I=1,10
      AQ=I
      F2CI=-ANN*ANORMG*(FI(BEA,ALA,ALA)-FI(ALA,ALA,BEA)+C1*(FI(BEA,ALA
     1,ALB)+FI(BEA,ALB,ALA)+FI(BBE,ALA,ALA)-FI(ALA,ALB,BEA)-FI(ALB
     2,ALA,BEA)-FI(ALA,ALA,BBE))+C2*(FI(BEA,ALB,ALB)+FI(BBE,ALA,ALB)+
     3FI(BBE,ALB,ALA)-FI(ALB,ALB,BEA)-FI(ALA,ALB,BBE)-FI(ALB,ALA,BBE))
     4+C3*(FI(BBE,ALB,ALB)-FI(ALB,ALB,BBE)))      /SQRT(6.0)
      F2CI=6.0*F2CI/ANN**2
      F2CI=-F2CI
C     IT SHOULD BE REMEMBERED THAT BECAUSE BETA IS LARGER THAN ALPHA
C     THE AMPLITUDE OF THE S PRIME STATE IS OPPOSITE THAT OF THE S STATE
      F2CI=F2CI*8.0*PI*PI
      PRINT20,F2CI
   20 FORMAT(F15.8)
   21 CONTINUE
    6 BP=BP+0.030
      IF(BP.LE.0.930) GO TO 3
   40 STOP
      END
```

```
C     OUR PROGRAM FOR THE R.M.S.RADIUS OF 3H AND 3HE
      PROGRAM RADII(INPUT,OUTPUT)
    1 READ2,A,B,C
    2 FORMAT(3F5.3)
      PRINT2,A,B,C
      IF(A.EQ.0) GO TO 40
      AB=A+B
      A=A+A
      B=B+B
      PI=3.14159
      C=-C
      C1=C
      C2=C1*C
      C3=C2*C
      C4=C3*C
      C5=C4*C
      C6=C5*C
      BP=0.870
    3 AP=0.74
   31 ALA=(A+AP)/2.0
      BBE=(B+BP)/2.0
      ALB=(B+AP)/2.0
      BEA=(A+BP)/2.0
      BA=AP+BP
    5 ANORM=AI(A,A,A)+3.0*C2*AI(A,A,B)+6.0*C1*AI(A,A,AB)+3.0*C4*AI(B,B,A
     1)+C6*AI(B,B,B)+6.0*C5*AI(B,B,AB)+12.0*C2*AI(AB,AB,A)+12.0*C4*AI(
     2AB,AB,B)+8.0*C3*AI(AB,AB,AB)+12.0*C3*AI(A,B,AB)
      ANORMG=BA*BA*SQRT(BA)*AP*SQRT(AP)/(SQRT(8.0*PI*PI*(8.0*AP*AP+5.0
     1*AP*BP+BP*BP)))
      ANN=SQRT(8.0*PI*PI*ANORM)
      RADS=3.0*2.89
      AI1=3.0*RADS/2.0
      BA=(AP+BP)/2.0
      AI11=ANORMG*ANORMG*(2.0*A4(AP,AP,BP)+A4(BP,AP,AP)-2.0*A4(BA,BA,
     1AP)-A4(AP,BA,BA))
      AI11=AI11*8.0*PI*PI
      PRINT 71,AI11
      AI1=AI1+AI11
      PRINT 71,AI1
      PRINT 71,ANN
   71 FORMAT(F15.8)
      AI2=ANN*ANORMG*SQRT(3.0/2.0)*(A4(BEA,ALA,ALA)+C1*A4(BEA,ALA,ALB)
     1+C1*A4(BEA,ALB,ALA)+C2*A4(BEA,ALB,ALB)+C1*A4(BBE,ALA,ALA)
     2+C2*A4(BBE,ALA,ALB)+C2*A4(BBE,ALB,ALA)+C3*A4(BBE,ALB,ALB)
     3-A4(ALA,ALA,BEA)-C1*A4(ALA,ALA,BBE)-C1*A4(ALA,ALB,BEA)-C2*A4(ALA
     4,ALB,BBE)-C1*A4(ALB,ALA,BEA)-C2*A4(ALB,ALA,BBE)-C2*A4(ALB,ALB,BEA)
     5-C3*A4(ALB,ALB,BBE))
      AI2=AI2*8.0*PI*PI
      AI2=AI2/(ANN*ANN)
      PRINT71,AI2
      AI3=AI2
      H3=2.0*AI1/9.0-(AI2+AI3)/9.0
      HE=2.0*AI1/9.0+2.0*(AI2+AI3)/9.0
      3H=SQRT(H3)
      3HE=SQRT(HE)
      PRINT 72, 3H,3HE
   72 FORMAT(2F15.8)
    6 BP=BP+0.03
        IF(BP.LE.0.930) GO TO 3
      GO TO 1
```

```
40 STOP
   END
   FUNCTION A4(C4,D4,E4)
   A41=96.0*(C4+D4+E4)/((C4+D4)**5.0*(C4+E4)**2.0*(D4+E4)**2.0)
     A42=96.0*(C4+D4+E4)/((C4+E4)**5.0*(C4+D4)**2.0*(D4+E4)**2.0)
   A43=72.0*(2.0*C4+D4+E4)/((C4+D4)**4.0*(C4+E4)**4.0*(D4+E4))
   A44=24.0/((C4+D4)**4.0*(C4+E4)*(D4+E4)**3.0)
   A45=24.0/((C4+D4)*(C4+E4)**4.0*(D4+E4)**3.0)
   A46=24.0*(3.0*D4+3.0*E4+2.0*C4)/((C4+D4)*(C4+E4)*(D4+E4))**3.0
   A4=A41+A42+A43+A44+A45+A46
   RETURN
   END
```

```
C       PROGRAM FOR THE COMPUTATION OF G SQUARED AND THE MAXIMUM CROSS-
C       SECTION IN THE REACTION 160(3H,P)180
        PROGRAM OXY(INPUT,OUTPUT)
        A=0.40
        B=2.00
        C=0.4
        AB=A+B
        A=A+A
        B=B+B
        PI=3.14159
        C=-C
        C1=C
        C2=C*C
        C3=C2*C
        C4=C3*C
        C5=C4*C
        C6=C5*C
      5 ANORM=AI(A,A,A)+3.0*C2*AI(A,A,B)+6.0*C1*AI(A,A,AB)+3.0*C4*AI(B,B,A
     1)+C6*AI(B,B,B)+6.0*C5*AI(B,B,AB)+12.0*C2*AI(AB,AB,A)+12.0*C4*AI(
     2AB,AB,B)+8.0*C3*AI(AB,AB,AB)+12.0*C3*AI(A,B,AB)
        ANN=SQRT(ANORM)
        A=A/2.0
        B=B/2.0
        AE=A+1.25
        BE=B+1.25
        G=0.5*(556.0*(AI(AE,A,A)+2.0*C1*AI(AE,A,B)+C1*AI(BE,A,A)+C2*AI
     1(AE,B,B)+2.0*C2*AI(BE,A,B)+C3*AI(BE,B,B))+4.0*41.47*V(A,B,2.80)
     2+4.0*17.21*V(A,B,2.00))/ANN
          G=G*SQRT(8.0*PI*PI)
        G2=G*G
        CSEC=32.5*G2/(31.4*1000000.0)
        PRINT 111, G,G2,CSEC
    111 FORMAT(3F20.5)
        AP=0.74
        BP=0.90
        BA=AP+BP
        ANORMG=BA*BA*SQRT(BA)*AP*SQRT(AP)/(SQRT(8.0*PI*PI*(8.0*AP*AP+5.0
     1*AP*BP+BP*BP)))
        PRINT 42, ANORMG,ANORM
     42 FORMAT(2F15.8)
        AP=AP/2.0
        BP=BP/2.0
        GSP=0.5*ANORMG*SQRT(8.0*PI*PI)*(556.0*(AI(BP+1.25,AP,AP)-AI(
     1AP+1.25,AP,BP))+4.0*(41.47*VS(AP,BP,2.80)+17.21*VS(AP,BP,2.80)))
     2/SQRT(6.0)
        GSP=GSP*SQRT(8.0*PI*PI)
        PRINT 100,GSP
    100 FORMAT(F20.5)
        G=SQRT(0.985)*G-GSP
        G2=G*G
        CSEC=32.5*G2/(31.4*1000000.0)
        PRINT 112,G,G2,CSEC
    112 FORMAT(3F20.5)
        STOP
        END
        FUNCTION V(X,Y,Z)
        C=-0.4
        C1=C
        C2=C1*C
        C3=C2*C
```

```
      V=-(2.0*X*X+2.0*X*Z+Z*Z)*(AI(X+Z,X,X)+2.0*C1*AI(X+Z,X,Y)
     1+C2*AI(X+Z,Y,Y))-(2.0*Y*Y+2.0*Y*Z+Z*Z)*C1*(AI(Y+Z,X,X)+2.0
     3*C1*AI(Y+Z,X,Y)+C2*AI(Y+Z,Y,Y))+(4.0*X+2.0*Z)*(A1(X+Z,X,X)+2.0
     4*C1*A1(X+Z,X,Y)+C2*A1(X+Z,Y,Y))+(4.0*Y+2.0*Z)*(A1(Y+Z,X,X)+2.
