Wavelength Assignment for Parallel FFT Communication Pattern on Linear Arrays by Lattice Embedding[*]

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Abstract

Fast Fourier Transform(FFT) represents a common communication pattern shared by a large class of scientific and engineering problems and wavelength assignment is a key issue to increase efficiency and reduce cost in Wavelength Division Multiplexing (WDM) optical networks. In this paper, we propose a new scheme for the wavelength assignment of parallel FFT communication pattern on WDM linear arrays. By lattice embedding, the number of wavelengths required to realize parallel FFT communication pattern on WDM linear arrays significantly improves the known result. Our proposed embedding method also provides a new approach to the hypercube layout problem considering connections dimension by dimension rather than all connections as in the traditional approach.

1. Introduction

Fast Fourier Transform (FFT) plays an important role in numerous scientific and technical applications. While the application fields of FFT are growing rapidly, the amount of data to be transformed is also increasing tremendously. Hence, there has been a great interest in implementing FFT on parallel computers and some parallel computers have been specially designed to perform FFT computations [2]. With the increasing computation power of parallel computers, interprocessor communication has become an important factor that limits the performance of supercomputing systems. Optical communication, in particular, Wavelength Division Multiplexing (WDM) technique, has become a promising technology for many emerging networking and parallel/distributed computing applications because of its huge bandwidth [7].

As we know, parallel FFT is often implemented on dense interconnection networks such as hypercube and shuffle-exchange networks due to its topological properties [2].

Since WDM divides the bandwidth of an optical fiber into multiple wavelength channels so that multiple devices can transmit on distinct wavelengths through the same fiber concurrently, these dense networks can be simplified to simple regular topologies by realizing connections of parallel FFT communication patterns in optical lightpaths.

To efficiently utilize the bandwidth resources and to eliminate the high cost and bottleneck caused by opto-electronic conversion and processing at intermediate nodes, end-to-end lightpaths are usually set up between each pair of source-destination nodes. A connection or a lightpath in a WDM network is an ordered pair of nodes (*x, y*) corresponding to that a packet is sent from source *x* to destination *y*. In this paper, we assume that no wavelength converter facility is available in the network. Thus, a connection must use the same wavelength throughout its path. In this case, the lightpath is said to satisfy the *wavelength-continuity constraint*.

RWA tries to minimize the number of channels to realize a communication requirement by taking into consideration both routing options and channel assignment options which can be described as follows [6]: Given a set of all-optical connections, the problem is to **(a)** find routes from the source nodes to their respective destinations, and **(b)** assign channels to these routes so that the same channel is assigned to all the links of a particular route. **(c)** The goal of RWA is to minimize the number of assigned channels. Numerous research studies have been conducted on the RWA problem. A review of routing and wavelength assignment approaches for wavelength-routed optical WDM networks was given in [8]. In [6], optimal routing and channel assignments for hypercube communication on optical mesh-like processor arrays were studied. In [5], multicasting in multi-hop optical WDM networks with limited wavelength conversion was surveyed. In [4], routing and wavelength assignment for parallel LU decomposition communication pattern on WDM ring was studied.

In [3], the problem of wavelength assignment for realizing parallel FFT communication pattern on a class of regu-

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lar optical WDM networks was addressed and two methods, sequential mapping and shift-reversal mapping, were proposed. In this paper, we design a new scheme for the wavelength assignment of parallel FFT communication pattern on WDM linear arrays. By lattice embedding, the number of wavelengths required to realize parallel FFT communication pattern on WDM linear arrays significantly improves the known result in [3]. Our results have a clear significance for applications because FFT represents a common communication pattern shared by a large class of scientific and engineering problems and WDM optical networks as a promising technology in networking has an increasing popularity. Our proposed embedding method also provides a new approach to the hypercube layout problem considering connections dimension by dimension rather than all connections as in the traditional approach.

The rest of this paper is organized as follows. In Section 2, we define the problem of wavelength assignment for parallel FFT communication pattern on WDM linear arrays. We then design lattice embedding and provide some results of wavelength assignment for parallel FFT communication pattern on linear arrays in Sections 3. Finally, we conclude this paper in Section 4.

