PROCEEDINGS OF SCIENCE

Baryon resonances and hadronic interactions in a finite volume

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In a finite volume, resonances and multi-hadron states are identified by discrete energy levels. When comparing the results of lattice QCD calculations to scattering experiments, it is important to have a way of associating the energy spectrum of the finite-volume lattice with the asymptotic behaviour of the S-matrix. A new technique for comparing energy eigenvalues with scattering phase shifts is introduced, which involves the construction of an exactly solvable matrix Hamiltonian model. The model framework is applied to the case of $\Delta \to N\pi$ decay, but is easily generalized to include multi-channel scattering. Extracting resonance parameters involves matching the energy spectrum of the model to that of a lattice QCD calculation. The resulting fit parameters are then used to generate phase shifts. Using a sample set of pseudodata, it is found that the extraction of the resonance position is stable with respect to volume for a variety of regularization schemes, and compares favorably with the well-known Lüscher method. The model-dependence of the result is briefly investigated.

The 30 International Symposium on Lattice Field Theory - Lattice 2012,
June 24-29, 2012
Cairns, Australia

*Speaker.

PoS(Lattice 2012)145

1. Introduction

Lüscher's method [1] constitutes the principal method for relating the discrete energy spectrum calculated in lattice QCD with the continuous asymptotic states measured in hadron scattering experiments [2, 3, 4, 5, 6, 7, 8, 9]. By matching the asymptotic behaviour of the S-matrix onto the energy spectrum of the toroidal topology of the lattice, a geometric equation is obtained, which is valid only for the scattering of two well-separated particles in a finite volume. In extending to more complicated cases e.g. multi-channel scattering, the interpretation of Lüscher's method becomes more difficult; however, such a development remains a promising and challenging area of ongoing research [2, 10, 11, 12, 13].

An alternative method for identifying resonance parameters in finite-volume scattering is proposed, which has the compelling property of being easily generalized to include more complicated interactions and additional channels. The method involves the construction of a matrix Hamiltonian model in a finite volume, such that its eigenvalue equation matches directly onto chiral effective field theory in the low-energy limit.

As a test example, the matrix Hamiltonian approach is applied to $\Delta \to N\pi$ decay. An energy spectrum can be generated from the model, and these energy levels can be matched directly to those of a lattice QCD calculation, fitting the free parameters of the model. With these fit parameters, the position of the resonance pole may then be obtained from the standard methods of scattering theory. The robustness of this Hamiltonian technique is tested by generating a finite-volume energy spectrum as 'pseudodata', and matching this spectrum to that of an alternative version of the model. Thus, a picture of the model-dependence is developed.

2. The finite-volume matrix Hamiltonian model

Consider a matrix Hamiltonian model for the $\Delta N\pi$ interaction, such that the finite-volume energy spectrum can be solved exactly. The Hamiltonian may be written as separate free and interaction parts, $H = H_0 + H_I$, where the free Hamiltonian includes the energies of the pion-nucleon system

$$H_0 = \begin{pmatrix} \Delta_0 & 0 & 0 & \cdots \\ 0 & \omega_\pi(k_1) & 0 & \cdots \\ 0 & 0 & \omega_\pi(k_2) & \\ \vdots & \vdots & & \ddots \end{pmatrix}, \tag{2.1}$$

where $\omega_\pi(k_n) = \sqrt{k_n^2 + m_\pi^2}$, and Δ_0 is the bare mass of the Δ baryon. The rows and columns of H represent the momentum states of the pion relative to the nucleon. The values of momentum-squared available in a finite volume, L^3, are $k_n^2 = (\frac{2\pi}{L})^2(n_x^2+n_y^2+n_z^2) \equiv (\frac{2\pi}{L})^2 n$, where n is a squared integer.