     50*C1*A1(Y+Z,X,Y)+C2*A1(Y+Z,Y,Y))*C1
      RETURN
      END
      FUNCTION VS(X,Y,Z)
      VS=-(2.0*Y*Y+2.0*Y*Z+Z*Z)*AI(Y+Z,X,Y)+(4.0*Y+2.0*Z)*A1(Y+Z,X,X)
     4+(2.0*X*X+2.0*X*Z+Z*Z)*AI(X+Z,X,Y)-(4.0*X+2.0*Z)*A1(X+Z,X,Y)
      RETURN
      END
```

```
C       THIS PROGRAM IS USED IN CHAPTER EIGHT
C       THE COMPLETE CALCULATIONS OF THE PROPERTIES OF THE ALPHA-
C       PARTICLE USING THE TWO-PARAMETER IRVING  FUNCTION
        PROGRAM ALPHA(INPUT,OUTPUT)
        DIMENSION Q(20),FORM(20)
        PI=3.14159
        PI=PI*PI*PI*PI
        READ 71, A
     71 FORMAT(F6.4)
     12 B=A+0.01
     11 A1=3.0*PI/(64.0*A**9)
        A2=3.0*PI/(64.0*B**9)
        A3=6.0*PI/(64.0*((A+B)/2)**9)
        ANORM =A1+A2-A3
        PRINT 50,A,B
     50 FORMAT(2F10.5)
        RADS=45.0*PI*(1.0/(512.0*A**11)+1.0/(512.0*B**11)
       1-2.0/(512.0*((A+B)/2)**11))
        RADS=3.0*RADS/(ANORM*4.0)
        RAD=SQRT(RADS)
        IF(RAD.LT.1.39) GO TO 2
        IF(RAD.GT.1.51) GO TO 1
        RAD1=ABS(RAD-1.40)
        RAD2=ABS(RAD-1.45)
        RAD3=ABS(RAD-1.50)
        IF(RAD1.LT.0.005) GO TO 55
        IF(RAD2.LT.0.005 ) GO TO 55
        IF(RAD3.LT.0.005) 55,1
     55 SIGB=1.28*RADS
        PRINT 3,RAD,SIGB
      3 FORMAT(2F10.5)
        DO 21 I=1,10
        Q(I)=I
        FORM(I)=A1/(1.0+(3.0*Q(I))/(64.0*A**2))**5+A2/(1.0+(3.0*Q(I))/
       1(64.0*B**2))**5-A3/(1.0+(3.0*Q(I))/(64.0*((A+B)/2)**2))**5
        FORM(I)=FORM(I)/ANORM
        PRINT 4,FORM(I)
      4 FORMAT(F10.5)
     21 CONTINUE
C       A AND B ARE DETERMINED BY FITS TO THE RADIUS AND FORM FACTOR
        AKE=20.74*(3.0/(16.0*A**7)+3.0/(16.0*B**7)-6.0*A*B/(16.0*((A+B)/2)
       1**9))
        AKE=AKE*PI
        AKE=AKE/ANORM
        POTL=-3.0*197.328*4.0*PI*56.0*30.0*24.0*(VO(A)+VO(B)-2.0*VO
       1((A+B)/2))/(1.156*1.414)**9
        POTL=POTL/ANORM
        BE=AKE+POTL
        PRINT 5,AKE,POTL,BE
      5 FORMAT(3F10.5)
C       THE QUANTITIES ENDING IN S REFER TO THE EXPECTATION VALUES
C       OF THE OPERATORS APPEARING  WHEN WE USE OUR VELOCITY DEPENDENT
C       POTENTIAL
        POTLS=-3.0*252.0*4.0*PI*56.0*30.0*24.0*(WO(A)+WO(B)-2.0*WO((A+B)/2
       1))/(1.414*1.25)**9
        POTLS=POTLS/ANORM
        AK1=1.414*2.80
        AK2=1.414*2.00
        VVEL=(VDEP(A,A,AK1)+VDEP(B,B,AK1)-VDEP(A,B,AK1)-VDEP(B,A,AK1))
       1*82.94+(VDEP(A,A,AK2)+VDEP(B,B,AK2)-VDEP(A,B,AK2)-VDEP(B,A,AK2))
```

```
     2*82.94*0.415
      VVEL=VVEL*3.0/ANORM
      BES=AKE+POTLS+VVEL
      PRINT61,POTLS,VVEL,BES
   61 FORMAT(3F15.10)
      VSQR=-123.33*0.8*8.0*PI*90.0*56.0*30.0*24.0*(VR(A)+VR(B)-2.0*VR
     1 ((A+B)/2))/(1.414*1.156)**11
      VSQR=VSQR/ANORM
      SGIN=60.0*(1.0-0.5*VSQR*4.0/(3.0*41.47))
      VSQR=VSQR*4.0/(3.0*41.47)
      PRINT 8,    VSQR, SGIN
    8 FORMAT(2F10.5)
      COEN=CO(A)+CO(B)-2.0*CO((A+B)/2)
      VSQRS=-126.0*8.0*PI*90.0*56.0*30.0*24.0*(WR(A)+WR(B)-2.0*WR
     1 ((A+B)/2))/(1.414*1.25)**11
      VSQRS=VSQRS/ANORM
      VSQRV=2.0*4.0*PI*56.0*30.0*24.0*(SO(A)+SO(B)-2.0*SO((A+B)/2))/
     1(2.80*1.414)**9
      VSQRV=VSQRV+2.0*4.0*PI*56.0*30.0*24.0*0.415*(TO(A)+TO(B)-2.0*TO
     1((A+B)/2))/(2.0*1.414)**9
      VSQRV=VSQRV/ANORM
      SGINS=60.0*(1.0-0.5*VSQRS*4.0/(3.0*41.47)+VSQRV)
      VSQRS=VSQRS*4.0/(3.0*41.47)
      PRINT 81,VSQRS,VSQRV,SGINS,COEN
   81 FORMAT(4F15.8)
    1 B=B+0.01
      GO TO 11
    2 A=A+0.02
      IF(A.LE.1.8) GO TO 12
      STOP
      END
      FUNCTION CO(Z)
      PI=3.14159
      CO=4.0*PI**4*42.0*20.0/(1.414*(4.0*Z)**8)
      RETURN
      END
      FUNCTION VO(X)
      VO1=64.0*(2.828*X/1.156)**3
      VO2=69.0*(2.828*X/1.156)**2
      VO3=30.0*(2.828*X/1.156)
      VO4=5.0
      VO5=840.0*(2.828*X/1.156)**6*(1.0+(2.828*X)/1.156)**6
      VO=(VO1+VO2+VO3+VO4)/VO5
      RETURN
      END
      FUNCTION WO(X)
      WO1=64.0*(2.828*X/1.25)**3
      WO2=69.0*(2.828*X/1.25)**2
      WO3=30.0*(2.828*X/1.25)
      WO4=5.0
      WO5=840.0*(2.828*X/1.25)**6*(1.0+(2.828*X)/1.25)**6
      WO=(WO1+WO2+WO3+WO4)/WO5
      RETURN
      END
      FUNCTION TO(X)
      TO1=64.0*(2.828*X/2.0)**3
      TO2=69.0*(2.828*X/2.0)**2
      TO3=30.0*(2.828*X/2.0)
      TO4=5.0
      TO5=840.0*(2.828*X/2.0)**6*(1.0+(2.828*X)/2.0)**6
```

```
      TO=(TO1+TO2+TO3+TO4)/TO5
      RETURN
      END
      FUNCTION VR(Y)
      VR1=1.0
      VR2=8.0*(4.0*Y)/(1.414*1.156)
      VR3=25.0*(4.0*Y/(1.414*1.156))**2
      VR4=32.0*(4.0*Y/(1.414*1.156))**3
      VR5=1260.0*(4.0*Y/(1.414*1.156))**6*(1.0+(4.0*Y)/(1.414*1.156))**8
      VR=(VR1+VR2+VR3+VR4)/VR5
      RETURN
      END
      FUNCTION WR(X)
      WR1=1.0
      WR2=8.0*(4.0*X)/(1.414*1.25)
      WR3=25.0*(4.0*X/(1.414*1.25))**2
      WR4=32.0*(4.0*X/(1.414*1.25))**3
      WR5=1260.0*(4.0*X/(1.414*1.25))**6*(1.0+(4.0*X)/(1.414*1.25))**8
      WR=(WR1+WR2+WR3+WR4)/WR5
      RETURN
      END
      FUNCTION SO(X)
      SO1=64.0*(2.828*X/2.80)**3
      SO2=69.0*(2.828*X/2.80)**2
      SO3=30.0*(2.828*X/2.80)
      SO4=5.0
      SO5=840.0*(2.828*X/2.80)**6*(1.0+(2.828*X)/2.80)**6
      SO=(SO1+SO2+SO3+SO4)/SO5
      RETURN
      END
      FUNCTION VDEP(X,Y,AK)
      DIMENSION WM1(50), WHAF(50), WTHR(50)
C       THE THREE FUNCTIONS NOT EVALUATED EXPLICITLY ARE INTEGRATED
C       USING SIMPSONS RULE
      AM=0.02
      SEV=7*6*5*4*3*2
      EIG=SEV*8
      VA=0.0
      VAI=0.0
      VB=0.0
      VBI=0.0
      VC=0.0
      VCI=0.0
      Z=2.0*X+2.0*Y
      ZAK=Z/AK
      DO 105 I=1,49,2
      WM1(I)=EIG*(1.0-(I*AM)**2)**2*(I*AM)**4/(Z+AK*AM*I)**9
      WHAF(I)=EIG*(1.0-(I*AM)**2)**2*(I*AM)**3/(Z+AK*AM*I)**9
      WTHR(I)=SEV*(1.0-(I*AM)**2)**2*(I*AM)**4/(Z+AK*AM*I)**8
      VA=VA+WM1(I)
      VB=VB+WHAF(I)
      VC=VC+WTHR(I)
  105 CONTINUE
      DO 106 I=2,48,2
      WM1(I)=EIG*(1.0-(I*AM)**2)**2*(I*AM)**4/(Z+AK*AM*I)**9
      WHAF(I)=EIG*(1.0-(I*AM)**2)**2*(I*AM)**3/(Z+AK*AM*I)**9
      WTHR(I)=SEV*(1.0-(I*AM)**2)**2*(I*AM)**4/(Z+AK*AM*I)**8
      VAI=VAI+WM1(I)
      VBI=VBI+WHAF(I)
      VCI=VCI+WTHR(I)
```

```
106 CONTINUE
    A=(AM*(4.0*VA+2.0*VAI)/3.0)
    B=(AM*(4.0*VB+2.0*VBI)/3.0)
    C=(AM*(4.0*VC+2.0*VCI)/3.0)
    D=SEV*(8.0*ZAK**2+5.0*ZAK+1.0)/(AK**8*105*ZAK**5*(1.0+ZAK)**5)
    E=SEV*(35.0*ZAK**3+47.0*ZAK**2+25.0*ZAK+5.0)/(AK**8*210.0*ZAK**6*
   1(1.0+ZAK)**5)
    F=EIG*(64.0*ZAK**3+69.0*ZAK**2+30.0*ZAK+5.0)/(AK**9*840.0*ZAK**6*
   1(1.0+ZAK)**6)
    PI=3.14159
    PI=PI*PI*PI*PI
    VDEP=-4.0*PI*(4.0*Y**2*A-6.0*Y*D+2.0*Y*C+(AK**2/2.0)*F+2.0*AK*Y*B
   1-AK*E)
    RETURN
    END
```

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THE S′ STATE IN THE TRINUCLEON

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Abstract: The relative probability ($P_{S'}$) of the mixed-symmetry S-state (the S′ state) in the ground state of the trinucleon, is determined by fitting the charge form factors and the muon capture rate of the three-particle system. The spatial wave functions used in this work are of exponential form with two-body correlations in the predominant totally symmetric S-state and have been proved sufficiently flexible in our recent binding energy calculations. Our results indicate that $P_{S'}$ cannot exceed 1.5 %.

1. Introduction

New experimental data on the high-energy scattering of electrons by ^{3}He and ^{3}H [ref. [1])], and the photodisintegration cross-sections [2]) and muon-capture rate of the three-nucleon system [3,4]), have led to a number of fresh attempts to explain the detailed structure of the ground state of the trinucleon [5–8]). Since the three-particle system is the simplest many-body problem, it presents the best starting point for the study of general nuclear forces and therefore its structure has immense significance. A classification based on symmetry properties of the three-body configuration [4]), predicts that there are likely to be fourteen states present in the ground state. However it is expected that only a small number contribute to any appreciable extent. Thus from measurements of the magnetic moments and the capture of neutrons by deuterium and various other evidence, including variational calculations of the binding energy, we believe these states to be the predominant S-state, which is completely symmetric under spatial exchange of the nucleons, together with small admixtures of the S-state of mixed symmetry (the so-called S′ state) and the D-states (which we label collectively by D). The authors in refs. [2,4–9]) have made estimates of the relative probabilities of the various states by fitting experimental data using consistent sets of wave functions. Unfortunately these estimates span a large range; for example the probability of the S′ state $P_{S'}$ assumes values from 0.5 to 4.0 %. Lately two groups have come to agree among themselves on the magnitude of $P_{S'}$. Thus Schiff and his co-workers take $P_{S'}$ to be 2 % while Mitra *et al.*, Levinger and Srivastava and Davies have obtained a value of about 1 %. Both sets of investigations possess shortcomings however; the variational calculations of Schiff's group are obscured by their use of simple wave functions that do not exhibit the correct asymptotic as well as close-in behaviour for the three-particle system while the separable non-local potential method of Mitra gives rise to values of the trineutron binding energy and the Coulomb energy of ^{3}He which are inconsistent with the general variational method and with experiment [10]). As some of the expectation values are extremely sensitive to the form of the wave function assumed, these doubts on the accuracy of the results of both groups do not allow us to make any firm conclusions about the actual probabilities of the states, hence the motivation for this investigation. Our aim in this work is to obtain a careful estimate of $P_{S'}$, assuming for simplicity that only the S and S′ states are present in the ground state. Insofar as it can affect our conclusions the D-state is also considered qualitatively. As short-range two-body correlations are known to be fairly important in three-body calculations [11]), we use as trial functions the product-form spatial functions from our recent investigation into the binding energy of the trinucleon. These are exponential wave functions suitably modified by short-range correlations. Satisfactory fits to the static properties of the trinucleon make us believe these functions are a good representation of the three-body system. Two sets of experimental data are selected to determine $P_{S'}$; these are the charge form factors and the muon capture rate. We have refrained from fitting the inelastic scattering cross section of electrons from ^{3}He and the two body photodisintegration cross section of the trinucleon since uncertainties about the deuteron wave function and the final-state interactions are likely to cloud the issue.

2.1. THE TRINUCLEON WAVE FUNCTION

Our totally antisymmetric wave function for the ground state of the trinucleon is

$$\Psi = \phi_0 u + (\phi_1 u_2 - \phi_2 u_1), \tag{1}$$

where the spatial functions u, u_1, and u_2 take the form

$$u = A \prod_{i<j=1}^{3} f(r_{ij}), \tag{2}$$

$$u_1 = \frac{1}{\sqrt{6}}[g(12,3) + g(13,2) - 2g(23,1)],$$

$$u_2 = \frac{1}{\sqrt{2}}[g(12,3) - g(13,2)], \tag{3}$$

with

$$f(r_{ij}) = \exp(-ar_{ij}) + c\exp(-br_{ij}), \tag{4}$$

$$g(ij,k) = Bg(ik)g(jk)h(ik), \tag{5}$$

and

$$g(ij) = \exp(-\alpha r_{ij}), \qquad \alpha = 0.37\ \mathrm{fm}^{-1},$$
$$h(ij) = \exp(-\beta r_{ij}). \tag{6}$$

The ϕ are the spin-isospin functions listed in Schiff's paper. The parameters of our fully symmetric function u are

$$a = 0.40\ \mathrm{fm}^{-1}, \qquad b = 2.00\ \mathrm{fm}^{-1}, \qquad c = -0.4, \tag{7}$$

which were used in our binding-energy calculations and were found to make our trial function sufficiently flexible. We have not included correlations in the S′ state functions as they are unlikely to be significant [see the discussion on this point in ref. [8])]. The value assumed for α stems from form-factor and binding-energy fits with unmodified exponentials for u [refs. [7,12])]; $P_{S'}$ determines β. Because our trial functions manifest the correct behaviour for small as well as large interparticle separations, we expect our values of $P_{S'}$ to be more acceptable than those of refs. [5,8]).