2. Problem Definition

FFT developed by Cooley and Tukey in the mid-60s is a method of computing the discrete Fourier transform which reduces the number of operations for an N-point complex vector from $O(N^2)$ to $O(Nlog_2N)$. The data-flow graph induced by an N-point FFT computation is usually described by means of the so-called butterfly representation [2]. The butterfly representation of FFT algorithm is a diagram made up of blocks representing identical computational units (butterflies) connected by arrows that show the flow of data between the blocks. Assuming that N is the length of the sequence to be transformed (N is an integer power of two), then the diagram with $N(log_2N + 1)$ nodes arranged in N rows and $log_2N + 1$ columns is made of log_2N stages of $N/2$ butterflies each.

Generally, FFT is implemented stage by stage, i.e. any stage of calculation cannot proceed until all the results of its previous stage have been completed. In this paper, we consider one dimensional data sequence of size $N = 2^n$. If each data is assigned a binary representation, the communications during the ith ($1 \leq i \leq n$) stage of the butterfly must take place between the nodes whose binary representations differ in the ith bit. If the butterfly representation is viewed as a process graph, i.e. each row of the butterfly is implemented by a process and each arrow by a communication channel, it is apparent that the butterfly communication pattern can map onto a WDM hypercube perfectly those links connecting the nodes having an address that differs by only one bit at each stage. However, if a WDM hypercube is used, only the ith dimensional links are used with one wavelength during the ith stage whereas other $(n-1) \times 2^{n-1}$ links are vacant during this stage, which may lead to wasting of wavelength channels. As we know, a connection in the hypercube communication pattern is called a dimensional i connection [6] if it connects two nodes that differ in the ith bit position, where $1 \leq i \leq n$. In a network of size 2^n, the set DIM_i is defined as the set of all dimension i connections and H_n is defined as the hypercube communication pattern which contains all connections in the hypercube. That is, $H_n = \bigcup_{i=1}^{n} DIM_i$ and $DIM_i = \{(j, j + (-1)^{\lfloor j/2^{n-i} \rfloor} \times 2^{n-i}) | 0 \leq j \leq 2^n - 1\}$. With input data distributed on processors, the set of all communications during n stages of parallel FFT is equivalent to H_n, and the set of communications during the ith stage is equivalent to DIM_i. Clearly, parallel FFT has a regular communication pattern which we denote by $FFT_n(n \geq 2)$.

We model a network as a directed graph $G(V, E)$. Nodes in V are switches and edges in E are links. In general, an optical WDM network consists of routing nodes interconnected by point-to-point fiber links, which can support a certain number of wavelengths. In this paper, we assume each link in the network is bidirectional and composed of a pair of unidirectional links with one link in each direction. For FFT_n, if $(x, y) \in FFT_n$, then $(y, x) \in FFT_n$. Assuming that these two communications can be realized by two lightpaths in the same path of opposite directions passing through different fiber links, the same wavelength can be assigned to these two lightpaths. In this case, we can ignore the problem of communication directions in FFT_n.

It should be noted that $FFT_n \neq H_n$ because the number of wavelengths required to realize FFT_n on the WDM optical networks is not equal to that required to realize hypercube communications of H_n. Since the n stages of parallel FFT should be implemented stage by stage, the number of wavelengths required to realize FFT_n on optical WDM networks is the maximum number among the wavelengths required by the n stages. Let $W_e(P, line)$ denote the number of wavelengths required to realize communication pattern P on linear arrays by embedding scheme e. Thus, $W_e(FFT_n, line) = \max_{1 \leq i \leq n} (W_e(DIM_i, line))$.

