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Scaling analysis of fat-link irrelevant clover fermion actions

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The fat-link irrelevant clover fermion action is a variant of the $O(a)$-improved Wilson action where the irrelevant operators are constructed using smeared links. While the use of such smearing allows for the use of highly improved definitions of the field strength tensor $F_{\mu\nu}$, we show that the standard 1-loop clover term with a mean field improved coefficient $c_{\rm sw}$ is sufficient to remove the $O(a)$ errors, avoiding the need for nonperturbative tuning. This result enables efficient dynamical simulations in QCD with the fat-link irrelevant clover fermion action

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I. INTRODUCTION

The fat-link irrelevant clover (FLIC) fermion action [1] is an efficient [2] Wilson-style nearest-neighbor lattice fermion action which incorporates both the thin gauge-field links of the Markov chain and fat links—links created via APE [3–6], HYP [7], or stout-link [8] smearing. Through the use of fat links in the irrelevant operators of the action, one achieves significant improvement in the chiral properties of the action reflected in a narrowing of the distribution of the critical Wilson mass [9]. One also bypasses the fine-tuning problem typically encountered in $\mathcal{O}(a)$ improvement, as the use of fat links in both the irrelevant Wilson and clover terms suppresses the otherwise large renormalizations of the improvement coefficients. At the same time, short-distance physics is preserved completely in the action as the relevant operators are constructed with thin links.

Previous work [10] established the good scaling properties of the FLIC fermion action when a highly improved definition of the lattice field strength tensor $F_{\mu\nu}$ is used in the clover term. In this work we demonstrate that the use of the standard 1-loop definition of $F_{\mu\nu}$ with fat links in the clover term is sufficient to provide $O(a^2)$ scaling for FLIC fermions. The 1-loop variant has the advantage of maintaining a simple force term when performing the molecular dynamics portion of a hybrid Monte Carlo algorithm to generate dynamical configurations.

In Sec. II we highlight the essential features of the FLIC action with a particular emphasis on the various lattice field strength tensors used in the simulations. In addition the $SU(3)$-projection method used to create the fat links is outlined. In Sec. III we describe the methods used to obtain an accurate scale determination on each lattice considered. Simulation parameters and scaling results are presented in Sec. IV while correlation function properties are examined in Sec. V. Conclusions are summarized in Sec. VI.

II. FLIC FERMIONS

The FLIC fermion action [1] is a variant of the clover action where the irrelevant operators are constructed using smeared links [3,4], and mean field improvement [11] is performed. The key point is that short-distance physics is suppressed in the irrelevant operators. This allows an effective mean-field improved calculation of the clover coefficient, required to match the Wilson and clover terms such that $O(a)$ errors are eliminated [10]. Further, the improved chiral properties of FLIC fermion action allow efficient access to the light quark regime [9].

The FLIC operator is given by

$$D_{\rm FLIC} = \nabla\!\!\!\!/_{\rm mfi} + \tfrac{1}{2}(\Delta^{\rm fl}_{\rm mfi} - \tfrac{1}{2}\sigma\cdot F^{\rm fl}_{\rm mfi}) - m, \tag{1}$$

where the presence of fat (or smeared) links and/or mean field improvement has been indicated by the super- and subscripts. The mean field improved lattice gauge covariant derivative is defined by

$$\nabla\!\!\!\!/_{\rm mfi} = \sum_\mu \frac{1}{2u_0}\gamma_\mu(U_\mu(x)\delta_{x+\hat\mu,y} - U^\dagger_\mu(x-\hat\mu)\delta_{x-\hat\mu,y}), \tag{2}$$

and likewise the (smeared link) lattice Laplacian is such that

$$\Delta^{\rm fl}_{\rm mfi} = \sum_\mu 2 - \frac{1}{u_0^{\rm fl}}(U^{\rm fl}_\mu(x)\delta_{x+\hat\mu,y} + U^{\rm fl\dagger}_\mu(x-\hat\mu)\delta_{x-\hat\mu,y}). \tag{3}$$