2.2. CHARGE FORM FACTORS

The relevant charge form factors are obtained from the three-dimensional Fourier transform of the expectation values of the nuclear charge density and are given by the expressions

$$\begin{aligned} 2F_{ch}(^3\mathrm{He}) &= (F^n_{ch}+2F^p_{ch})F_1+(F^n_{ch}-F^p_{ch})F_2\,,\\ F_{ch}(^3\mathrm{H}) &= (F^p_{ch}+2F^n_{ch})F_1+(F^p_{ch}-F^n_{ch})F_2\,, \end{aligned} \tag{8}$$

where $F_{ch}(^3\mathrm{He})$, $F_{ch}(^3\mathrm{H})$, F^p_{ch} and F^n_{ch} are the charge form factors of ^{3}He, ^{3}H, the proton and the neutron, respectively. In the notation of ref. [8]),

$$\begin{aligned} F_1 &= F(\mathrm{S},\mathrm{S})+F_1(\mathrm{D},\mathrm{D}),\\ F_2 &= F(\mathrm{S},\mathrm{S}')+F_2(\mathrm{D},\mathrm{D}), \end{aligned} \tag{9}$$

which reduce to

$$\begin{aligned} F_1 &= F(\mathrm{S},\mathrm{S}) = F_1(q^2),\\ F_2 &= F(\mathrm{S},\mathrm{S}') = \tfrac{2}{3}F_2(q^2), \end{aligned} \tag{10}$$

since we do not include D-state contributions. Using our wave functions,

$$F_1(q^2) = \int \exp(iq\cdot x_1)A^2 \prod_{i<j=1}^{3} f^2(r_{ij})\mathrm{d}\tau, \tag{11}$$

$$F_2(q^2) = -\frac{1}{\sqrt{6}}\int [\exp(iq\cdot x_1)-\exp(iq\cdot x_3)]AB \prod_{i<j=1}^{3} f(r_{ij})g(12,3)\mathrm{d}\tau, \tag{12}$$

where the volume element

$$\mathrm{d}\tau = 8\pi^2 r_1 r_2 r_3\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3, \qquad r_i = r_{jk}, \tag{13}$$

and x_i is the distance between the centre of mass of the trinucleon and the nucleon i.

The evaluation of these integrals is easily accomplished with the help of the universal function [11])

$$\begin{aligned} F(q^2) &= \int \exp(-\alpha_1 r_1-\alpha_2 r_2-\alpha_3 r_3)\exp(iq\cdot x_1) r_1 r_2 r_3\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r_3\\ &= \frac{1024\alpha_1\alpha_2\alpha_3}{\pi}\int_0^\infty \left[\frac{A(k_1)}{B(k_1)}\ln\{C(k_1)\times D(k_1)\}+\frac{2}{E(k_1)}+\frac{1}{2G(k_1)}\right]\mathrm{d}k_1\,, \end{aligned} \tag{14}$$

with

$$A(k_1) = \frac{k_1^2}{(k_1^2+\alpha_2^2)^2(2k_1^2+4\alpha_3^2-2\alpha_1^2-\frac{4}{9}q^2)^2}, \tag{15}$$

$$B(k_1) = -\tfrac{4}{3}qk_1(2k_1^2+4\alpha_3^2-2\alpha_1^2-\tfrac{4}{9}q^2)^2, \tag{16}$$

$$C(k_1) = \frac{k_1^2+\alpha_1^2+\frac{4}{9}q^2-\frac{4}{3}qk_1}{k_1^2+\alpha_1^2+\frac{4}{9}q^2+\frac{4}{3}qk_1}, \tag{17}$$

$$D(k_1) = \frac{4k_1^2+4\alpha_3^2+\frac{4}{9}q^2+\frac{8}{3}qk_1}{4k_1^2+4\alpha_3^2+\frac{4}{9}q^2-\frac{8}{3}qk_1}, \tag{18}$$

$$E(k_1) = (4k_1^2+4\alpha_3^2+\tfrac{4}{9}q^2)^2-(\tfrac{8}{3}qk_1)^2, \tag{19}$$

$$G(k_1) = (k_1^2+\alpha_1^2-\tfrac{4}{9}q^2)^2-(\tfrac{4}{3}qk_1)^2. \tag{20}$$

Our function u gives an excellent fit to $F_1(q^2)$ [ref. [11])]. In fig. 1, we have plotted the curves for $F_2(q^2)$ for three values of $P_{S'}$. (It is worthwhile mentioning here that for

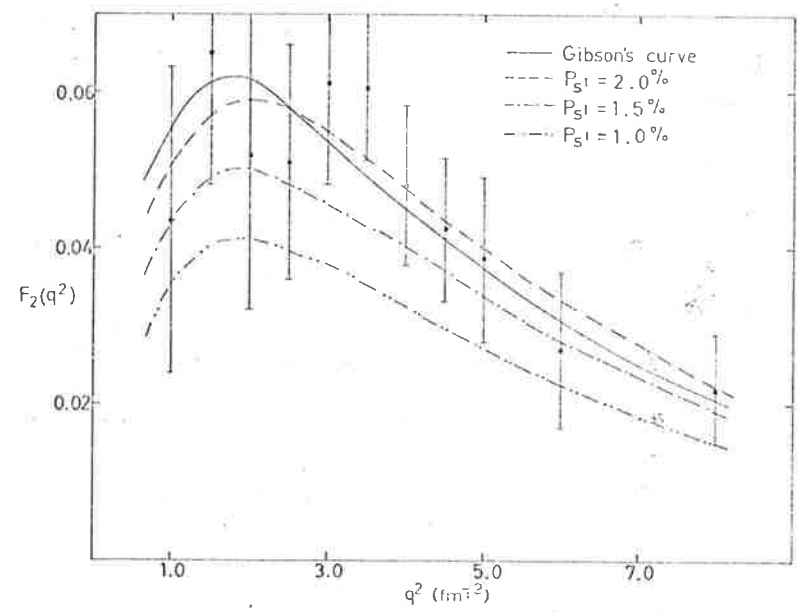

Fig. 1. A plot of $F_2(q^2)$ against the square momentum transfer q^2. The solid curve is that from Gibson's most recent paper [8] whilst the dashed curves are those of this work for the three values of $P_{S'}$.

$F_2(q^2)$ to be positive i.e. in agreement with experiment, β must be larger than α; this means the amplitudes of the S and S′ states must have opposite signs, which occurs in the calculations of Davies too). It is obvious from fig. 1 that the best fit to the experimental data of Collard *et al.* is given by $P_{S'}$ equal to 2 %. However it is known from the work of Gibson that the D-D contribution increases $F_2(q^2)$ [which is the quantity $X(q^2)$ in ref. [8])] for small momentum transfer q^2. Since there are no cross

terms between the S and D states in $F_2(q^2)$ we can use Gibson's result as a reasonable estimate of the contribution of the D-state. Therefore assuming P_D to be about 6 %, the increment in $F_2(q^2)$ due to the presence of the D-state will be of the order of 0.01. This will spoil the fit for $P_{S'}$ equal to 2 % but correspondingly will improve that for the other two values of $P_{S'}$. We feel that we can safely conclude that $P_{S'}$ cannot exceed 1.5 % on this evidence. It appears that the values of $F_2(q^2)$ derived from our correlated functions suggest that $P_{S'}$ is near 1.5 %.

2.3. MUON-CAPTURE RATE

Assuming the V-A theory and neglecting the relativistic effects, the capture rate for the reaction

$$^3\mathrm{He}+\mu^- \rightarrow {}^3\mathrm{H}+\nu$$

can be shown to be

$$\Lambda_\mu = \frac{1}{(2\pi)^3}\left[\frac{Zm_\mu}{137}\right]^3 \frac{\nu^2}{1+\dfrac{|\nu|}{M_{^3\mathrm{H}}}} \int \frac{\mathrm{d}\nu}{|\nu|} \sum_{m^3\mathrm{H}=\pm\frac{1}{2}} \sum_{m^3\mathrm{He}=\pm\frac{1}{2}} \times \left\{ G_\mathrm{V}^2 \left| \int 1 \right|^2 + G_\mathrm{A}^2 \left| \int \sigma \right|^2 + (G_\mathrm{P}^2 - 2G_\mathrm{A} G_\mathrm{P}) \left| \frac{\nu}{|\nu|} \int \sigma \right|^2 \right\}, \tag{21}$$

with

$$\begin{aligned} G_\mathrm{V} &\equiv g_\mathrm{V}\left(1+\frac{|\nu|}{2M}\right)+g_\mathrm{S}, \\ G_\mathrm{A} &\equiv g_\mathrm{A}-(g_\mathrm{V}+g_\mathrm{M})\frac{|\nu|}{2M}, \\ G_\mathrm{P} &\equiv \frac{|\nu|}{2M}(g_\mathrm{P}-g_\mathrm{A}-g_\mathrm{V}-g_\mathrm{M}+g_\mathrm{T}), \\ \int 1 &\equiv \langle \Psi_{^3\mathrm{H}} | \sum_{i=1}^{3} \exp[-i\nu\cdot x_i]\varphi_\mu(x_i)\tau_i^{(-)} | \Psi_{^3\mathrm{He}} \rangle, \\ \int \sigma &\equiv \langle \Psi_{^3\mathrm{H}} | \sum_{i=1}^{3} \exp[-i\nu\cdot x_i]\varphi_\mu(x_i)\tau_i^{(-)}\sigma_i | \Psi_{^3\mathrm{He}} \rangle, \end{aligned} \tag{22}$$

where $|\nu|$ is the neutrino energy (taken to be 102.505 MeV), $\nu/|\nu|$ is the unit vector in the direction of the emitted neutrino, M the nucleon mass, $M_{^3\mathrm{H}}$ the triton mass, m_μ the reduced mass of the muon and $\varphi_\mu(x_i)$ the muon wave function. $\varphi_\mu(x_i)$ is very close to 1.0 over nuclear dimensions and in our calculations is assumed to have that value. Using the relations

$$\langle \Psi_{^3\mathrm{H}} | \sum_{i=1}^{3} \exp[-i\nu\cdot x_i]O_i | \Psi_{^3\mathrm{He}} \rangle = 3\langle \Psi_{^3\mathrm{H}} | \exp[-i\nu\cdot x_i]O_i | \Psi_{^3\mathrm{He}} \rangle, \tag{23}$$

where O_i is 1 or σ_i.