The problem of wavelength assignment for FFT_n on WDM linear arrays can be regarded as the problem of embedding the hypercube communications on linear arrays, such that the maximum congestion of dimensional i connections, for $1 \leq i \leq n$, is minimized. This problem arising from the wavelength assignment of FFT_n, which focuses on minimizing the congestion of each dimensional connections, is different from the traditional embedding of hypercube on linear arrays to minimize congestion which considers all the connections on hypercube.
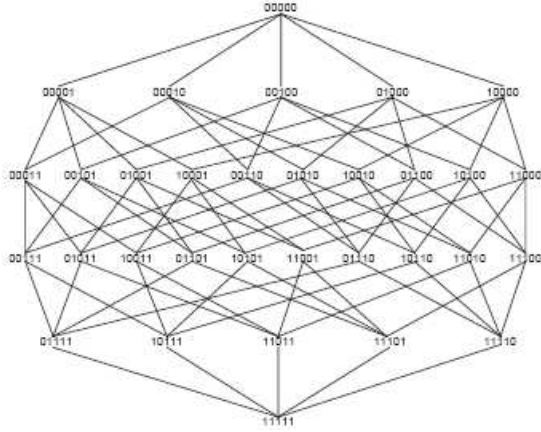
Figure 1. H_5 represented as a lattice

3. Lattice Embedding of Parallel FFT on Linear Arrays

3.1. Lattice Embedding

We design a new embedding scheme on linear arrays, lattice embedding, which is based on the lattice form of hypercube. Firstly, we introduce the following properties of hypercube in a form of lattice.

Hypercube can be represented in a form of lattice, as shown in Figure 1. As we know, for the hypercube with 2^n nodes, each node corresponds to an n-bit binary representation, and two nodes are linked with a connection if and only if their binary representations differ in precisely one bit. For the lattice form of 2^n-node hypercube, there are $n+1$ node rows connected by the hypercube connections. If the rows are numbered from top to bottom in ascending order starting from 0, the binary representations of the nodes on the kth row, for $0 \leq k \leq n$, have k 1s and $n - k$ 0s. In addition, hypercube connections only exist between the nodes on two neighborhood rows. Such a lattice form of hypercube also has the following properties:

Property 1: Hypercube with 2^n nodes can be represented in a form of lattice with $\binom{n}{k}$ nodes on the kth row for $0 \leq k \leq n$.

Proof: As the number of nodes whose binary representations have k 1s among n bits is $\binom{n}{k}$, it is easy to know the number of nodes on the kth row of the hypercube lattice is $\binom{n}{k}$.

Property 2: Hypercube with 2^n nodes can be represented in a form of lattice with $n \times \binom{n-1}{k}$ connections connecting the nodes of the kth row and the $(k+1)$th row, for $0 \leq k \leq n - 1$.

Proof: Since the number of 1s on the kth row and $(k+1)$th row are k and $k + 1$ respectively, node u on the kth row

connects with those nodes on the $(k+1)$th row whose binary representations have k 1s on the same positions with u. As the number of nodes on the $(k+1)$th row with k 1s on the same positions with u is $n - k$, each node on the kth row has $n - k$ connections with the nodes on the $(k+1)$th row. Therefore, the number of connections between the nodes of the kth row and $(k+1)$th row is $(n - k) \times \binom{n}{k} = n \times \binom{n-1}{k}$.

Property 3: Hypercube with 2^n nodes can be represented in a form of lattice with $\binom{n-1}{k}$ dimensional i connections, for $1 \leq i \leq n$, connecting the nodes of the kth row and the $(k + 1)$th row for $0 \leq k \leq n - 1$.

Proof: If node u is on the kth row whose ith bit is 0, then u must connect with one of the nodes on the $(k+1)$th row by dimensional i connection. It is easy to know the number of nodes on the kth row, whose binary representations have k 1s and ith bit is 0, is $\binom{n-1}{k}$. Therefore, the number of dimensional i connections, for $1 \leq i \leq n$, connecting the nodes of the kth row and the $(k + 1)$th row is $\binom{n-1}{k}$.

It can be seen from the above properties that the number of dimensional i connections is identical with each $1 \leq i \leq n$.