We choose $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. For the clover term, one usually selects a standard one-loop $F_{\mu\nu}$,

$$F_{\mu\nu}(x) = -\frac{i}{2}(C_{\mu\nu}(x) - C^\dagger_{\mu\nu}(x)), \tag{4}$$

$$C_{\mu\nu}(x) = \tfrac{1}{4}(U_{\mu,\nu}(x) + U_{-\nu,\mu}(x) + U_{\nu,-\mu}(x) + U_{-\mu,-\nu}(x)), \tag{5}$$

where $U_{\mu,\nu}(x) = U_\mu(x)U_\nu(x+\hat\mu)U^\dagger_\mu(x+\hat\nu)U^\dagger_\nu(x)$ is the elementary plaquette in the μ, ν plane. However, with the use of fat links, one is also able to choose highly improved definitions of $F_{\mu\nu}$ [12]. Let $C^{m\times n}_{\mu\nu}(x)$ correspond to the sum of the four $m\times n$ loops at the point x in the clover formation, and then define

$$F_{\mu\nu}^{m\times n}(x) = -\frac{i}{2}(C_{\mu\nu}^{m\times n}(x) - C_{\mu\nu}^{\dagger m\times n}(x)). \tag{6}$$

We can construct a 2-loop field strength tensor which is free of $O(a^2)$ errors,

$$F_{\mu\nu}^{2\mathrm{L}} = \tfrac{5}{3}F_{\mu\nu}^{1\times 1} - \tfrac{1}{6}(F_{\mu\nu}^{1\times 2} + F_{\mu\nu}^{2\times 1}), \tag{7}$$

or a 3-loop version which is free of $O(a^4)$ errors,

$$F_{\mu\nu}^{3\mathrm{L}} = \tfrac{3}{2}F_{\mu\nu}^{1\times 1} - \tfrac{3}{20}F_{\mu\nu}^{2\times 2} + \tfrac{1}{90}F_{\mu\nu}^{3\times 3}. \tag{8}$$

The smeared links in the FLIC action can be equally well constructed from standard APE smeared links, or the more novel stout-link method [8]. As the smeared links only appear in the irrelevant operators, the physics of the action are essentially independent of the choice of smearing method. The only requirement is that sufficient smearing is done such that the mean field improvement becomes an effective means of estimating the clover coefficient c_{sw}. We typically find that four sweeps of APE smearing at $\alpha = 0.7$ or four sweeps of stout smearing at $\rho = 0.1$ to be sufficient for lattices with a spacing between 0.1 and 0.165 fm.

In this work we use APE smeared links $U_\mu^{\mathrm{fl}}(x)$ constructed from $U_\mu(x)$ by performing 4 smearing sweeps, where in each sweep we first perform an APE blocking step (at $\alpha = 0.7$),

$$V_\mu^{(j)}(x) = (1-\alpha) \quad + \frac{\alpha}{6}\sum_{\nu\neq\mu} \quad + \quad , \tag{9}$$

followed by a projection back into $SU(3)$, $U_\mu^{(j)}(x) = \mathcal{P}(V_\mu^{(j)}(x))$. We follow the "unit-circle" projection method given in [13], which allows for dynamical simulations. The projection is defined by first performing a projection into $U(3)$

$$U'(V) = V[V^\dagger V]^{-1/2}, \tag{10}$$

followed by projection into $SU(3)$

$$\mathcal{P}(V) = \frac{1}{\sqrt[3]{\det U'(V)}}U'(V). \tag{11}$$

It should be noted that the principal value of the cube root (being that with the largest real part) is the appropriate branch of the cube root function to choose. As noted in [13] this choice provides the mean link which is closest to unity.

Mean field improvement is performed by making the replacements

$$U_\mu(x) \to \frac{U_\mu(x)}{u_0}, \qquad U_\mu^{\mathrm{fl}}(x) \to \frac{U_\mu^{\mathrm{fl}}(x)}{u_0^{\mathrm{fl}}}, \tag{12}$$

where u_0 and u_0^{fl} are the mean links for the standard and fattened links. We calculate the mean link via the fourth root of the average plaquette

$$u_0 = \left\langle \tfrac{1}{3}\mathrm{Re}\,\mathrm{Tr} U_{\mu\nu}(x)\right\rangle_{x,\mu<\nu}^{1/4}. \tag{13}$$

III. SCALE DETERMINATION

The scale is determined using a 4-parameter ansatz

$$V(\mathbf{r}) = V_0 + \sigma r - e\left[\frac{1}{\mathbf{r}}\right] + l\left(\left[\frac{1}{\mathbf{r}}\right] - \frac{1}{r}\right) \tag{14}$$

as in Ref. [14]. The tree-level lattice Coulomb term used in the ansatz is given by

$$\left[\frac{1}{\mathbf{r}}\right] = 4\pi \int \frac{d^3\mathbf{k}}{2\pi^3}\cos(\mathbf{k}\cdot\mathbf{r})D_{00}(0,\mathbf{k}). \tag{15}$$