$$\int \frac{\mathrm{d}\nu}{|\nu|} \left| \frac{\nu}{|\nu|} \int \sigma \right|^2 = \frac{1}{3}\int \frac{\mathrm{d}\nu}{|\nu|} \left| \int \sigma \right|^2, \tag{24}$$

and introducing the Gamow-Teller constant

$$\Gamma^2 = G_\mathrm{A}^2 + \tfrac{1}{3}(G_\mathrm{P}^2 - 2G_\mathrm{A}G_\mathrm{P}) \tag{25}$$

our expression for the capture rate becomes

$$\Lambda_\mu = \frac{1}{(2\pi)^3}\left[\frac{Zm_\mu}{137}\right]^3 \frac{\nu^2}{1+\dfrac{|\nu|}{M_{^3\mathrm{H}}}} \frac{9}{2}\int \frac{\mathrm{d}\nu}{|\nu|} \sum_{m^3\mathrm{H}=\pm\frac{1}{2}} \sum_{m^3\mathrm{He}=\pm\frac{1}{2}} \left\{ G_\mathrm{V}^2 \left| \int 1 \right|^2 + \Gamma^2 \left| \int \sigma \right|^2 \right\}. \tag{26}$$

Evaluating this explicitly, Λ_μ is

$$\Lambda_\mu = \frac{1}{(2\pi)^3}\left[\frac{Zm_\mu}{137}\right]^3 \frac{\nu^2}{1+\dfrac{|\nu|}{M_{^3\mathrm{H}}}} \tfrac{9}{2}4\pi[G_\mathrm{V}^2 F_\mathrm{F}^2 + \tfrac{3}{2}\Gamma^2 F_\mathrm{GT}^2], \tag{27}$$

where

$$\begin{aligned} F_\mathrm{F} = {} & -\tfrac{2}{3}P_\mathrm{S}\langle u| \exp[-i\nu\cdot x_1]|u\rangle - \tfrac{1}{3}P_{\mathrm{S}'}(\langle u_1| \exp[-i\nu\cdot x_1]|u_1\rangle \\ & + \langle u_2| \exp[-i\nu\cdot x_1]|u_2\rangle) - \frac{4}{3\sqrt{2}}\sqrt{P_\mathrm{S}P_{\mathrm{S}'}}\langle u| \exp[-i\nu\cdot x_1]|u_1\rangle \\ & - \frac{4}{\sqrt{6}}\sqrt{P_\mathrm{S}P_{\mathrm{S}'}}\langle u| \exp[-i\nu\cdot x_1]|u_2\rangle, \end{aligned} \tag{28}$$

and

$$\begin{aligned} F_\mathrm{GT} = {} & -\tfrac{2}{3}P_\mathrm{S}\langle u| \exp[-i\nu\cdot x_1]|u\rangle + \tfrac{1}{3}P_{\mathrm{S}'}\langle u_1| \exp[-i\nu\cdot x_1]|u_1\rangle \\ & - \frac{4}{3\sqrt{3}}P_{\mathrm{S}'}\langle u_1| \exp[-i\nu\cdot x_1]|u_2\rangle - \tfrac{1}{9}P_{\mathrm{S}'}\langle u_2| \exp[-i\nu\cdot x_1]|u_2\rangle. \end{aligned} \tag{29}$$

Using the second set of form factor coupling constants from ref. [13])

$$\begin{aligned} G_\mathrm{A} &= -1.39 g_\mathrm{V}^\beta \\ G_\mathrm{V} &= 1.02 g_\mathrm{V}^\beta \\ G_\mathrm{P} &= -0.59 g_\mathrm{V}^\beta \\ g_\mathrm{V}^\beta &= 1.4149\times 10^{-49}\ \mathrm{erg}\cdot\mathrm{cm}^3 \end{aligned}$$

we have calculated Λ_μ for the three values of $P_{S'}$ from sect. 2.2. Our results are displayed on table 1. On comparison with the experimental values quoted by Rood and Pascual and Pascual, it is clear that for our theoretical estimates of Λ_μ to agree with experiment, $P_{S'}$ cannot be more than 1.5 %, in complete accord with the findings of section 2.2.

There is, of course, a possibility that this statement is open to question since relativistic corrections will tend [14]) to increase Λ_μ. However this is more than offset by the inclusion of the D-state which will decrease it [4]). In this respect $P_{S'}$ equal to 2 % will be discriminated against more than the other two values of $P_{S'}$. Uncertainties in the coupling constants are present of course but on latest evidence [15]), large variations

TABLE 1

Values of the muon-capture rate of ^{3}He (sec^{-1})

This work	100 % S 1464	1 % S′ 1446	1.5 % S′ 1434	2.0 % S′ 1420
Experimental				
3	1485±40			
4	1468±40			

away from the set we have used are most unlikely. All these considerations suggest that we are still entitled to claim that our calculations of Λ_μ indicate that $P_{S'}$ cannot be larger than 1.5 %.

3. Conclusion

Using flexible wave functions that include two-body short-range correlations, we have obtained estimates of the probability of the S′ state in the ground state of the trinucleon by fitting two sets of experimental data. Our analyses concur and suggest that $P_{S'}$ is smaller than 1.5 %. It appears that the figure of 2 % estimated by Schiff and Gibson is too large and leads to substantial disagreement with the form factors and the muon-capture rate. Our results support the calculations of Davies who found in his extensive work on the binding energy of the three-nucleon system that $P_{S'}$ is 1.2 %.

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SHORT-RANGE CORRELATIONS AND THE TRI-NUCLEON

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Abstract: The effect of short-range correlations in the tri-nucleon is considered by employing suitable forms for the spatial part of the totally antisymmetric S-state wave function. The parameters of our trial wave functions, which are of product form with two-particle correlation functions introduced to simulate the presence of a soft repulsive core, are obtained by fitting the r.m.s. radius and the body form factor of the three-nucleon system. Using a modification of Srivastava's central velocity-dependent potential, we are able to obtain good agreement with the binding energy of ^{3}H and the photodisintegration cross section of ^{3}He, with the exponential wave forms.

1. Introduction

The fairly large binding energies of the 1s shell nuclei ^{3}H, ^{3}He and ^{4}He together with their relatively compact structures suggest that the component nucleons are most of the time well within the range of their mutual nuclear forces. Such a physical situation is indicative of the importance of short-range two-body correlations between the nucleons for these light nuclei. However the effect of such correlations has been little considered until the recent work of Rosati *et al.* [1]), Tang *et al.* [2]), Okamoto *et al.* [3]) and Khanna [4]). Khanna used as wave forms for the spatial part of the totally antisymmetric S-state wave function of the triton, simple two-body Gaussians and exponentials, which were suitably modified by two-body correlatton functions, to obtain reasonably good agreement with the experimental cross sections for the inelastic scattering of electrons on ^{3}He. These wave forms have been successfully exploited by Dalitz and Downs [5]), Rajasekaran and Dalitz [6]) and Murphy and Bodmer [7]) in their analysis of the light hypernuclei and have been shown to have sufficient flexibility to account for both close- in and asymptotic regions at the same time. It seems therefore, that if the two-particle interaction is described by a central velocity-dependent potential, these trial wave functions will be particularly suitable to represent the ground state of the bound system. In this work, we evaluate the triton binding energy and the Coulomb energy and photodisintegration cross sections of ^{3}He using trial functions similar to those of Khanna and an improvement of Srivastava's [8]) velocity-dependent potential. The parameters of our product radial wave functions are obtained by fitting the body form factor and the r.m.s. radius of the three-body system.

In sect. 2, we discuss the trial functions used, whilst in sect. 3, we present the procedure for evaluating the triplet nucleon-nucleon potential. Sect. 4 contains the triton binding energy calculations and sect. 5 an explicit derivation of the photodisintegration cross-section expressions.

2. Tri-nucleon wave function

The ground state of the three-nucleon system with $J = T = \frac{1}{2}$ is approximated by the predominant, spatially symmetric S-state wave function Ψ, where

$$\Psi = \psi_s(\chi'\zeta'' - \chi''\zeta'), \tag{1}$$

with

$$\chi' = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \alpha_2\beta_1)\alpha_3,$$

$$\chi'' = \frac{1}{\sqrt{6}}(2\alpha_1\alpha_2\beta_3 - \alpha_1\beta_2\alpha_3 - \beta_1\alpha_2\alpha_3), \tag{2}$$

ζ' and ζ'' are the corresponding isospin functions. The totally symmetric spatial wave function ψ_s is assumed to have the form

$$\psi_s = \prod_{i<j}^{3} g(r_{ij}) f(r_{ij}), \tag{3}$$

where $g(r_{ij})$ and $f(r_{ij})$ are chosen to have one of the forms

$$\begin{aligned} &\text{(i)} \quad g(r_{ij}) = \exp(-a' r_{ij}^2), \qquad f(r_{ij}) = 1 - c' \exp(-b' r_{ij}^2) \\ &\text{(ii)} \quad g(r_{ij}) = \exp(-a|r_{ij}|), \qquad f(r_{ij}) = 1 - c \exp(-b|r_{ij}|). \end{aligned} \tag{4}$$

These functions have considerable flexibility and should be capable of giving a good representation of the principal features of the bound system with short-range forces. The parameters a', b', c' and a, b, c are obtained by fitting the body form factor $F_1(q^2)$ given by

$$F_1(q^2) = \int \psi_s^2 \exp(i\boldsymbol{q} \cdot \tfrac{2}{3}\boldsymbol{R}) 8\pi^2 r_1 r_2 r_3 \, dr_1 \, dr_2 \, dr_3, \tag{5}$$

where $\boldsymbol{R} = \boldsymbol{x}_1 - \frac{1}{2}(\boldsymbol{x}_2 + \boldsymbol{x}_3)$ and r_1, r_2, r_3 are the interparticle separations, and the r.m.s. radius $R_{\text{r.m.s.}}$ is given by

$$R_{\text{r.m.s.}}^2 = \int \psi_s^* \{\tfrac{1}{9}[2(r_1^2 + r_2^2) - r_3^2]\} \psi_s 8\pi^2 r_1 r_2 r_3 \, dr_1 \, dr_2 \, dr_3. \tag{6}$$

These integrals and the others which follow are taken over the domain consistent with the triangular inequalities $r_1 + r_2 \geqq r_3$, $r_1 + r_3 \geqq r_2$ and $r_2 + r_3 \geqq r_1$. They can be evaluated explicitly and are given in the appendix for completeness. The experimental values of $F_1(q^2)$ selected are those of Levinger–Srivastava [9]), whilst the r.m.s. radius

is taken to be 1.66 fm for the Gaussian to facilitate comparison with Okamoto's results) and 1.70 fm for the exponential. It should be remembered that these are two independent procedures since the fit to $R_{r.m.s.}$ only determine $F_1(q^2)$ for very small q.

For the Gaussian, the parameters b' and c' are varied from 0.2 to 6.0 fm^{-2} and 0 to 1.0, respectively, with $a' = 0.058$, 0.062, 0.066 and 0.070 fm^{-2}. The best fit to $R_{r.m.s.}$ and $F_1(q^2)$ (see fig. 1) is given by $a' = 0.062$ fm^{-2}, $b' = 2.00$ fm^{-2} and $c' = 0.4$.

For the exponential, the parameters b and c are varied from 0.5 to 6.0 fm^{-1} and 0 to 1.0, respectively, with $a = 0.36$, 0.38, 0.40 and 0.42 fm^{-1}. The best fit to the selected $R_{r.m.s.}$ and $F_1(q^2)$ is given by $a = 0.40$ fm^{-1}, $b = 2.00$ fm^{-1} and $c = 0.4$.

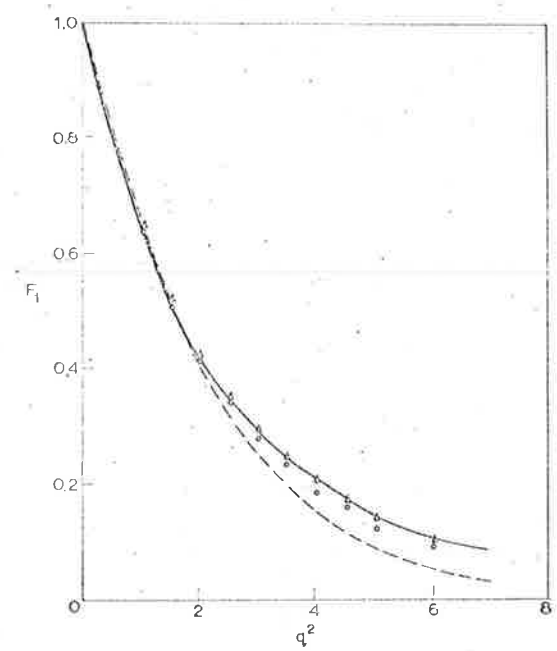

Fig. 1. Comparison of theoretical and experimental form factors. Circles are averages of the two sets of experimental data analysed by Levinger and Srivastava [8]), triangles data from the analysis of Okamota-Lucas [3]), dashed-curve obtained from Gaussian and continuous curve obtained from exponential.