Assume that the nodes of WDM linear arrays are numbered from left to right in ascending order starting from 0, and that the links are numbered from left to right starting from 1. Define each node numbering of H_n by a function $\eta : V_{H_n} \mapsto \{0, 1, ..., 2^n - 1\}$ which is a one-to-one mapping from the nodes of H_n to the nodes of an 2^n-node linear array. Let $R(u) = k$ if node u is on the kth row of hypercube lattice. Embed the nodes of the 2^n-node hypercube lattice from row 0 to row n onto the 2^n-node linear array from left to right node by node. That is to say, if $R(u) < R(v)$, then $\eta(u) < \eta(v)$. We call the above embedding way *lattice embedding*. Let $e^n(k) = \sum_{i=0}^{k} \binom{n}{i} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{k}$. By the definition of lattice embedding, the node on the row 0 of hypercube lattice is mapped onto node 0 of the linear array, and nodes on the kth row of 2^n-node hypercube lattice, for $1 \leq k \leq n$, are mapped between node $e^n(k - 1)$ and node $e^n(k) - 1$ on the 2^n-node linear array. That is to say, if $R(u) = k$, then $e^n(k-1) \leq \eta(u) \leq e^n(k) - 1$.

3.2. Wavelength Assignment of Parallel FFT on Linear Arrays

Let ω_l^{ij} be the number of wavelengths required by lattice embedding to realize FFT_n on the jth link of the linear arrays during the ith stage and ω_l^i be the maximum number of wavelengths required among all the links of the linear arrays during the ith stage. Therefore, the number of wavelengths required to realize FFT_n by lattice embedding, denoted by ω_l, is

$$\omega_l = \max_{1 \leq i \leq n} \omega_l^i = \max_{1 \leq i \leq n} \left(\max_{1 \leq j \leq 2^{n-1}} \omega_l^{ij} \right)$$

By lattice embedding, the following results can be obtained.

Theorem 1: By lattice embedding, the number of wavelengths required to realize FFT_n is not less than $\binom{n-1}{\lfloor(n-1)/2\rfloor}$.

Proof: By Property 2 and 3, all the hypercube connections are only between the nodes of the kth row and the $(k+1)$th row with $\binom{n-1}{k}$ dimensional i connections, for $0 \leq k \leq n-1$ and $1 \leq i \leq n$. So, the number of dimensional i connections passing through link $e^n(k)$ are $\binom{n-1}{k}$. Therefore, the number of wavelengths required to realize FFT_n by lattice embedding satisfies, $\omega_l = \max_{1\leq i\leq n}\omega_l^i = \max_{1\leq i\leq n}(\max_{1\leq j\leq 2^{n-1}}\omega_l^{ij}) \geq \max_{0\leq k\leq n-1}\binom{n-1}{k} = \binom{n-1}{\lfloor(n-1)/2\rfloor}$.

Theorem 2: By lattice embedding, the number of wavelengths required to realize FFT_n is not more than $\binom{n}{\lfloor n/2\rfloor}$.

Proof: On the kth row of the hypercube lattice for $1 \leq k \leq n-1$, there are $\binom{n-1}{k-1}$ nodes, whose ith bit is 1, connecting the nodes on the $(k-1)$th row by dimensional i connections and $\binom{n-1}{k}$ nodes, whose ith bit is 0, connecting the nodes on the $(k+1)$th row by dimension i connections. Therefore, the number of dimensional i connections passing thought the links from $e^n(k-1)+1$ to $e^n(k)-1$ is not more than the sum of the maximum number of dimensional i connections between the kth row with its two neighborhood rows. Therefore, the number of wavelengths required to realize FFT_n satisfies, $\omega_l = \max_{1\leq i\leq n}(\max_{1\leq j\leq 2^{n-1}}\omega_l^{ij}) \leq \max_{1\leq k\leq n-1}(\binom{n-1}{k}+\binom{n-1}{k-1}) = \max_{1\leq k\leq n-1}\binom{n}{k} = \binom{n}{\lfloor n/2\rfloor}$

As $n! \sim \sqrt{2\pi n}(n/e)^n$ for large n according to Stirling's Formula, it can be calculated from Theorem 1 and 2 that realizing FFT_n requires $O(2^n/\sqrt{n})$ wavelengths for large n. It can be concluded that lattice embedding outperforms the known embedding approaches in [3]. Theorem 1 and Theorem 2 give a rough estimation of the number of wavelengths required to realize FFT_n on linear arrays. Furthermore, we discuss a computation method for the lattice embedding.