Here $D_{00}(0,\mathbf{k})$ comes from the tree-level gluon propagator for the appropriate gluon action. For the Wilson gluon action, we have at tree-level,

$$D_{00}^{-1}(0,\mathbf{k}) = 4\sum_{\mu=1}^{3}\sin^2\frac{k_\mu}{2}, \tag{16}$$

where on a lattice with extents L_μ the allowed momenta are

$$k_\mu = \frac{2\pi n_\mu}{L_\mu}, \qquad -\frac{L_\mu}{2} < n_\mu \le \frac{L_\mu}{2}. \tag{17}$$

For the Lüscher-Weisz gluon action, we have at tree-level,

$$D_{00}^{-1}(0,\mathbf{k}) = 4\sum_\mu\left(\sin^2\frac{k_\mu}{2} + \frac{1}{3}\sin^4\frac{k_\mu}{2}\right). \tag{18}$$

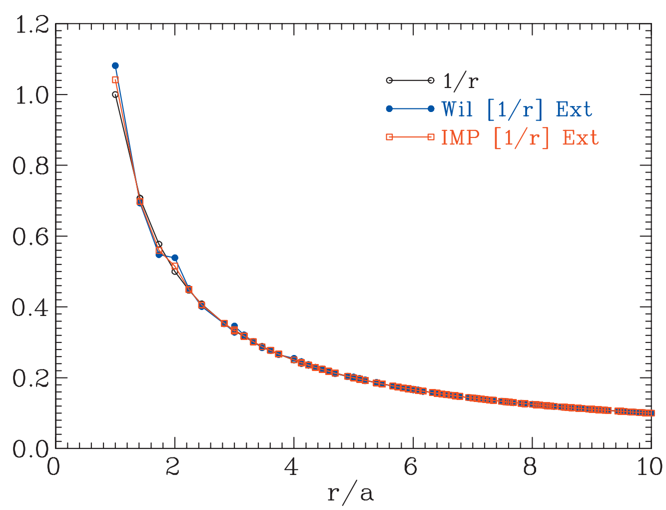

FIG. 1 (color online). The (infinite volume) tree-level lattice Coulomb term (in lattice units) for the Wilson and Lüscher-Weisz (IMP) gauge action.

The lattice Coulomb term is constructed by calculating on large lattice volumes and then extrapolating to infinite volume. Explicitly, we choose $L = 128$ and $L = 256$ and calculate $[\frac{1}{\mathbf{r}}]_L$ for an L^3 spatial volume. On a finite volume, the Coulomb term takes the form [15]

$$\frac{1}{r} + \frac{1}{L-r} = \frac{1}{r} + \frac{1}{L} + O\left(\frac{r}{L^2}\right). \tag{19}$$

In order to calculate the infinite volume tree-level lattice Coulomb term $[\frac{1}{\mathbf{r}}]$, we extrapolate $[\frac{1}{\mathbf{r}}]_L$ linearly in $\frac{1}{L}$ to $\frac{1}{L} = 0$.

The tree-level lattice Coulomb term $[\frac{1}{\mathbf{r}}]$ for the Wilson and Lüscher-Weisz gauge action is shown in Fig. 1. The important finite lattice spacing artefacts are revealed at small $r \lesssim 3a$. The $\mathcal{O}(a^2)$ improvement in the Lüscher-Weisz Coulomb term is also readily apparent.

IV. SCALING RESULTS

Calculations are performed on quenched mean-field improved plaquette plus rectangle $SU(3)$ Lüscher-Weisz lattices. Lattice spacings determined using fits to Eq. (14) above are given in Table I.

Gauge configurations are generated using the Cabibbo-Marinari pseudo heat-bath algorithm with three diagonal $SU(2)$ subgroups looped over twice. Simulations are performed using a parallel algorithm with appropriate link partitioning [16].

For each of the lattices we calculate quark propagators using the FLIC fermion action with a 1, 2 and 3-loop clover term as described in Sec. II. We use ensembles of 50 configurations at a lattice size of $16^3 \times 32$.