3. The nucleon-nucleon interaction

For the exponential wave function, the general two-body interaction is taken to be

$$V(r_{ij}) = (wP_{ij}^{W} + bP_{ij}^{B} + hP_{ij}^{H} + mP_{ij}^{M})(V)_{static}(r_{ij}) + V_{vel.\,dep.}(r_{ij}), \tag{7}$$

where w, b, h, m are the exchange force constants and P_{ij}^{W}, P_{ij}^{H}, P_{ij}^{B} and P_{ij}^{M} the usual interchange operators. Here $(V)_{static}(r_{ij})$ is given by

$$(V)_{static}(r_{ij}) = X_{static}(V_0)_{static} \exp(-2r_{ij}/\beta_s), \tag{8}$$

where (the ratio of the triplet static potential to the single static potential)

$$X_{static} = \frac{w+b+h+m}{w-b-h+m}; \tag{9}$$

$$(V)_{vel.\,dep.}(r_{ij}) = (V_s)_{vel.\,dep.} + (V_t)_{vel.\,dep.}, \tag{10}$$

$$(V_s)_{vel.\,dep.}(r_{ij}) = [(V_0)_{vel.\,dep.}/2][p_{ij}^2 w_s(r_{ij}) + w_s(r_{ij})p_{ij}^2], \tag{11}$$

$$(V_t)_{vel.\,dep.}(r_{ij}) = [X_{vel.}(V_0)_{vel.\,dep.}/2][p_{ij}^2 w_t(r_{ij}) + w_t(r_{ij})p_{ij}^2], \tag{12}$$

with

$$w(r_{ij}) = \exp(-2r_{ij}/\beta').$$

(Subscripts s and t refer to the singlet and triplet states, respectively.) The effective nucleon-nucleon potential in the triton is hence

$$V_{eff}(r_{ij}) = -\tfrac{1}{2}(1 + X_{static})(V_0)_{static} \exp(-2r_{ij}/\beta_s) + (V)_{vel.\,dep.}(r_{ij}). \tag{13}$$

The values of the potential parameters assumed by Srivastava were

$$\begin{aligned} (V_0)_{static} &= -100 \text{ MeV}, & 1/\beta_s &= 0.625 \text{ fm}^{-1}, \\ (V_0)_{vel.\,dep.} &= 82.94 \text{ MeV}, & 1/\beta_s' &= 1.40 \text{ fm}^{-1}, \quad 1/\beta_t' = 1.0 \text{ fm}^{-1}, \\ X_{static} &= 1.84, & X_{vel.} &= 0.55. \end{aligned} \tag{14}$$

However, his parameters X_{static} and $X_{vel.}$ calculated from a single parameter deuteron trial function have been shown by Lovitch and Rosati [1]) to give an overbound deuteron. We, therefore, re-evaluate X_{static} and $X_{vel.}$ using a three parameter trial function of the form $\exp(-xr_{ij}) - z\exp(-yr_{ij})$ for the deuteron. With Srivastava's potential, we determine x, y and z which give a binding energy of 3.49 MeV (the result obtained by Lovitch-Rosati through a direct numerical evaluation of the two-body Schroedinger equation). With these values of x, y and z, we plot the sets of X_{static} and $X_{vel.}$ which give the true deuteron binding energy against those of Srivastava, which give the correct ^3S phase shifts at $E_{lab} = 270$ MeV (see fig. 2). The point of intersection gives our values of X_{static} and X_{vel}. These are

$$X_{static} = 1.52, \qquad X_{vel} = 0.415. \tag{15}$$

This potential will be used with our exponential wave function, whilst Tang's will be used with the Gaussian.

It is pertinent to point out that there are two interpretations of p in the velocity-dependent potential. Srivastava's calculations are based on the assumption that for the three-body system p is in each instance the momentum conjugate to the inter-particle separations, r_1, r_2 and r_3 (to be referred to as case 1). Lovitch and Rosati, however, interpret p to be the momentum canonically conjugate to the two-particle relative coordinates in the centre-of-mass frames of the particles taken two at a time as if the third one did not exist (case 2). Although we believe Lovitch and Rosati's

interpretation to be the correct one, we, nevertheless, use both in our exponential calculations so as to derive a quantitative comparison of the two.

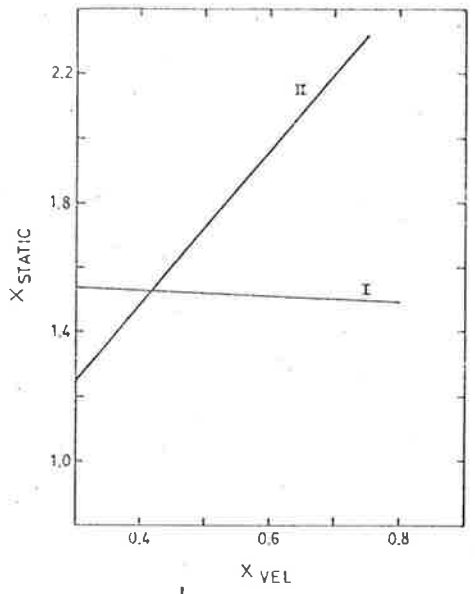

Fig. 2. Plot of X_{static} against X_{vel}. Line I from our deuteron calculations, line II from Srivastava's [8] phase-shift fit.

4. Triton binding energy and Coulomb energy of ^{3}He

Following the notation of Dalitz and Downs, we can write the binding energy of the triton as

$$E(^3\text{H}) = \frac{T(\psi_s, \psi_s) + V(\psi_s, \psi_s)}{N(\psi_s, \psi_s)}, \tag{16}$$

where

$$T(\phi, \xi) = -\int \left[\frac{\hbar^2}{M}\left(\frac{\partial\phi^*}{\partial r_1}\frac{\partial\xi}{\partial r_1} + \frac{\partial\phi^*}{\partial r_2}\frac{\partial\xi}{\partial r_2} + \frac{\partial\phi^*}{\partial r_3}\frac{\partial\xi}{\partial r_3}\right)\right.$$
$$\left. + \frac{\hbar^2}{M}[t(231) + t(312) + t(123)]\right] 8\pi^2 r_1 r_2 r_3 \, dr_1 \, dr_2 \, dr_3 , \tag{17}$$

$$V(\phi, \xi) = \int \phi^* [3V_{\text{eff}}(r_{ij})] \xi 8\pi^2 r_1 r_2 r_3 \, dr_1 \, dr_2 \, dr_3 , \tag{18}$$

$$N(\phi, \xi) = \int \phi^* \xi 8\pi^2 r_1 r_2 r_3 \, dr_1 \, dr_2 \, dr_3 , \tag{19}$$

with

$$t(ijk) = \frac{r_i^2 + r_j^2 - r_k^2}{4 r_i r_j}\left(\frac{\partial\phi^*}{\partial r_i}\frac{\partial\xi}{\partial r_j} + \frac{\partial\phi^*}{\partial r_j}\frac{\partial\xi}{\partial r_i}\right). \tag{20}$$

These integrals although complicated are straightforward and are given in the appendix. The two interpretations of p result only in a difference in the expectation value of the velocity-dependent part of the potential energy. The expression obtained by Srivastava becomes

$$V_{\text{vel. dep.}}(\psi_s, \psi_s) = \frac{3}{2}\int \psi_s^* \{(V_0)_{\text{vel. dep.}}\}\{[p_{12}^2 w_s(r_3) + w_s(r_3) p_{12}^2]$$
$$+ X_{\text{vel.}}[p_{12}^2 w_t(r_3) + w_t(r_3) p_{12}^2]\}\psi_s 8\pi^2 r_1 r_2 r_3 \, dr_1 \, dr_2 \, dr_3 , \tag{21}$$

where

$$p_{12}^2 = -\left(\frac{\partial^2}{\partial r_3^2} + \frac{2}{r_3}\frac{\partial}{\partial r_3}\right),$$

and while that of Lovitch–Rosati is

$$V_{\text{vel. dep.}}(\psi_s, \psi_s) = 3\int \{[w_1 + \tfrac{1}{4}w_2 + \tfrac{1}{4}w_3] t_1(g_1 f_1) g_2^2 f_2^2 g_3^2 f_3^2\}$$
$$8\pi^2 r_1 r_2 r_3 \, dr_1 \, dr_2 \, dr_3 , \tag{22}$$

where

$$t_1(u) = \left(\frac{\partial u}{\partial r_1}\right)^2 - \frac{u}{r_1}\frac{\partial^2 (r_1 u)}{\partial r_1^2}, \qquad w_i = (V_{0i})_{\text{vel. dep.}} \exp(-2r_i/\beta'). \tag{23}$$

For the Coulomb energy of ^{3}He, we use the potential of Schneider and Thaler [10] which takes account of the finite nucleon size. Thus

$$\text{C.E.} = C(\psi_s, \psi_s), \tag{24}$$

with

$$C(\phi, \xi) = \int \phi^* \left\{\frac{e^2}{r_3}[1 - e^{-3.36 r_3}(0.582 r_3 - 2.776)\right.$$
$$\left. - e^{-2.97 r_3}(0.644 r_3 + 3.639)]\right\} \xi 8\pi^2 r_1 r_2 r_3 \, dr_1 \, dr_2 \, dr_3 . \tag{25}$$

5. The cross-sections σ_b and σ_{int} for ^{3}H and ^{3}He

The bremsstrahlung-weighted cross-section σ_b for a nuclear ground state wave function that is fully spatially symmetric is simply related to the r.m.s. radius through the expression [11]

$$\sigma_b = \tfrac{4}{3}\pi^2 \frac{e^2}{\hbar c}\frac{NZ}{A-1} R_{\text{rms}}^2 . \tag{26}$$

Thus to evaluate σ_b (^{3}He) we need only multiply our mean square radii by the constant in eq. (26). The integrated cross section σ_{int} for the photodisintegration of ^{3}He, is

$$\sigma_{\text{int}} = \left(\frac{2\pi^2 e^2 \hbar}{Mc}\right)[(\sum_n f_{0n})_T + (\sum_n f_{0n})_{\text{static}} + (\sum_n f_{0n})_{\text{vel. dep.}}], \tag{27}$$

where $\sum_n f_{0n}$ is the summed oscillator strength, and T, static and vel. dep. refer to the contributions of the kinetic energy, the static potential and the velocity-dependent part of the potential. For ^{3}He and using the interpretation of Srivastava [12])

$$(\sum_n f_{0n})_{\mathrm{T}} = \frac{NZ}{A} = \frac{2}{3}, \tag{28}$$

$$(\sum_n f_{0n})_{\mathrm{static}} = -\left(\frac{M}{3\hbar^2}\right)\left\{\int \psi_s^*[\sum_i \sum_j r_{ij}^2 (V)_{\mathrm{static}}(r_{ij}) \right.$$

$$\left. (mP_{ij}^{\mathrm{M}} + hP_{ij}^{\mathrm{H}})]\psi_s 8\pi^2 r_1 r_2 r_3 \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}r_3\right\}, \tag{29}$$

$$(\sum f_{0n})_{\mathrm{vel.\,dep.}} = \tfrac{4}{9}(V_0)_{\mathrm{vel.\,dep.}}\left\{\int \psi_s^*[w_s(r_{ij}) + X_{\mathrm{vel.}} w_t(r_{ij})]\right.$$

$$\left. 8\pi^2 r_1 r_2 r_3 \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}r_3\right\}. \tag{30}$$

In eq. (29), i and j denote protons and neutron, respectively. Application of the Thomas-Reiche-Kuhn sum rules eliminates the terms in w and b in $(\sum_n f_{0n})_{\mathrm{static}}$ since the corresponding Wigner and Bartlett operators commute with the space coordinates. The expectation value of the Heisenberg operator with our S-state wave function is

$$\langle P_{ij}^{\mathrm{H}} \rangle = \tfrac{1}{2}\langle P_{ij}^{\mathrm{M}} \rangle.$$

Since

$$(V)_{\mathrm{static}}(r_{ij}) = \frac{1}{(w+m)}\,[V_{\mathrm{eff}}(r_{ij})]_{\mathrm{static}}, \tag{31}$$

our expression for σ_{int} reduces to

$$\sigma_{\mathrm{int}} = \frac{4\pi^2 e^2 \hbar}{3Mc}\left[1 - \frac{M(m+\frac{1}{2}h)}{(w+m)\hbar^2}\int \psi_s^*\{\sum_i \sum_j [V_{\mathrm{eff}}(r_{ij})]_{\mathrm{static}} r_{ij}^2 P_{ij}^{\mathrm{M}}\}\psi_s \mathrm{d}\tau + \tfrac{2}{3}(V_0)_{\mathrm{vel.\,dep.}}\right.$$

$$\left. \times\left|\int \psi_s^*[w_s(r_{ij}) + X_{\mathrm{vel.}} w_t(r_{ij})]\psi_s \mathrm{d}\tau\right|\right],$$

$$\mathrm{d}\tau = 8\pi^2 r_1 r_2 r_3 \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}r_3, \tag{32}$$

which differs from eq. (27) of Srivastava [12,13]). Using the force mixtures from Laskar [14]) and X_{static}, we calculate $\sigma_{\mathrm{int}}(^3\mathrm{He})$ for the Serber and Biel forces (where the forms of the singlet and triplet static potentials differ as in Tang's potential, we take X_{static} to be equal to 1.67). If case 2 is used instead of case 1, $\sigma_{\mathrm{int}}(^3\mathrm{He})$ is not greatly affected since the velocity-dependent contribution is small. Besides, the difference between Srivastava and Lovitch-Rosati's velocity dependent calculations, is likely to be at most 15 %, as discussed in the next section.