Theorem 3: The minimum number of wavelengths required to realize FFT_n on 2^n-node linear array by lattice embedding is $\binom{n-1}{\lfloor(n-1)/2\rfloor}+1$.

Proof: If $e^n(k-1) \leq j \leq e^n(k)-1$ for some $1 \leq k \leq n-1$, let ω_{l1}^{ij} be the number of dimensional i connections between nodes in $U = \{u|e^n(k-1) \leq \eta(u) \leq j-1\}$ and nodes in $U' = \{u'|R(u')=k+1\}$, and ω_{l2}^{ij} be the number of dimensional i connections between nodes in $V = \{v|j \leq \eta(v) \leq e^n(k)-1\}$ and nodes in $V' = \{v'|R(v')=k-1\}$. As the connections only take place between the neighborhood rows, the number of dimension i connections passing through the jth link on the linear array is $\omega_{l1}^{ij}+\omega_{l2}^{ij}$. Define $\eta^{-1}(j)$ as the node of FFT_n which is mapped onto the jth node of the linear array. Let $\theta_i(u)=1$ if the ith

bit of u is 1 and $\theta_i(\neg u)=1$ if the ith bit of u is 0. From the properties of hypercube lattice, it is easy to know that u connects with a node in $V' = \{v'|R(v')=k-1\}$ by dimensional i connection if $\theta_i(u)=1$, or with a node in $U' = \{u'|R(u')=k+1\}$ by dimensional i connection if $\theta_i(\neg u)=1$. Therefore, $\omega_{l1}^{ij} = \sum_{u=\eta^{-1}(e^n(k-1))}^{\eta^{-1}(j-1)}\theta_i(\neg u)$ and $\omega_{l2}^{ij} = \sum_{u=\eta^{-1}(j)}^{\eta^{-1}(e^n(k)-1)}\theta_i(u)$.

As the number of nodes whose ith bit is 0 in $U \bigcup V$ is $\binom{n-1}{k}$, and that in U is ω_{l1}^{ij}, the number of nodes in V whose ith bit is 1 can also be calculated by $\omega_{l2}^{ij} = (e^n(k)-j)-(\binom{n-1}{k}-\omega_{l1}^{ij}) = e^n(k)-\binom{n-1}{k}+\omega_{l1}^{ij}-j$.

Therefore, the number of wavelengths required to realize FFT_n on linear arrays can be calculated by the following equation: $\omega_l = \max_{1\leq i\leq n,1\leq k\leq n-1}(\max_{e^n(k-1)\leq j\leq e^n(k)-1}(\omega_{l1}^{ij}+\omega_{l2}^{ij})) = \max_{1\leq i\leq n,1\leq k\leq n-1}(\max_{e^n(k-1)\leq j\leq e^n(k)-1}(2\omega_{l1}^{ij}+e^n(k)-\binom{n-1}{k}-j)) = \max_{1\leq i\leq n,1\leq k\leq n-1}(\max_{e^n(k-1)\leq j\leq e^n(k)-1}(2\omega_{l1}^{ij}+e^n(k)-\binom{n-1}{k}-j))$.

For n is even, $\omega_l = \max\{2\sum_{u=\eta^{-1}(e^n(n/2-1))}^{\eta^{-1}(j-1)}\theta_i(\neg u)+e^n(n/2)-\binom{n-1}{n/2}-j : 1 \leq i \leq n, e^n(n/2-1) \leq j \leq e^n(n/2)-1\}$. Since each node has n/2 0s on the $(n/2)$th row, $\sum_{i=1}^{n}\sum_{u=\eta^{-1}(e^n(n/2-1))}^{\eta^{-1}(j-1)}\theta_i(\neg u) = \frac{n}{2}\times(j-e^n(n/2-1))$. So, $\max_{1\leq i\leq n}(\sum_{u=\eta^{-1}(e^n(n/2-1))}^{\eta^{-1}(j-1)}\theta_i(\neg u)) \geq \lceil\frac{\frac{n}{2}\times(j-e^n(n/2-1))}{n}\rceil$. By the above equations, it can be calculated that $\omega_l \geq \max\{2\times\lceil\frac{\frac{n}{2}\times(j-e^n(n/2-1))}{n}\rceil+e^n(n/2)-\binom{n-1}{n/2}-j, e^n(n/2-1) \leq j \leq e^n(n/2)-1\} = \binom{n-1}{n/2-1}+1$.