The π, ρ, and N masses are calculated and interpolated to a π/ρ mass ratio of 0.7, shown in Table II. The interpolation is performed by first calculating at two different quark masses that are close to but either side of the desired π/ρ mass ratio of 0.7 (typically the calculated mass have π/ρ ratios between 0.67–0.73). The ρ and N mass are then fitted linearly in m_π^2 (a valid ansatz in the heavy quark mass region that we are studying). Having obtained m_ρ and m_N as functions of m_π^2 it is then trivial to interpolate the appropriate value of m_π to achieve the desired π/ρ mass

TABLE I. The lattice spacing for pure Lüscher-Weisz glue determined by the string tension $\sqrt{\sigma} = 440$ MeV and the Sommer scale $r_0 = 0.49$ fm for various couplings β.

β	$a[\sigma]$(fm)	$a[r_0]$(fm)
4.60	0.120(1)	0.113(1)
4.53	0.132(1)	0.124(1)
4.38	0.164(1)	0.152(1)

TABLE II. Results for the N and ρ masses on the three lattices, for the scale determined by the string tension σ and the Sommer scale r_0.

	β	$M_\rho/\sqrt{\sigma}$	$M_N/\sqrt{\sigma}$	$M_\rho r_0$	$M_N r_0$
FLIC-1L	4.60	2.278(26)	3.347(33)	2.638(30)	3.875(39)
	4.53	2.313(27)	3.368(41)	2.662(31)	3.876(47)
	4.38	2.299(21)	3.323(32)	2.688(25)	3.886(38)
FLIC-2L	4.60	2.347(26)	3.394(33)	2.717(30)	3.929(39)
	4.53	2.390(30)	3.453(44)	2.751(35)	3.974(51)
	4.38	2.410(24)	3.450(35)	2.818(28)	4.034(41)
FLIC-3L	4.60	2.365(30)	3.461(37)	2.738(34)	4.006(43)
	4.53	2.413(37)	3.478(48)	2.776(43)	4.003(55)
	4.38	2.435(27)	3.474(38)	2.847(32)	4.062(44)

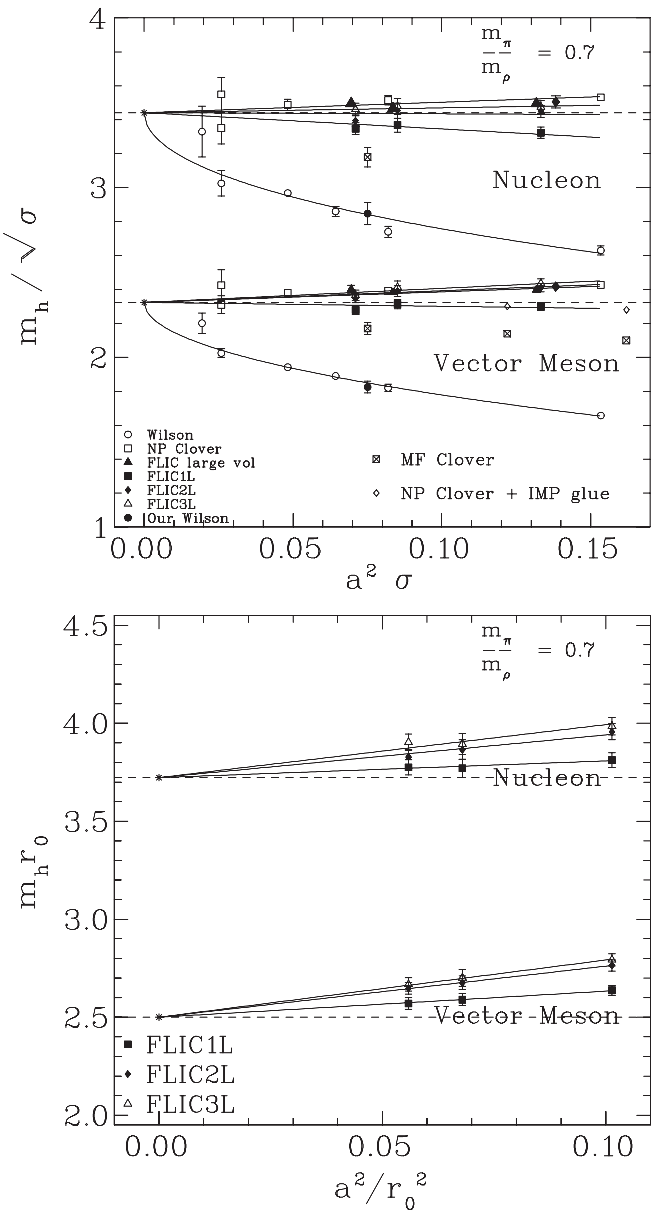

FIG. 2. The scaling of the N and ρ masses for various quark actions in the quenched approximation according to the string tension (upper) and the Sommer scale (lower).