6. Results and discussion

For the Gaussian wave function, our Coulomb energy results are in agreement with those of Okamoto-Lucas. This suggests little difference between their trial function and ours. However, an advantage with our modified Gaussian is that it is perhaps more tractable for binding energy calculations. The triton binding energy obtained with Tang's potential is much smaller than that of case A and case B of Lovitch-Rosati. This may be expected since our fit to $F_1(q^2)$ is not good for large momentum

TABLE 1

The binding energy, Coulomb energy and the r.m.s. radius of the three-nucleon system

Ref.	Wave function	Potential	B.E. (^{3}H) (MeV)	C.E. (point) (MeV)	C.E. (finite) (MeV)	r.m.s. radius (fm)
Okamoto-Lucas [3])	modified Gaussian			0.69	0.64	1.66
	uncorrelated Gaussian	Tang (case 2)	2.50	0.708	0.66	1.62
Lovitch-Rosati [1])	case A	Tang (case 2)	7.623	0.771		1.63
	case B	Tang (case 2)	8.240	0.717		1.67
present work	modified Gaussian	Tang (case 2)	3.30	0.67	0.62	1.66
	modified exponential	Srivastava (case 1)	9.30	0.74	0.69	1.70
Lovitch-Rosati	case A	Srivastava (case 2)	10.20	0.74	0.69	1.70
	case B	Srivastava (case 2)	10.903	0.748		1.65
Srivastava [8])	uncorrelated exponential	Srivastava (case 2)	11.201	0.706		1.69
	three-parameter exponential	Srivastava (case 1)	6.08	0.597		
		Srivastava (case 1)	7.17	0.663		1.92
present work	modified exponential	modified Srivastava (case 1)	5.92	0.74	0.69	1.70
		modified Srivastava (case 2)	6.76	0.74	0.69	1.70
experimental			8.48		0.764	1.50(^{3}H) 1.97(^{3}He)

transfers; the Gaussian form is known to be unsuitable for asymptotic regions and this is partially confirmed by the smallness of σ_{int}. For the exponential wave function, we can expect our results to be more accurate. Although the contributions of S′ and D states at large momentum transfers are likely to alter our theoretical curves for $F_1(q^2)$, the exponential wave form should still present the better fit. Our binding

energy values for Srivastava's potential are sufficiently near Lovitch–Rosati's to encourage us to believe that our trial function is close to the best obtained variationally. The excellent agreement between our σ_{int} and experiment suggests that our velocity-dependent potential is quite accurate. As a check, we calculated σ_{int} for ^{4}He using Srivastava's analysis and our potential parameters. The agreement with experiment is substantially improved; σ_{int} is little affected by our use of Srivastava's interpretation of ϕ because it was found in our binding energy calculation that there was only

TABLE 2

σ_b and σ_{int} of ^{3}H and ^{4}He

Nucleus	Author	$\sigma_{int}/(4\pi^2e^2h/3Mc)$	σ_b (mb)	σ_{int} (Serber) (MeV·mb)	σ_{int} (Biel) (MeV·mb)
^{3}He	Srivastava	$1.038+0.724(m+\frac{1}{2}h)$	3.5	56.0	
		$1.038+0.724\left(\frac{m+\frac{1}{2}h}{\omega+m}\right)$	3.5	58.1	62.2
	(corrected to our potential)	$1.03\ +0.642\left(\frac{m+\frac{1}{2}h}{\omega+m}\right)$	3.5	56.0	59.6
	Mathur *et al.*	$1.0\ +0.72\ (m+\frac{1}{2}h)$	2.85	54.0	
		$1.0\ +0.72\ \left(\frac{m+\frac{1}{2}h}{\omega+m}\right)$	2.85	56.2	60.4
	present work (Gaussian)	$1.03\ +0.485\left(\frac{m+\frac{1}{2}h}{\omega+m}\right)$	2.62	52.0	54.8
	(exponential)	$1.05\ +0.713\left(\frac{m+\frac{1}{2}h}{\omega+m}\right)$	2.74	57.8	62.0
	experimental		2.53 ± 0.19		62±6
^{4}He	Srivastava-Jain	$\frac{3}{2}(1.130+1.239)m+\frac{1}{2}h))$	2.4	105.0	
		$\frac{3}{2}\left(1.130+1.239\left(\frac{m+\frac{1}{2}h}{\omega+m}\right)\right)$	2.4	110.5	121.0
	(corrected to our potential)	$\frac{3}{2}\left(1.113+1.099\left(\frac{m+\frac{1}{2}h}{\omega+m}\right)\right)$	2.4	103.3	113.0
	experimental		2.4 ± 0.15		95±7

a small quantitative difference between case 1 and case 2. It can be concluded that the discrepancy between Srivastava and Lovitch–Rosati's binding energy results is almost wholly attributable to the poor trial function of Srivastava. A comparison of the results (tables 1 and 2) obtained by the uncorrelated and correlated functions for both wave forms, indicates the importance of short-range correlations when the two-body interaction is described by a velocity-dependent potential. The Coulomb energies in our calculations are nowhere near the difference between the experimental binding energy of ^{3}H and ^{3}He. This tends to support the claim of Okamoto that charge asymmetry must be present to the order ot 0.1 MeV.

7. Conclusion

Using radial wave functions of product form with two-body correlations we have been able to obtain good agreement with the results of other authors. Our calculations indicate that short-range correlations are important as their introduction improves results considerably. Our trial functions, in particular the modified exponential, provide excellent approximations for the S-state structure of the tri-nucleon and are extremely tractable. Work is now in progress using these trial functions to deter-mine the μ-capture rate of ^{3}He.

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Appendix

The integrals appearing in this paper are presented here for completeness. For the Gaussian

(i) $$\int \exp(-\alpha_1 r_1^2-\alpha_2 r_2^2-\alpha_3 r_3^2) r_1 r_2 r_3 \,dr_1\,dr_2\,dr_3 = \tfrac{1}{8}\pi(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{-\frac{3}{2}}$$

(ii) $$\int \exp(-\alpha_1 r_1^2-\alpha_2 r_2^2-\alpha_3 r_3^2) r_1 r_2 \,dr_1\,dr_2\,dr_3 = \left(\frac{\pi}{\alpha_1+\alpha_2}\right)^{\frac{1}{2}} \times (4[\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3])^{-1},$$

(iii) $$\int \exp(-\alpha_1 r_1^2-\alpha_2 r_2^2-\alpha_3 r_3^2) r_1 r_2 r_3^3 \,dr_1\,dr_2\,dr_3 = \tfrac{3}{16}\pi(\alpha_1+\alpha_2) \times (\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{-\frac{5}{2}},$$

(iv) $$\int \exp(-\alpha_1 r_1^2-\alpha_2 r_2^2-\alpha_3 r_3^2)\exp(\tfrac{2}{3}iq\cdot R) r_1 r_2 r_3 \,dr_1\,dr_2\,dr_3 = \pi(4[\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3])^{-\frac{3}{2}} \exp\left(-\frac{q^2[4\alpha_1+\alpha_2+\alpha_3]}{36[\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3]}\right),$$

(v) if $\phi = \exp(-\alpha_1 r_1^2-\alpha_2 r_2^2-\alpha_3 r_3^2)$,

$\xi = \exp(-\alpha_1^1 r_1^2-\alpha_2^1 r_2^2-\alpha_3^1 r_3^2)$,

$$T(\phi\xi) = 3\pi^3\hbar^2\left\{\sum_c \frac{2\alpha_1\alpha_1^1(\alpha_2+\alpha_2^1+\alpha_3+\alpha_3^1)}{M} + \sum_c \frac{(\alpha_1+\alpha_1^1)(\alpha_2\alpha_3^1+\alpha_3\alpha_2^1)}{M}\right\} \times\{\sum_c(\alpha_1+\alpha_1^1)(\alpha_2+\alpha_2^1)\}^{-\frac{5}{2}},$$

where $\sum_c$ denotes sum over cyclic permutation.

For the exponential

(i) $$\int \exp(-\alpha_1 r_1-\alpha_2 r_2-\alpha_3 r_3) r_1 r_2 r_3 \,dr_1\,dr_2\,dr_3 = 8\{\alpha_1(\alpha_1+\alpha_2)(\alpha_1+\alpha_3) + \alpha_2\alpha_2+\alpha_1)(\alpha_2+\alpha_3)+\alpha_3(\alpha_3+\alpha_1)(\alpha_3+\alpha_2)+2(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\} \times\{(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\}^{-3} = I(\alpha_1,\alpha_2,\alpha_3),$$

(ii) $$\int \exp(-\alpha_1 r_1-\alpha_2 r_2-\alpha_3 r_3) r_1 r_2\,dr_1\,dr_2\,dr_3 = 4\{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3) +(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\}\{(\alpha_1+\alpha_2)^{-3}(\alpha_1+\alpha_3)^{-2}(\alpha_2+\alpha_3)^{-2}\},$$

(iii) $$\int \exp(-\alpha_1 r_1-\alpha_2 r_2-\alpha_3 r_3) r_1 r_2 r_3^3\,dr_1\,dr_2\,dr_3$$
$$= 8\left\{\frac{12(\alpha_1+\alpha_2+\alpha_3)}{(\alpha_2+\alpha_3)^5(\alpha_1+\alpha_3)^2(\alpha_1+\alpha_2)^2} + \frac{12(\alpha_1+\alpha_2+\alpha_3)}{(\alpha_1+\alpha_3)^5(\alpha_1+\alpha_2)^2(\alpha_2+\alpha_3)^2} + \frac{9(2\alpha_3+\alpha_1+\alpha_2)}{(\alpha_2+\alpha_3)^4(\alpha_1+\alpha_3)^4(\alpha_1+\alpha_2)} + \frac{3}{(\alpha_2+\alpha_3)^4(\alpha_1+\alpha_3)(\alpha_1+\alpha_2)^3} + \frac{3}{(\alpha_1+\alpha_3)^4(\alpha_2+\alpha_3)(\alpha_1+\alpha_2)^3} + \frac{3(3\alpha_1+3\alpha_2+2\alpha_3)}{(\alpha_1+\alpha_2)^3(\alpha_1+\alpha_3)^3(\alpha_2+\alpha_3)^3}\right\},$$

(iv) $$\int \exp(-\alpha_1 r_1-\alpha_2 r_2-\alpha_3 r_3)\exp(i\tfrac{3}{2}q\cdot R) r_1 r_2 r_3\,dr_1\,dr_2\,dr_3$$
$$= \frac{1024\alpha_1\alpha_2\alpha_3}{\pi}\int_0^\infty \frac{k_1^2}{(k_1^2+\alpha_2^2)^2(2k_1^2+4\alpha_3^2-2\alpha_1^2-\frac{4}{9}q^2)^2} \times[\tfrac{4}{3}qk_1(2k_1^2+4\alpha_3^2-2\alpha_1^2-\tfrac{4}{9}q^2)]^{-1}\ln\left[\left(\frac{k_1^2+\alpha_1^2+\frac{4}{9}q^2-\frac{4}{3}qk_1}{k_1^2+\alpha_1^2+\frac{4}{9}q^2+\frac{4}{3}qk_1}\right) \times\left(\frac{4k_1^2+4\alpha_3^2+\frac{4}{9}q^2+\frac{8}{3}qk_1}{4k_1^2+4\alpha_3^2+\frac{4}{9}q^2-\frac{8}{3}qk_1}\right)\right] + \frac{2}{[(4k_1^2+4\alpha_3^2+\frac{4}{9}q^2)^2-(\frac{8}{3}qk_1)^2]} + \frac{1}{2[(k_1^2+\alpha_1^2+\frac{4}{9}q^2)^2-(\frac{4}{3}qk_1)^2]}\right\}dk_1,$$

(v) if $\phi = \exp(-\alpha_1 r_1-\alpha_2 r_2-\alpha_3 r_3)$,
$\xi = \exp(-\alpha_1^2 r_1-\alpha_2^1 r_2-\alpha_3^1 r_3)$,

$$T(\phi,\xi) = \frac{\hbar^2}{M}(\alpha_1\alpha_1^1+\alpha_2\alpha_2^1+\alpha_3\alpha_3^1)I(\alpha_1+\alpha_1^1,\alpha_2+\alpha_2^1,\alpha_3+\alpha_3^1) + \frac{\hbar^2}{4M}(\alpha_2\alpha_3^1+\alpha_2\alpha_3^1)J(\alpha_2+\alpha_2^1,\alpha_3+\alpha_3^1,\alpha_1+\alpha_1^1) + \frac{\hbar^2}{4M}(\alpha_3\alpha_1^1+\alpha_1\alpha_3^1)J(\alpha_3+\alpha_3^1,\alpha_1+\alpha_1^1,\alpha_2+\alpha_2^1) + \frac{\hbar^2}{4M}(\alpha_1\alpha_2^1+\alpha_2\alpha_1^1)J(\alpha_1+\alpha_1^1,\alpha_2+\alpha_2^1,\alpha_3+\alpha_3^1),$$

where $I(\alpha,\beta,\gamma)$ is integral (i) above and

$$J(\alpha,\beta,\gamma) = 16[(\alpha^3+\beta^3)\gamma^2+(\alpha^2+\beta^2)(4\gamma^3+\alpha\beta\gamma) +(\alpha+\beta)(4\gamma^4+7\alpha\beta\gamma^2)+\gamma^5+3\alpha^2\beta^2\gamma+10\alpha\beta\gamma^3][(\alpha+\beta)^3(\beta+\gamma)^4(\alpha+\gamma)^4]^{-1}.$$

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THE PHOTODISINTEGRATION CROSS SECTION OF ^{4}He

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Abstract: The integrated cross section σ_{int} for the photodisintegration of helium (^{4}He) is calculated by applying the sum rules of Levinger and Bethe and using a two-parameter Irving wave function. The two-body interactions are assumed to be that obtained from our modification of the central velocity-dependent potential of Srivastava and the Rarita-Present potential. Our results give considerably better agreement with the experimental results of Gorbunov and Spiridonov than do those of other authors and indicate that Serber and Biel force mixtures are more suitable forms of interaction.