Similarly, it can be calculated that for n is odd, $\omega_l \geq \binom{n-1}{(n-1)/2}+1$.

From the proving of Theorem 3, it can be observed that the minimum number of wavelengths required by lattice embedding can be achieved if the 0s are distributed as much as evenly among the n bits of the number of 0s on the nodes before each node j for $e^n(n/2-1) \leq j \leq e^n(n/2)-1$.

For n is even, the number of 0s of the nodes in $U = \{u|e^n(n/2-1) \leq \eta(u) \leq e^n(n/2)-1\}$ is $n/2$. If u is in U, $\neg u$ must be in U. So, $U = \{u_i \bigvee \neg u_i|1 \leq i \leq \binom{n}{n/2}/2\}$. Thus, there are $\binom{n}{n/2}/2$ such node pairs in U. The minimum number of wavelengths required by lattice embedding can be archived by mapping the nodes in U on the 2^n-node linear array pair by pair. That is to say, the number of wavelengths required is $\binom{n-1}{n/2-1}+1$ if we map $u_1,\neg u_1,u_2,\neg u_2,...,u_{\binom{n}{n/2}/2},\neg u_{\binom{n}{n/2}/2}$ onto the node from $e^n(n/2-1)$ to $e^n(n/2)-1$ respectively. For example, the number of wavelengths required to realize FFT_4 on 16-node linear array is 4, if the nodes 0011, 1100, 0101, 1010, 1001, 0110 on FFT_4 are mapped onto the
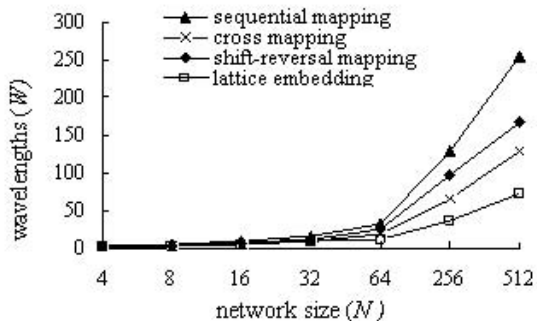
Figure 2. Comparisons on the number of wavelengths

nodes of 5,6,7,8,9,10 on the 16-node linear array respectively. The minimum number by lattice embedding can be discussed similarly for *n* is odd. For example, if n=5, the nodes of FFT_5 which are mapped onto the nodes of 6,7,8,9,10,11,12,13,14,15 on the 32-node linear array can be 00011, 01100, 10001, 00110, 11000, 00101, 01010, 10100, 01001, 10010 respectively.

It can be seen that lattice mapping outperforms the known embedding approaches to realize FFT communication pattern on linear arrays on the number of wavelengths, as shown in Figure 2.

4. Concluding Remarks

In this paper, we propose a new scheme for the wavelength assignment of parallel FFT communication pattern on WDM linear arrays. By lattice embedding, the number of wavelengths required to realize parallel FFT communication pattern on WDM linear arrays significantly improves the known result in [3]. Our proposed embedding method also provides a new approach to the hypercube layout problem considering connections dimension by dimension rather than all connections as in the traditional approach. Our results have a clear significance for applications because FFT represents a common communication pattern shared by a large class of scientific and engineering problems and WDM optical networks as a promising technology in networking has an increasing popularity.

Since there are different communication patterns according to different parallel algorithms, how to realize these communication patterns on optical networks is a hot research field. Future work may include other type of optical networks and other RWA problems. More researches on RWA considering various parallel communication patterns for parallel algorithms may be a worthwhile effort.

Another interesting issue is to find the number of wave-

lengths required on other types of WDM optical networks such as ring, mesh and torus based on the result in this paper. An open problem is to find the lower bounds for this problem and the improving schemes which can achieve the lower bounds.

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