Including a direct coupling to a Δ baryon, the interaction Hamiltonian takes the form

$$H_I = \begin{pmatrix} 0 & g_{\Delta N}^{\text{fin}}(k_1) & g_{\Delta N}^{\text{fin}}(k_2) & \cdots \\ g_{\Delta N}^{\text{fin}}(k_1) & 0 & 0 & \cdots \\ g_{\Delta N}^{\text{fin}}(k_2) & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.2}$$

The coupling, $g^{\rm fin}_{\Delta N}(k_n)$, is obtained from chiral effective field theory, and includes appropriate dimensional factors for a finite-volume calculation

$$g^{\rm fin}_{\Delta N}(k_n) = \sqrt{\frac{C_3(n)}{4\pi}}\left(\frac{2\pi}{L}\right)^{3/2} g_{\Delta N}(k_n) \tag{2.3}$$

$$= \sqrt{\chi_\Delta \frac{C_3(n)}{2\pi^2}}\left(\frac{2\pi}{L}\right)^{3/2} \frac{k_n u(k_n)}{\sqrt{\omega_\pi(k_n)}}, \tag{2.4}$$

$$\text{for} \quad \chi_\Delta = \frac{3}{32\pi f_\pi^2}\frac{2}{9}\mathscr{C}^2, \tag{2.5}$$

using $f_\pi = 92.4$ MeV, and the SU(6) value $\mathscr{C} = -1.52$. The normalization $C_3(n)$ represents the number of ways of summing three squared integers to equal n. The introduction of a regulator function, $u(k_n)$, into the model serves to keep the range of the interaction finite. In generating a sample set of pseudodata, a dipole regulator with a mass of $\Lambda = 0.8$ GeV is chosen, being well-matched to phenomenology in the nucleon-pion sector [14, 15, 16]. However, it will be demonstrated that the regularization scheme has limited impact on the consistency of the final extraction of the resonance position.

The eigenvalue equation of the Hamiltonian, $\det(H - \lambda\mathbb{I}) = 0$, takes the form

$$\lambda = \Delta_0 - \sum_{n=1}^{\infty} \frac{\left(g^{\rm fin}_{\Delta N}(k_n)\right)^2}{\omega_\pi(k_n) - \lambda} \tag{2.6}$$

$$= \Delta_0 - \frac{\chi_\Delta}{2\pi^2}\left(\frac{2\pi}{L}\right)^3 \sum_{n=1}^{\infty} C_3(n) \frac{k_n^2 u^2(k_n)}{\omega_\pi(k_n)[\omega_\pi(k_n) - \lambda]}, \tag{2.7}$$

and the lowest-lying energy levels from the discrete spectrum of eigenvalues, $\lambda = E_j$, are shown as a function of lattice box size in Fig. 1. Note that the formulae in Eqs. (2.6) & (2.7) match the one-loop $N\pi$ contribution to the Δ baryon self-energy in effective field theory near the pole position: $\lambda \simeq E_\Delta \equiv 292$ MeV.

At infinite volume, the real part of the one-pion loop integral takes the following form

$$\mathrm{Re}\,\Sigma_{\Delta N}(k) = \mathscr{P}\int_0^\infty \mathrm{d}k' \frac{k'^2 g^2_{\Delta N}(k')}{\omega_\pi(k) - \omega_\pi(k')} \tag{2.8}$$

$$= \chi_\Delta \frac{2}{\pi}\mathscr{P}\int_0^\infty \mathrm{d}k' \frac{k'^4 u^2(k')}{\omega_\pi(k')\left[\omega_\pi(k) - \omega_\pi(k')\right]}, \tag{2.9}$$

where $\mathscr{P}$ indicates that a principal value integral must be performed. This integral contributes to the phase shift via the t-matrix for elastic $N\pi$ scattering (with a Δ baryon intermediate). The relationship between the phase shift and the on-shell quantity, $T = t(k,k;E^+)$, is

$$T = \frac{g^2_{\Delta N}(k)}{\omega_\pi(k) - \Delta_0 - \Sigma_{\Delta N}(k)} = -\frac{1}{\pi k\,\omega_\pi(k)} e^{i\delta(k)} \sin\delta(k). \tag{2.10}$$

By solving Eq. (2.10), the phase shift, δ, may be plotted as a function of energy, $E = \omega_\pi(k)$, to obtain the curve shown in Fig. 2. The bare resonance mass, Δ_0, may be tuned so that the final resonance energy matches the physical ΔN mass-splitting, $E_\Delta \equiv 292$ MeV, and hence serves as an input for solving the finite-volume spectrum, via Eq. (2.1).