1. Introduction

The bremsstrahlung-weighted cross section and the integrated cross section in the photodisintegration of ^{4}He have recently been evaluated by Goldhammer and Valk, Rustgi and Mukherjee, Srivastava and Jain and Davey and Valk [1–4]) by applying the sum rules of Levinger and Bethe [5]). The results of these groups are found to be unsatisfactory on close comparison with the measurements of Gorbunov and Spiridonov [6]). It appears that the disagreement with experiment cannot be attributed to the failure of the sum rules which have been successful in other applications. Our view is that the discrepancy must be associated with the particular choices of interaction potentials and ^{4}He ground-state wave functions made by these workers, since the expression for σ_{int} depends on both the assumed potential and the ground-state wave function of the nucleus. The disagreement probably arises from their use of either an insufficiently realistic nucleon-nucleon potential or a simple wave function whose parameters are obtained via the uncertainties of a variational calculation. In particular, the modified Irving wave function of Rustgi and Mukherjee is derived from a variational calculation that leads to an underbound alpha particle, whilst the central velocity-dependent potential employed by Srivastava and Jain is over-attractive. [7]) In this work, our purpose is to see if a fairly flexible trial function whose parameters are fixed by fitting the body form factor and the r.m.s. radius of ^{4}He together with a moderately realistic nucleon-nucleon potential (i.e. one that explains most of the existing two-body data) can resolve the disagreement between theory and experiment. We also investigate the nature of the exchange forces that are most suitable for the interactions used.

In sect. 2 we consider the two forms of internucleonic force and the alpha-particle ground-state wave function that are the basis of this work, and in sect. 3 we present an explicit derivation of the sum-rule formulae for the integrated and bremsstrahlung-weighted cross sections of ^{4}He.

2. The nucleon-nucleon interaction and the ^{4}He ground-state wave function

One of our two choices for the nuclear potential is the Rarita-Present [8]) potential, which takes the form

$$V(r_{ij}) = V_t(r_{ij}) = -123.3 \exp(-1.156 r_{ij})\{\omega + bP_{ij}^B + hP_{ij}^H + mP_{ij}^M\}, \quad (1)$$

$$V_s(r_{ij}) = qV_t(r_{ij}), \quad (2)$$

(where the ratio q, of the singlet static to triplet static potentials is 0.6, and the subscripts s and t refer to singlet and triplet, respectively). The second potential is a modification of the central velocity-dependent potential of Srivastava [3])

$$V(r_{ij}) = (\omega + bP_{ij}^B + hP_{ij}^H + mP_{ij}^M)(V)_{static}(r_{ij}) + V_{vel.\,dep.}(r_{ij}). \quad (3)$$

Here $(V)_{static}(r_{ij})$ is given by

$$V_{static}(r_{ij}) = \frac{(V_0)_{static}}{q} \exp\left(-\frac{2r_{ij}}{\beta_s}\right), \quad (4)$$

$$V_{vel.\,dep.}(r_{ij}) = (V_s)_{vel.\,dep.}(r_{ij}) + (V_t)_{vel.\,dep.}(r_{ij}), \quad (5)$$

where

$$(V_s)_{vel.\,dep.}(r_{ij}) = \tfrac{1}{2}(V_0)_{vel.\,dep.}\{p_{ij}^2\omega_s(r_{ij}) + \omega_s(r_{ij})p_{ij}^2\}, \quad (6)$$

$$(V_t)_{vel.\,dep.}(r_{ij}) = \tfrac{1}{2}X_{vel.}(V_0)_{vel.\,dep.}\{p_{ij}^2\omega_t(r_{ij}) + \omega_t(r_{ij})p_{ij}^2\}, \quad (7)$$

with

$$\omega(r_{ij}) = \exp\left(-\frac{2r_{ij}}{\beta'}\right). \quad (8)$$

The values of our potential parameters are

$$(V_0)_{static} = -100 \text{ MeV}, \quad \frac{1}{\beta_s} = 0.625 \text{ fm}^{-1},$$

$$(V_0)_{vel.\,dep.} = 82.94 \text{ MeV}, \quad \frac{1}{\beta_s'} = 1.40 \text{ fm}^{-1}, \quad \frac{1}{\beta_t'} = 1.0 \text{ fm}^{-1},$$

$$X_{vel.} = 0.415, \quad q = 0.655. \quad (9)$$

It is obvious from refs. [7–9]) that our two potentials satisfy the criteria for a realistic potential. We approximate the ground state of ^{4}He by the predominant spatially symmetric $^1S_0$ state. Thus our wave function can be written as

$$\Psi(^1S_0) = \psi_s(\chi'\zeta'' - \chi''\zeta'), \quad (10)$$

where

$$\chi' = \tfrac{1}{2}(\alpha_1\beta_2-\alpha_2\beta_1)(\alpha_3\beta_4-\alpha_4\beta_3),$$
$$\chi'' = \frac{1}{\sqrt{3}}\sigma_1\cdot\sigma_3\chi' \qquad (11)$$

are orthogonal spin functions and ζ' and ζ'' the corresponding isospin functions for the four-nucleon system. The spatially symmetric part of the wave function ψ_s is taken to be the two-parameter Irving wave function [9])

$$\psi_s = N\{\exp[-a(\sum_{i<j} r_{ij}^2)^{\frac{1}{2}}]-\exp[-b(\sum_{i<j} r_{ij}^2)^{\frac{1}{2}}]\}, \qquad i,j = 1,2,3,4, \qquad (12)$$

where the normalization constant is

$$N^2 = \frac{2^6}{3\pi^4}\left\{\frac{1}{a^9}+\frac{1}{b^9}-\frac{2}{\{\frac{1}{2}(a+b)\}^9}\right\}. \qquad (13)$$

This wave function is expected to be inadequate for the velocity-dependent potential in binding-energy calculations if only on the basis of our work on the triton [7]) where short-range two-body correlations were found to be necessary. It should nevertheless give as in ^{3}He photodisintegration cross-section calculations, a value of σ_{int} (^{4}He) that is near that obtainable from a correlated function. This follows since it is known that σ_{int} depends critically only on the asymptotic behaviour of the wave function used, and our procedure for selecting the constants a and b makes ψ_s good at large radial distances. Our parameters a and b are found by the method of ref. [7]), which allows us to avoid the uncertainties introduced by the interaction in any variational calculation. This is important as wave functions chosen by variational methods to give the correct binding energy give too concentrated a nucleus [9]). Thus a and b are evaluated by fitting the form factor of ^{4}He given by our wave function

$$F_B(q^2) = \frac{3\pi^4}{2^6}N^2\left\{\left(\frac{16}{16+\frac{3q^2}{4a^2}}\right)^5\frac{1}{a^9}+\left(\frac{16}{16+\frac{3q^2}{4b^2}}\right)^5\frac{1}{b^9}-2\left(\frac{16}{16+\frac{3q^2}{4\{\frac{1}{2}(a+b)\}^2}}\right)^5\frac{1}{\{\frac{1}{2}(a+b)\}^9}\right\} \qquad (14)$$

to that obtained from the expression [10])

$$F_B(q^2) = \frac{F_{ch}(^4\mathrm{He})}{F_{ch}(\mathrm{n})+F_{ch}(\mathrm{p})}. \qquad (15)$$

It should be remembered that this fit is for high momentum transfers since the r.m.s.-radius already determines $F_B(q^2)$ for small q. In this calculation, the charge form factor $F_{ch}(^4\mathrm{He})$ is taken from the experimental work of Frosch [11]), Repellin [12]) and for

the neutron and proton charge form factors $F_{ch}(n)$, $F_{ch}(p)$, we use the values of De Vries [13]) and Janssens [14]). We take three values for the r.m.s. radius given by

$$(R_{rms})^2 = \frac{135N^2\pi^4}{2^{\frac{9}{2}}}\left\{\frac{1}{a^{\frac{9}{2}}}+\frac{1}{b^{\frac{9}{2}}}-\frac{2}{\{\frac{1}{2}(a+b)\}^{\frac{9}{2}}}\right\}. \qquad (16)$$

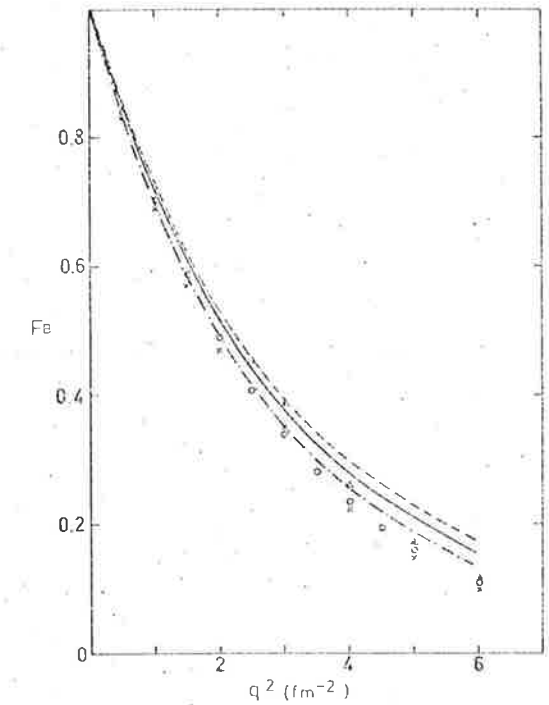

Fig. 1. Comparison of theoretical and experimental form factors. Circles experimental data analysed by Tang and Herndon, crosses our analysis with the data of Frosch and Janssens and triangles our analysis with the data of Repellin and Janssens.

TABLE 1

The binding energy, Coulomb energy and the r.m.s. radius of ^{4}He

	Wave-function parameters a(fm)	b(fm^{-1})	r.m.s. radius (fm)	Potential	B.E. (MeV)	C.E. (MeV)
(i)	0.99	1.06	1.40	a)	30.39	0.910
(ii)	0.95	1.04	1.45	a)	29.67	0.881
(iii)	0.91	1.01	1.50	a)	28.72	0.849
(iv)	0.99	1.06	1.40	b)	19.14	0.910
(v)	0.95	1.04	1.45	b)	20.38	0.881
(vi)	0.91	1.01	1.50	b)	21.40	0.849

a) Rarita-Present.
b) Modified Srivastava.