TABLE III. Continuum extrapolated values for the ρ and N mass obtained from the fits, for the string tension (σ) and Sommer scale (r_0). Listed are the fit value, total χ^2, degrees of freedom and χ^2 per degree of freedom.

	Fit	χ^2	DoF	χ^2/DoF
$m_\rho/\sqrt{\sigma}$	2.32(2)	8.76	11	0.796
$m_N/\sqrt{\sigma}$	3.44(3)	13.4	11	1.22
$m_\rho r_0$	2.50(4)	0.0693	5	0.0139
$m_N r_0$	3.72(5)	0.980	5	0.195

ratio and obtain the corresponding values of the nucleon and ρ masses.

Scaling results are presented in Fig. 2. The lines of fit are extrapolations in a^2 constrained to pass through a single point at the continuum limit. As this is quenched QCD at nonphysical quark masses, we do not know *a priori* the continuum value of the ρ and N masses and thus the continuum values are fit parameters. The points labeled FLIC1L, FLIC2L and FLIC3L are the result of this work,

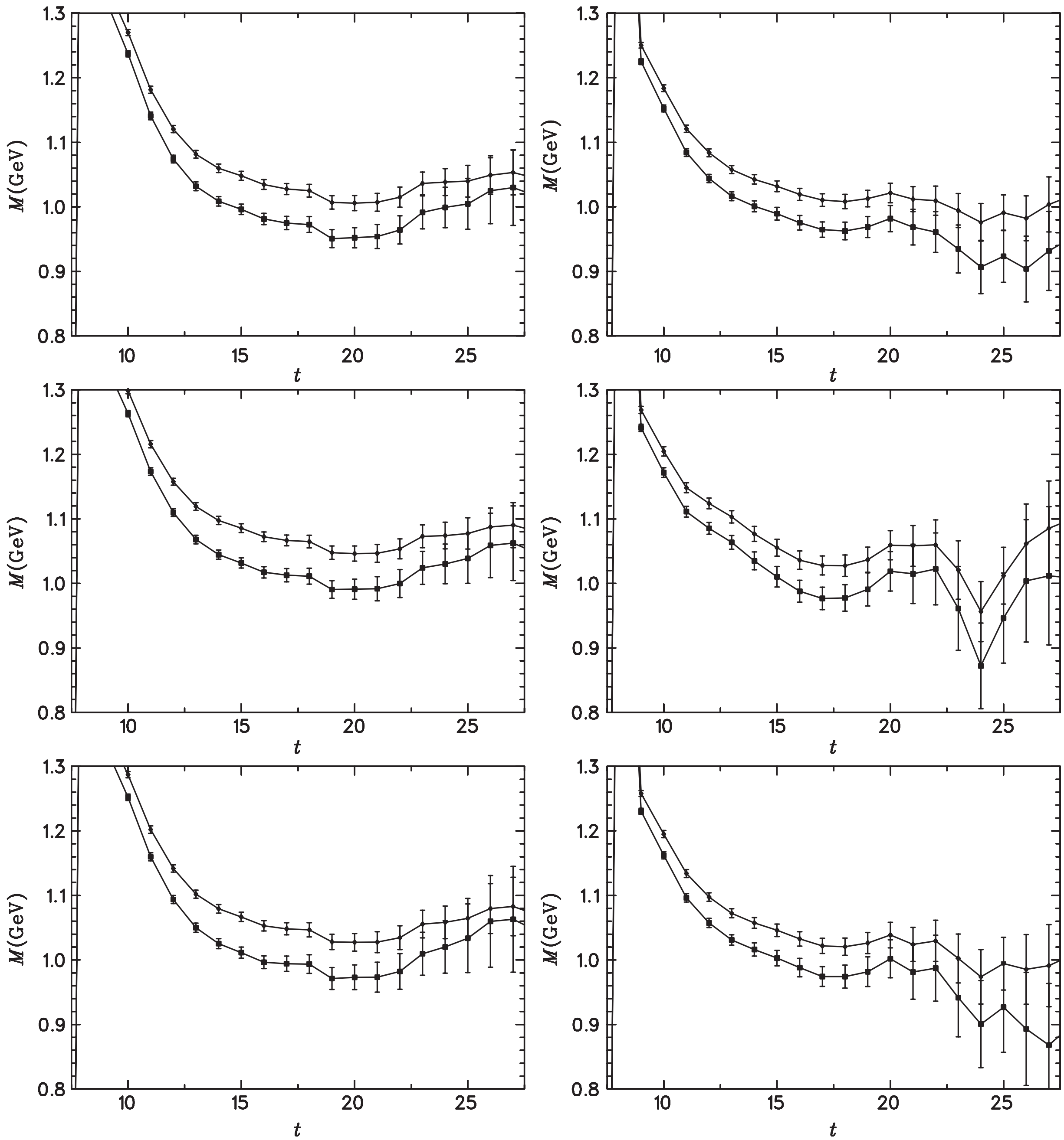

FIG. 3. ρ-meson effective mass functions at two approximately matched quark masses for the 1-loop (top), 2-loop (middle), and 3-loop (bottom) FLIC actions. Two lattices at $16^3 \times 32$ are shown, $\beta = 4.60$ (left) and $\beta = 4.53$ (right).

all other points are obtained from the literature or previous work [10].

Examining the string tension plot in Fig. 2, we can see from the Wilson and MF-clover points that the presence of any $O(a)$ errors causes significant deviation from the continuum result. Given the proximity of the FLIC results to the NP-clover results, we fit the FLIC results using an $O(a^2)$ term as the leading error. The subsequent extrapolated masses and corresponding χ^2 values are given in Table III. We successfully fit straight lines for the nonperturbatively improved clover action and all FLIC actions, indicating $O(a^2)$ scaling, that is the effective elimination of $O(a)$ errors.

The second plot in Fig. 2 shows the scaling of the three FLIC actions against the Sommer scale, r_0. This is interesting because r_0 is the preferred measure of setting the scale in dynamical simulations. The presence of the string breaking in full QCD makes using the string tension to set the scale undesirable. We see that on this plot the $O(a^2)$ errors are smallest for the 1-loop FLIC action for both the ρ and N masses.

Thus, 1-, 2- and 3-loop fat-link formulations of $F_{\mu\nu}$ in the FLIC fermion action all provide $O(a)$ improvement as expected. The different formulations differ at the level of $O(a^2)$. Remarkably, the 1-loop action is actually the preferred action. First, it is the cheapest to perform molecular dynamics with, which is important for Hybrid Monte Carlo dynamical simulations. Second, when using r_0 to set the scale, the 1-loop action has the smallest residual $O(a^2)$ errors in the quantities we have studied here. We will also see that correlation functions have smaller fluctuations.