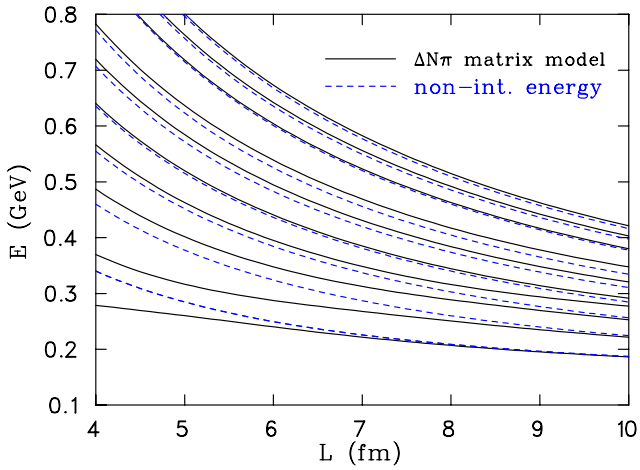

Figure 1: (color online). The lowest-lying energy levels from the $\Delta N\pi$ model (solid lines), and the corresponding non-interacting energies (dotted lines).

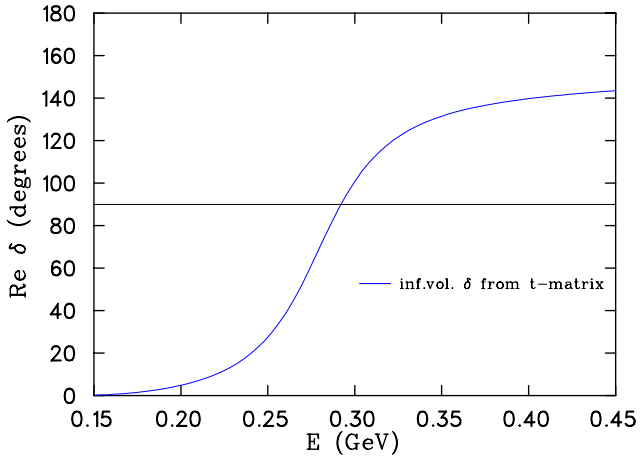

Figure 2: (color online). The infinite-volume phase shift, δ, associated with elastic $N\pi$ scattering via a Δ baryon intermediate state, plotted against the external pion energy, E (where $E_{\rm tot} = M_N + E$), as calculated from the on-shell t-matrix.

3. Lüscher's Method

Lüscher's formula describes a fixed relationship between the scattering phase shift, δ, and the energy levels in a finite volume

$$\delta(k_j;L) = j\pi - \phi\left(\frac{k_j L}{2\pi}\right), \tag{3.1}$$

where j is an integer indexing the energy levels, $E_j = \sqrt{k_j^2 + m^2}$. Lüscher's formula is derived assuming that two identical particles of mass m scatter from a finite-range interaction (potential $V(r) = 0$ for range $R < r$), and are well-separated ($R < r < L/2$), i.e. the particle wavefunction is in the asymptotic region. The angle function, $\phi(q)$, takes the form of a three-dimensional Zeta-like function (which must be regularized), defined in terms of dimensionless lattice momenta $q \equiv$

$kL/(2\pi)$

$$\phi(q) = -\arctan\left(\frac{\pi^{3/2}q}{\mathscr{Z}_{00}(1,q^2)}\right), \tag{3.2}$$

$$\mathscr{Z}_{00}(1,q^2) = \frac{1}{\sqrt{4\pi}}\sum_{\vec{n}\in\mathbb{Z}^3}\frac{1}{\vec{n}^2-q^2}. \tag{3.3}$$

The momenta corresponding to a lattice QCD spectrum, $k_j = \sqrt{E_j^2 - m_\pi^2}$, may be input into Lüscher's formula to obtain phase shifts, $\delta(k_j;L)$. Alternatively, the energy levels may be fit directly to the effective hadronic model described above, and the phase shift (and resonance position) can then be extracted from the t-matrix in Eq. (2.10). Using a set of pseudodata generated from the $\Delta N\pi$ model, the two methods will be compared in the following section.

4. An alternative method of phase shift extraction

Motivated by the general result that the potential is separable near a resonance [17], a new method is proposed for obtaining a phase shift from discrete energy levels, such as those of lattice QCD. Using the $\Delta N\pi$ model, the parameters χ_Δ and Δ_0, are chosen to minimize the chi-square between the energy levels of the model and the energy levels of a lattice QCD calculation. An estimate of the phase shift can then be calculated from the t-matrix formula of Eq. (2.10), using the fitted values of the parameters.