(The curves are, from top to bottom, in ascending order of K_{rms}).

These are 1.40, 1.45 and 1.50 fm, respectively. From fig. 1 it is seen that the best fits between the calculated and experimental values for the body form factor are very satisfactory. The three sets of a and b are displayed in table 1. As possible checks on the accuracy of our wave functions, we have determined the Coulomb energy and the binding energy of ^{4}He. The Coulomb energy is obtained from the expression

$$\text{C.E.} = \frac{7!N^2\pi^4 e^2}{2^{15}3\sqrt{2}}\left\{\frac{1}{a^8}+\frac{1}{b^8}-\frac{1}{\{\frac{1}{2}(a+b)\}^8}\right\}, \tag{17}$$

(here we are assuming point nucleons).
The binding energy is found from

$$\text{B.E.} = \langle T\rangle+\langle V\rangle_{\text{static}}+\langle V\rangle_{\text{vel. dep.}}, \tag{18}$$

where

$$\langle T\rangle = \frac{3N^2\pi^4\hbar^2}{2^5M}\left\{\frac{1}{a^7}+\frac{1}{b^7}-\frac{2ab}{\{\frac{1}{2}(a+b)\}^7}\right\}, \tag{19}$$

$$\langle V\rangle_{\text{static}} = -24\,\frac{8!N^2\pi^4(V_0)_{\text{eff}}}{(\sqrt{2}k)^8}\{F(a,k)+F(b,k)-2F(\tfrac{1}{2}(a+b),k)\}, \tag{20}$$

$$F(\alpha,k) = \frac{64\beta^3+69\beta^2+30\beta+5}{840\beta^6(1+\beta)^6}, \tag{21}$$

$$\beta = \frac{2\sqrt{2}\alpha}{k}, \tag{22}$$

$$(V_{\text{eff}})_{\text{static}}(r_{ij}) = \tfrac{1}{2}\{(V_t)_{\text{static}}(r_{ij})+(V_s)_{\text{static}}(r_{ij})\} = -(V_0)_{\text{eff}}\exp(-kr_{ij}). \tag{23}$$

The velocity dependent contribution is

$$\begin{aligned}\langle V\rangle_{\text{vel. dep.}} = {}&(V_0)_{\text{vel. dep.}}\{G(a,a,k_s')+G(b,b,k_s')-G(a,b,k_s')\\ &-G(b,a,k_s')\}+X_{\text{vel.}}(V_0)_{\text{vel. dep.}}\{G(a,a,k_t')+G(b,b,k_t')\\ &-G(a,b,k_t')-G(b,a,k_t')\},\end{aligned} \tag{24}$$

where we have used

$$\begin{aligned}G(a_1,a_2,k) = -\int\exp[-4a_1(u^2+v^2+\omega^2)^{\frac{1}{2}}-\sqrt{2}K\omega]\Big\{&4a_2^2\omega^2(u^2+v^2+\omega^2)^{-1}\\ &-6a_2^4(u^2+v^2+\omega^2)^{-\frac{1}{2}}+2a_2\omega^2(u^2+v^2+\omega^2)^{-\frac{3}{2}}+K^2\\ &+2\sqrt{2}Ka_2\omega(u^2+v^2+\omega^2)^{-\frac{1}{2}}-\frac{\sqrt{2}K}{\omega}\Big\}\,du\,dv\,dw,\end{aligned} \tag{25}$$

$$k' = \frac{2}{\beta'},$$

and the transformation

$$u = \tfrac{1}{2}(r_3+r_4-r_1-r_2), \qquad v = \frac{1}{\sqrt{2}}(r_2-r_1),$$

$$\omega = \frac{1}{\sqrt{2}}(r_4-r_3), \qquad \sum_{i<j}^{4} r_{ij}^2 = 4(u^2+v^2+\omega^2). \tag{26}$$

The function $G(a_1, a_2, K)$ can be evaluated explicitly using the transformations of Irving. Our results for the binding energy and the Coulomb energy given in table 1 clearly show that our trial function has enough flexibility to describe properly the ground state of the four-body system, especially when the interaction is the Rarita-Present potential. The slightly underbound alpha particle when we use the velocity-dependent potential arises from the large value of ψ_s for small interparticle separations; ψ_s does not exhibit any correlations between pairs of particles. Writing ψ_s in product form with short-range correlations will improve our binding-energy calculations considerably, but the improvement in $\sigma_{\text{int}}(^4\text{He})$ is likely to be minimal (see table 2, where we have included our ^{3}He results). This point will be investigated in more detail when the present calculational programme is extended to include ψ_s in product form.

3. Sum-rule formulae for ^{4}He

The bremsstrahlung-weighted cross section σ_b for a nuclear ground-state wave function that is fully spatially symmetric is related to the r.m.s. radius through the expression [4])

$$\sigma_b = \int_0^\infty\left(\frac{\sigma}{W}\right)dW = \frac{4\pi^2e^2}{3hc}\left(\frac{NZ}{A-1}\right)R_{\text{rms}}^2. \tag{27}$$

Thus to evaluate σ_b (^{4}He), we need only multiply our mean square radii by the constant in eq. (27). In the electric dipole approximation, the integrated cross-section σ_{int} is defined as

$$\sigma_{\text{int}} = \int_0^\infty\sigma(W)dW = \frac{2\pi^2e^2\hbar}{Mc}\sum_n f_{0n}, \tag{28}$$

where the oscillator strength f_{0n} when summed over all states gives

$$\sum_n f_{0n} = -\frac{M}{\hbar^2}\langle[[H,D],D]\rangle. \tag{29}$$

For ^{4}He,

$$D = \tfrac{1}{2}(Z_{12}+Z_{34}), \tag{30}$$

and the nuclear Hamiltonian H is

$$H = \sum_i T_i+\sum_{i<j}V(r_{ij}); \qquad i,j = 1,2,3,4. \tag{31}$$

TABLE 2

σ_b and σ_{int} of ^{3}He and ^{4}He

Nucleus	Ref.	$\sigma_{int}/(4\pi^2e^2\hbar/Mc)$	σ_b (mb)	σ_{int}(S) (MeV·mb)	σ_{int}(B) (MeV·mb)	σ_{int}(I) (MeV·mb)	σ_{int}(R) (MeV·mb)
^{3}He							
	corrected to our potential [a])	$1.03+0.642\ \frac{(m+\frac{1}{2}h)}{(\omega+m)}$	3.5	56.0	59.6	66.9	66.9
	hard core [b])	$1.00\pm0.720\ \frac{(m+\frac{1}{2}h)}{(\omega+m)}$	2.85	56.2	60.4	68.8	68.8
	correlated exponential [c])	$1.05+0.713\ \frac{(m+\frac{1}{2}h)}{(\omega+m)}$	2.74	57.8	62.0	70.5	70.5
	experimental [d])		2.53 ± 0.19		62 ± 6		
^{4}He							
	corrected to our potential [e])	$\left(\frac{3}{2}(1.113\pm1.099\ \frac{(m+\frac{1}{2}h)}{(\omega+m)}\right)$	2.4	103.3	113.0	132.7	132.7
	[f])	$\frac{3}{2}(1.0+0.635)m+\frac{1}{2}h)$ $+0.124(m'+h')+0.014(m+h))$	2.70	83.2		97.1	96.0
	[g])	$\frac{3}{2}(1.0\pm1.57(m+\frac{1}{2}h))$	2.61	107.0			
	[h])	$\frac{3}{2}(1.0+0.970(m+\frac{1}{2}h))$	0.8	89.0		106.0	106.0
	[i])						
	(i)	$\left(\frac{3}{2}(1.0+1.032)\ \frac{(m+\frac{1}{2}h)}{(\omega+m)}\right)$	2.51	94.9	103.9	121.9	121.9
	(ii)	$\left(\frac{3}{2}(1.0+1.022)\ \frac{(m+\frac{1}{2}h)}{(\omega+m)}\right)$	2.69	94.5	103.5	121.3	121.3
	(iii)	$\left(\frac{3}{2}(1.0+1.009)\ \frac{(m+\frac{1}{2}h)}{(\omega+m)}\right)$	2.89	94.1	102.9	120.5	120.5
	(iv)	$\left(\frac{3}{2}(1.098\pm1.098)\ \frac{(m+\frac{1}{2}h)}{(\omega+m)}\right)$	2.51	102.4	112.1	131.8	131.8
	(v)	$\left(\frac{3}{2}(1.091+1.083\ \frac{(m+\frac{1}{2}h)}{(\omega+m)}\right)$	2.69	101.5	111.1	130.4	130.4
	(vi)	$\left(\frac{3}{2}(1.084+1.064)\ \frac{(m+\frac{1}{2}h)}{(\omega+m)}\right)$	2.89	100.4	109.8	128.9	128.9
	experimental [j])		2.4 ± 0.15		$95\ \pm7$		

[a]) Ref. [17]). [b]) Ref. [18]). [c]) Ref. [7]). [d]) Ref. [20]). [e]) Ref. [3]). [f]) Ref. [2]). [g]) Ref. [1]). [h])Ref. [19]).
[i]) Present work. [j]) Ref. [6]).

Using the general two-body force given by eq. (3) and evaluating $\sum_n f_{0n}$ explicitly, the expression for σ_{int} becomes

$$\sigma_{int} = \frac{2\pi^2e^2\hbar}{Mc}\left\{\frac{N'Z}{A} - \frac{M}{3\hbar^2}\left\langle\sum_i\sum_j r_{ij}^2(V)_{static}(r_{ij})(mP_{ij}^M + hP_{ij}^H)\right\rangle\right\}, \qquad (32)$$

where i and j stand for protons and neutrons, respectively, and the term $\langle\ldots\rangle$ is evaluated with the complete ground state wave function. The velocity-dependent contribution has been absorbed into the term $N'Z/A$ ($N' = N$ if $V_{vel.\ dep.}(r_{ij})$ is absent). With our ground state wave function and following the analysis of Srivastava and Jain, σ_{int} reduces to

$$\sigma_{int} = \frac{2\pi^2e^2\hbar}{Mc}\left\{1+(V_0)_{vel.\ dep.}[\langle\omega_s(r_{34})\rangle + X_{vel.}\langle\omega_t(r_{34})\rangle] - \frac{4M(m+\frac{1}{2}h)}{3\hbar^2}\langle(V)_{static}(r_{34})r_{34}^2\rangle\right\}, \qquad (32)$$

$$(V)_{static}(r_{ij}) = (V_{eff})_{static}(r_{ij})/(\omega+m). \qquad (33)$$

(It should be noted that $(V)_{static}(r_{ij})$ is not the average of triplet even and singlet even static nucleon-nucleon forces as was assumed by Srivastava and Jain, see eqs. (3) and (10) of ref. [3]), but it is in fact the spatial part of the triplet static force.) The expectation values are easily evaluated with the appendix of Srivastava and Jain. Table 2 shows our results together with those of other authors for the standard Serber, Biel, Inglis and Rosenfeld [15]), for mixtures (where the singlet and triplet spatial forms are different and for the Inglis and Rosenfeld calculations we have assumed q to be 0.6).

4. Results and discussion

We note from table 2 that our calculated values of σ_{int} are sensitive to the exchange mixture of the potential and give good agreement with the experimental data of Gorbunov and Spiridonov only for the Serber and Biel forces. This suggests that the trial function used in this paper is capable of giving an adequate description of the ground state of ^{4}He except in a small region of configuration space. Since correlations do not influence the σ_{int} value too much, it is reasonable to expect that the values of σ_{int} obtained from our velocity-dependent potential are excellent approximations to those which would be obtained with correlated functions. Even allowing for the small increment which our experience with short-range correlations in the trinucleon would suggest, the agreement between calculated and experimental values of σ_{int} will still remain. Although the large uncertainties in the experimental data are as yet too great to yield a critical test, it is our belief that the methods developed here can lead to improved results. Our success in this investigation confirms the conclusion of Davey and Valk that it should be possible to fit all of the photonuclear cross sections in the 1s shell with one form of the nucleon-nucleon interaction, i.e. the velocity-dependent potential with Serber exchange character. It can also be concluded that within experimental error the charge distributions as measured by high-energy electron scattering is the same charge distribution as gives rise to electric dipole absorption. The introduction of short-range correlations into product form wave

functions now appears feasible since the integrals encountered can be simplified using the recent technique of Roberts [16]). This is now being investigated.

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