V. CORRELATION FUNCTIONS

Finally, we compare the ρ-meson correlation function on the fine $\beta = 4.60$ and coarse $\beta = 4.53$ lattices at approximately matched pion masses for the three different FLIC actions. The source is at time slice 8. Two masses are calculated for each action on each lattice. The lighter mass has $m_\pi/m_\rho \approx 0.68$ and the heavier mass has $m_\pi/m_\rho \approx 0.72$.

The effective mass plots are given in Fig. 3. We note that the greater the degree of statistical fluctuation in the correlation function on a given time slice, the larger the error bar on the estimate of the central value at that point. The main effect that we observe from Fig. 3 is that as the Euclidean time index progresses into the latter half of the lattice, the 1-loop FLIC correlators show reduced fluctuations, as indicated by the reduced error bars when compared with the 2-loop and 3-loop FLIC results. The difference is particularly clear on the coarser $\beta = 4.53$ lattice. We understand this to be due to the 1-loop action having a more local field strength $F_{\mu\nu}(x)$ than the 2- and 3-loop actions making it less susceptible to large fluctuations. The reduced fluctuations in the more local action means that a larger range of time slices are potentially available to choose for a fit window.

VI. CONCLUSIONS

We have examined the role of improvement in the lattice field strength tensor of the FLIC fermion action. Our results demonstrate that the standard 1-loop choice of for the lattice clover term in the FLIC fermion action provides $O(a^2)$ scaling.

Remarkably the 1-loop action provides results that are preferable to those obtained from the 2-loop $\mathcal{O}(a^2)$-improved lattice field strength tensor or those obtained from the 3-loop $\mathcal{O}(a^4)$-improved definition. The 1-loop results provide

(1) Smaller residual $O(a^2)$ errors (using the r_0 scale),
(2) Stable hadron correlators with reduced fluctuations,
(3) Smaller statistical uncertainties, and
(4) A more efficient action suitable for dynamical fermion simulations.

This result enables efficient and effective dynamical QCD simulations with FLIC fermions. Simulations are currently under way.

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