In order to test this idea, energy levels for the $\Delta N\pi$ model are treated as pseudodata. A modified version of the model is then constructed using a different regulator function, $\tilde{u}(k_n)$, such as a Gaussian regulator. By matching the two sets of energy levels and obtaining fit values for χ_Δ and Δ_0, phase shift estimates may be calculated for a range of box sizes, L.

In order to visualize the comparison between Lüscher's method and the new method, the behaviour of the resonance energy, E_{res}, may be plotted as a function of $1/L$, as shown in Fig. 3. Using Lüscher's method, an interpolation function must be chosen in order to obtain the pole position from the phase shift. In the alternative method, two energy eigenvalues are chosen from the pseudodata, which are closest to the resonance energy, as estimated by Lüscher's formula. These eigenvalues are then used to constrain the parameters χ_Δ and Δ_0. Evidently, matching the pseudodata to a model with a different regulator function leads to a result that is at least comparable with Lüscher's method.

By varying the regularization scale, Λ, a systematic uncertainty of only a few MeV is observed. This is encouraging, because it suggests that approximating the underlying physics of a lattice calculation with a regulator, sensibly chosen, will lead to a result that is not highly dependent on the features of the particular regulator function. Furthermore, one may treat Λ as an additional fit parameter, and in this case, the closest three eigenvalues from the pseudodata are chosen for fitting. Fig. 3 indicates that a Gaussian regulator parameter of $\Lambda \simeq 0.6$ GeV provides the best matching with the pseudodata.

The result of using the $\Delta N\pi$ model with regularization scale removed (i.e. $\Lambda \to \infty$) is also displayed in Fig. 3. The pole extraction closely resembles that of Lüscher's method.

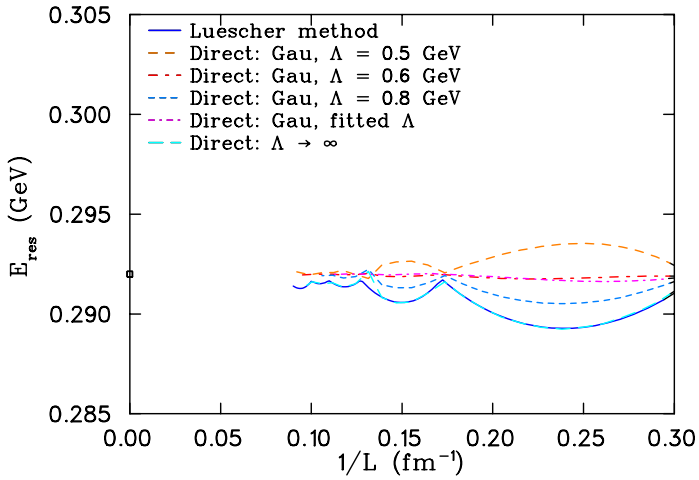

Figure 3: (color online). The resonance energy, E_{res}, plotted against $1/L$. The experimental value is marked with a square. The results from fitting pseudodata with a different regulator, a Gaussian with $\Lambda = 0.5$, 0.6 and 0.8 GeV, or treating Λ as a fit parameter, are plotted. The result of effectively removing the regulator (i.e. $\Lambda \to \infty$) is also shown. For comparison, the approach using Lüscher's method is marked with a solid line.

5. Summary

An alternative method for the extraction of resonance parameters in a finite volume is investigated. An exactly solvable matrix Hamiltonian is constructed to model $\Delta \to N\pi$ decay in a finite-volume, in anticipation of generalizing to more complicated multi-channel scattering problems. By matching the energy levels of the model to those of a lattice QCD calculation, the parameters of the model can be input into effective field theory in order to generate phase shifts. The model is tested by generating pseudodata, and extracting the resonance position using an alternative form of the model. The results are comparable with Lüscher's method, and the extraction of the phase shift is stable with respect to volume.

6. Acknowledgements

We would like to thank M. Döring and J. Dudek for helpful conversations. This research was supported by the Australian Research Council through the ARC Centre of Excellence for Particle Physics at the Terascale, and through grants DP110101265 (DBL & RDY), FT120100821 (RDY) and FL0992247 (AWT).

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